Tôhoku Math. Journ.
39 (1987), 365–372.

TORSION AND DEFORMATION OF CONTACT METRIC STRUCTURES ON 3-MANIFOLDS

SAMUEL I. GOLDBERG[1] AND GABOR TOTH

(Received June 9, 1986)

Abstract. S.-S. Chern raised the question of determining those compact 3-manifolds M admitting a contact metric structure whose characteristic vector field generates a one-parameter group of isometries. S. Tachibana showed that the first betti number of these spaces must be even, and H. Sato proved that the second homotopy group of M is zero unless M is homotopy equivalent to $S^1 \times S^2$. A. Weinstein pointed out that M is a Seifert fibre space over an orientable surface. In this paper, it is shown as a consequence of a more general theorem that if, in addition, the scalar curvature is suitably bounded below by a negative constant, then the metric may be deformed to a metric of positive constant sectional curvature. Thus, if the manifold is simply connected it is diffeomorphic with the 3-sphere.

1. Introduction. Lutz and Martinet [6] showed that every compact and oriented 3-manifold M possesses a contact structure, that is, M carries a globally defined 1-form ω with $\omega \wedge d\omega \neq 0$ everywhere. One can associate with ω a vector field X_0 (determined by $\omega(X_0) = 1$ and $d\omega(X_0, \cdot) = 0$), a linear transformation field φ (which is a complex structure on $B = \ker \omega$, and has kernel $\boldsymbol{R}X_0$) and a Riemannian metric g (with respect to which φ is skew-symmetric and $\omega = g(X_0, \cdot)$). The resulting *contact metric structure* $(\varphi, X_0, \omega, g)$ is said to be *K-contact* if X_0 is a Killing field with respect to g. Chern and Hamilton [3] introduced the torsion invariant $c = |\tau|$, where $\tau = L_{X_0} g$ is the Lie derivative of g with respect to X_0, and conjectured that for fixed ω, with X_0 inducing a Seifert foliation, there exists a complex structure $\varphi|B$ on B such that the 'Dirichlet energy'

$$E(\tau) = \frac{1}{2} \int_M c^2 \operatorname{vol}(M, g)$$

is critical over all CR-structures. Should this conjecture be true, $\nabla_{X_0} \tau$ must vanish, or equivalently, the sectional curvature of all planes at a

1980 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C20.

Key words and phrases. Contact Riemannian manifolds, torsion, curvature.

[1] Supported by the Natural Sciences and Engineering Research Council of Canada.

given point perpendicular to B are equal (cf. [3]). The torsion τ is then said to be *critical.*

We now state our main result.

THEOREM. *Let M be a compact oriented 3-manifold with contact metric structure $(\varphi, X_0, \omega, g)$ and critical torsion. If there exists a constant a, $0 < a < 1$, such that $c < 2a$ and*

$$|\sigma|^2 < 2\left(a^2 - \frac{c^2}{4}\right)\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}c\right) , \tag{1}$$

where $\sigma = (\iota_{X_0}S)|B$, S denotes the Ricci tensor and r the scalar curvature, then M admits a contact metric of positive Ricci curvature. If, in addition, M is simply connected, it is diffeomorphic with the 3-sphere.

COROLLARY. *Let M be a compact oriented 3-manifold with K-contact metric structure $(\varphi, X_0, \omega, g)$. If $r > -2$, then M admits a contact metric of positive Ricci curvature.*

If the torsion invariant c is critical, the Webster curvature (cf. [3]) $W = (r+4)/8$ is independent of c, and the condition $r > -2$ is equivalent to $W > 1/4$.

An analogous result restricting the Ricci curvature of g was obtained in [4].

We record our thanks to J.-P. Bourguignon for stimulating conversations on the subject.

2. Contact manifolds. A $(2n+1)$-dimensional C^∞ manifold is called a *contact manifold* if it carries a global 1-form ω with the property that $\omega \wedge (d\omega)^n \neq 0$ everywhere. It has an underlying *almost contact metric structure* $(\varphi, X_0, \omega, g)$, that is,

$$\omega(X_0) = 1,\ \varphi X_0 = 0,\ \varphi^2 = -I + \omega \otimes X_0,\ \omega = g(X_0, \cdot),\ g(\varphi X, Y) = -g(X, \varphi Y) ,$$

where I is the identity transformation. Moreover,

$$g(X, \varphi Y) = d\omega(X, Y) .$$

If the almost complex structure J on $M \times \boldsymbol{R}$ defined by $J(X, fd/dt) = (\phi X - fX_0, \omega(X)d/dt)$, where f is a real-valued function, is integrable, the contact structure is said to be *normal.* In this case, X_0 is a Killing vector field, that is $\tau = 0$. Conversely, if $n = 1$, and X_0 is a Killing field, then M is normal.

We introduce the φ-torsion ψ which is closely related to τ. It is defined by $\psi(X, Y) = g((L_{X_0}\varphi)X, Y)$, and is known to be symmetric (cf. [2]).

PROPOSITION 1. (i) $\tau(X_0, \cdot) = \psi(X_0, \cdot) = 0$,

(ii) $\psi(X, Y) = -\tau(X, \varphi Y)$, *or equivalently*, $\tau(X, Y) = \psi(X, \varphi Y)$, $X, Y \in C^\infty(TM)$.

(iii) φ *is symmetric with respect to both* τ *and* ψ,

(iv) $\tau(\varphi X, \varphi Y) = -\tau(X, Y)$ *and* $\psi(\varphi X, \varphi Y) = -\psi(X, Y)$, $X, Y \in C^\infty(TM)$,

(v) $\operatorname{trace} \tau = \operatorname{trace} \psi = 0$,

(vi) $\tau(X, Y) = \psi(\varphi^{1/2} X, \varphi^{1/2} Y)$, $X, Y \in C^\infty(TM)$,

(vii) $|\tau| = |\psi| (=c)$.

PROOF. (i) For contact metric structures, $\nabla_{X_0} X_0 = 0$ (cf. [2]). Hence,

$$\tau(X_0, X) = (L_{X_0} g)(X_0, X) = X_0 \cdot g(X_0, X) - g(X_0, [X_0, X]) = g(X_0, \nabla_X X_0)$$
$$= \frac{1}{2} X \cdot g(X_0, X_0) = 0 , \quad X \in C^\infty(TM) .$$

The statement for ψ follows from $(L_{X_0}\varphi) X_0 = 0$.

(ii)
$$\begin{aligned}\tau(X, \varphi Y) = (L_{X_0} g)(X, \varphi Y) &= X_0 \cdot g(X, \varphi Y) \\ &\quad - g([X_0, X], \varphi Y) - g(X, [X_0, \varphi Y]) \\ &= X_0 \cdot g(X, \varphi Y) - g([X_0, X], \varphi Y) \\ &\quad - g(X, \varphi[X_0, Y]) - \psi(X, Y) .\end{aligned}$$

On the other hand, $(d\omega)(X, Y) = g(X, \varphi Y)$, so

$$\begin{aligned}(L_{X_0}(d\omega))(X, Y) &= X_0 \cdot (d\omega)(X, Y) - d\omega([X_0, X], Y) - d\omega(X, [X_0, Y]) \\ &= X_0 \cdot g(X, \varphi Y) - g([X_0, X], \varphi Y) - g(X, \varphi[X_0, Y])\end{aligned}$$

which vanishes since $L_{X_0}(d\omega) = 0$.

(iii) Follows directly from (ii) since τ and ψ are symmetric in their arguments.

(iv) By repeated application of (ii), we obtain

$$\tau(\varphi X, \varphi Y) = -\psi(\varphi X, Y) = -\psi(Y, \varphi X) = -\tau(Y, X) = -\tau(X, Y) .$$

A similar proof holds for ψ.

(v) Choosing a φ-basis $\{E^i, \varphi E^i, X_0\}_{i=1}^n$,

$$\operatorname{trace} \tau = \sum_{i=1}^n \tau(E^i, E^i) + \sum_{i=1}^n \tau(\varphi E^i, \varphi E^i) + \tau(X_0, X_0) = 0$$

by (i) and (iv).

(vi) By (i), we may assume that $X, Y \in C^\infty(B)$, $B = \ker \omega$. Since $\varphi^{1/2} = (I + \varphi)/\sqrt{2}$ on B,

$$\psi(\varphi^{1/2} X, \varphi^{1/2} Y) = \frac{1}{2} \psi(X + \varphi X, Y + \varphi Y) = \psi(X, \varphi Y) = \tau(X, Y)$$

by (ii)-(iv).

(vii) Follows from (vi) since $\varphi^{1/2}$ is an isometry on B.

The integrability tensor $N^{(1)}$ occurring in the normality condition for contact metric structures in [2] is given by

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + 2d\omega(X, Y)X_0\ , \quad X, Y \in C^\infty(TM)\ ,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. For fixed $X \in C^\infty(TM)$, we consider the 2-tensor μ_X on M defined by

$$\mu_X(Y, Z) = g(N^{(1)}(X, Y), \varphi Z)\ , \quad Y, Z \in C^\infty(TM)\ .$$

Clearly, $\mu_X(\cdot, X_0) = 0$ and

$$g((\nabla_X\varphi)Y, Z) = \frac{1}{2}\mu_Y(Z, X) + g(Y, X)\omega(Z) - g(Z, X)\omega(Y)\ , \qquad X, Y, Z \in C^\infty(TM) \tag{2}$$

(see [2]).

PROPOSITION 2. (i) $\mu_{X_0} = -\psi$,
(ii) $\mu_X(\varphi Y, \varphi Z) = -\mu_X(Y, Z)$, $Y, Z \in C^\infty(B)$, $B = \ker \omega$,
(iii) trace $\mu_{X_0} = 0$.

PROOF. (i) For $Y, Z \in C^\infty(TM)$,

$$\begin{aligned}\mu_{X_0}(Y, Z) &= g([\varphi, \varphi](X_0, Y), \varphi Z) = g(\varphi^2[X_0, Y], \varphi Z) - g(\varphi[X_0, \varphi Y], \varphi Z)\\ &= g(\varphi[X_0, Y], Z) - g([X_0, \varphi Y], Z) + \omega([X_0, \varphi Y])g(X_0, Z)\\ &= -g((L_{X_0}\varphi)Y, Z) + \omega((L_{X_0}\varphi)Y)g(X_0, Z) = -\psi(Y, Z)\ ,\end{aligned}$$

since $\omega((L_{X_0}\varphi)Y) = g(X_0, (L_{X_0}\varphi)Y) = \tau(X_0, Y) = 0$ by (i) of Proposition 1.

(ii) By the previous step and (iv) of Proposition 1, we may assume that $X \in C^\infty(B)$. Then,

$$\begin{aligned}\mu_X(\varphi Y, \varphi Z) + \mu_X(Y, Z) &= -g([\varphi, \varphi](X, \varphi Y), Z) + g([\varphi, \varphi](X, Y), \varphi Z)\\ &= 0\ .\end{aligned}$$

(iii) As in (v) of Proposition 1, we choose a φ-basis and apply (i) and (ii).

3. Proof of the Theorem. We first replace g by the new metric

$$\tilde{g} = ag + b\omega \otimes \omega\ , \tag{3}$$

where $a, b \in \boldsymbol{R}$ with $a > 0$, $a + b > 0$. Then, the corresponding Ricci tensors $\tilde{S}$ and S are related by the formula

$$\tilde{S} = S - \frac{2b}{a}g + \frac{2b}{a^2}[(2n+1)a + nb]\omega \otimes \omega \tag{4}$$

$$+\frac{b}{a+b}\psi+\frac{b}{2(a+b)}\nabla_{X_0}\tau .$$

To see this, let W be the tensor field defined by $W^i_{jk}=\tilde{\Gamma}^i_{jk}-\Gamma^i_{jk}$. Then, by (3),

$$W^i_{jk}=-\frac{b}{a}(\varphi^i_{\cdot j}\omega_k+\varphi^i_{\cdot k}\omega_j)+\frac{b}{2(a+b)}X^i_0\tau_{jk} ,$$

where $\tau_{jk}=\nabla_j\omega_k+\nabla_k\omega_j$ (see [4]). Now,

$$\begin{aligned}\tilde{S}_{jk}-S_{jk}=\tilde{R}^i_{\cdot jki}-R^i_{\cdot jki}&=\nabla_iW^i_{jk}-\nabla_kW^i_{ji}+W^i_{ri}W^r_{jk}-W^i_{rk}W^r_{ji}\\&=-\frac{b}{a}\{\omega_k\nabla_i\varphi^i_{\cdot j}+\omega_j\nabla_i\varphi^i_{\cdot k}+\varphi^i_{\cdot j}\nabla_i\omega_k+\varphi^i_{\cdot k}\nabla_i\omega_j\}\\&\quad+\frac{b}{2(a+b)}X^i_0\nabla_i\tau_{jk}+\frac{2nb^2}{a^2}\omega_j\omega_k-\frac{b^2}{a(a+b)}\psi_{jk} ,\end{aligned}$$

where we used $\operatorname{div} X_0=\operatorname{trace}\nabla\omega=(1/2)\operatorname{trace}\tau=0$ (by (v) of Proposition 1), Proposition 1 (ii), as well as various well-known identities for contact metric structures. Since

$$\begin{aligned}\varphi^i_{\cdot j}\nabla_i\omega_k+\varphi^i_{\cdot k}\nabla_i\omega_j&=\varphi^i_{\cdot j}\tau_{ik}-\varphi^i_{\cdot j}\nabla_k\omega_i+\varphi^i_{\cdot k}\tau_{ij}-\varphi^i_{\cdot k}\nabla_j\omega_i\\&=\varphi^i_{\cdot j}\tau_{ik}+\omega_i\nabla_k\varphi^i_{\cdot j}+\varphi^i_{\cdot k}\tau_{ij}+\omega_i\nabla_j\varphi^i_{\cdot k}\\&=-2\psi_{jk}+\omega_i(\nabla_k\varphi^i_{\cdot j}+\nabla_j\varphi^i_{\cdot k}) ,\end{aligned}$$

we obtain

$$\begin{aligned}\tilde{S}_{jk}-S_{jk}=&-\frac{b}{a}\{\omega_k\nabla_i\varphi^i_{\cdot j}+\omega_j\nabla_i\varphi^i_{\cdot k}+\omega_i(\nabla_k\varphi^i_{\cdot j}+\nabla_j\varphi^i_{\cdot k})\}\\&+\frac{b}{2(a+b)}\nabla_{X_0}\tau_{jk}+\frac{2nb^2}{a^2}\omega_j\omega_k+\frac{2b}{a}\Big(1-\frac{b}{2(a+b)}\Big)\psi_{jk} .\end{aligned}$$

To simplify the terms in $\{\cdots\}$, we use (2) and the properties of μ_X given in Proposition 2. Thus,

$$\begin{aligned}\{\cdots\}&=\frac{1}{2}\omega_k\operatorname{trace}\mu_{\partial/\partial x^j}+\frac{1}{2}\omega_j\operatorname{trace}\mu_{\partial/\partial x^k}-\frac{1}{2}\mu_{X_0}(\partial/\partial x^j,\partial/\partial x^k)\\&\quad-\frac{1}{2}\mu_{X_0}(\partial/\partial x^k,\partial/\partial x^j)+2g_{jk}-2(2n+1)\omega_j\omega_k\\&=\psi_{jk}+2g_{jk}-2(2n+1)\omega_j\omega_k .\end{aligned}$$

To see this, we first re-write formula (2):

(2′) $$g((\nabla_{\partial/\partial x^i}\varphi)\partial/\partial x^j,\partial/\partial x^k)=\frac{1}{2}\mu_{\partial/\partial x^j}(\partial/\partial x^k,\partial/\partial x^i)+g_{ij}\omega_k-g_{ik}\omega_j ,$$

that is,

$$g_{lk}\nabla_i\varphi^l_{\cdot j} = \frac{1}{2}\mu_{jki} + g_{ij}\omega_k - g_{ik}\omega_j ,$$

where $\mu_{jki} = \mu_{\partial/\partial x^j}(\partial/\partial x^k, \partial/\partial x^i)$, from which

$$\nabla_i\varphi^r_{\cdot j} = \frac{1}{2}g^{rs}\mu_{jsi} + g_{ij}X_0^r - \delta_i^r\omega_j .$$

It follows that

$$\omega_k\nabla_i\varphi^i_{\cdot j} = \frac{1}{2}\omega_k g^{is}\mu_{\partial/\partial x^j}(\partial/\partial x^s, \partial/\partial x^i) - 2n\omega_j\omega_k = \frac{1}{2}\omega_k \operatorname{trace} \mu_{\partial/\partial x^j} - 2n\omega_j\omega_k ,$$

and

$$\omega_i\nabla_k\varphi^i_{\cdot j} = \omega_i\Big(\frac{1}{2}g^{is}\mu_{jsk} + g_{kj}X_0^i - \delta_k^i\omega_j\Big) = \frac{1}{2}X_0^s\mu_{jsk} + g_{kj} - \omega_k\omega_j ,$$

from which $\{\cdots\}$ follows. This yields (4).

Now, consider the case $n = 1$, and assume that τ is critical, i.e. $\nabla_{X_0}\tau = 0$. Then, choosing $b = a^2 - a$, (4) reduces to

$$(5) \qquad \tilde{S} = S + 2(1-a)g + 2(a-1)(a+2)\omega\otimes\omega + \frac{a-1}{a}\psi .$$

To ensure that $\tilde{S} > 0$ we determine, at each point $x \in M$, the entries of the matrix of the r.h.s. of (5) with respect to a suitable φ-basis $\{E, \varphi E, X_0\}$ of T_xM, and compute the respective subdeterminants along the main diagonal. First, assume that $\sigma_x \neq 0$ and choose $E \in \ker\sigma_x$, $|E| = 1$, such that $\sigma(\varphi E) = |\sigma|$. Then,

$$\tilde{S}(X_0, X_0) = S(X_0, X_0) - 2(1-a^2) = 2\Big(a^2 - \frac{c^2}{4}\Big)$$

since

$$S(X_0, X_0) = 2 - \operatorname{trace}\Big(\frac{1}{2}L_{X_0}\varphi\Big)^2 = 2\Big(1 - \frac{c^2}{4}\Big)$$

by [2]. Since τ is critical,

$$g(R(E, X_0)X_0, E) = g(R(\varphi E, X_0)X_0, \varphi E) .$$

This implies that $S(E, E) = S(\varphi E, \varphi E)$, and by polarization, $S(E, \varphi E) = 0$. It follows that

$$S(E, E) = S(\varphi E, \varphi E) = \frac{r}{2} + \frac{c^2}{4} - 1 .$$

Hence,

$$\tilde{S} = \begin{bmatrix} \tilde{S}(E, E) & \frac{a-1}{a}\psi(E, \varphi E) & 0 \\ \frac{a-1}{a}\psi(E, \varphi E) & \tilde{S}(\varphi E, \varphi E) & |\sigma| \\ 0 & |\sigma| & 2\left(a^2 - \frac{c^2}{4}\right) \end{bmatrix},$$

where

$$\tilde{S}(E, E) = \frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}\psi(E, E)$$

and

$$\tilde{S}(\varphi E, \varphi E) = \frac{r}{2} + \frac{c^2}{4} + 1 - 2a + \frac{1-a}{a}\psi(E, E),$$

Now, we claim that $c < 2a$ together with (1) ensures that $\tilde{S} > 0$ at $x \in M$. Indeed, since $c^2 = \psi(E, E)^2 + \psi(E, \varphi E)^2$, the subdeterminants along the main diagonal of $\tilde{S}$ can be estimated as

$$\begin{aligned} \tilde{S}(E, E) &= \frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}\psi(E, E) \\ &\geqq \frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}c > 0, \end{aligned}$$

$$\begin{aligned} &\tilde{S}(E, E)\tilde{S}(\varphi E, \varphi E) - \left(\frac{a-1}{a}\right)^2\psi(E, \varphi E)^2 \\ &\quad = \left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - \left(\frac{1-a}{a}\right)^2 c^2 > 0, \end{aligned}$$

and

$$\begin{aligned} \det \tilde{S} &= 2\left(a^2 - \frac{c^2}{4}\right)\left\{\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - \left(\frac{1-a}{a}\right)^2 c^2\right\} \\ &\quad - |\sigma|^2\left\{\frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}\psi(E, E)\right\} \\ &\geqq \left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a + \frac{1-a}{a}c\right) \\ &\quad \times \left\{2\left(a^2 - \frac{c^2}{4}\right)\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \frac{1-a}{a}c\right) - |\sigma|^2\right\} > 0. \end{aligned}$$

For $\sigma_x = 0$ we choose an arbitrary φ-basis, and apply the above argument. Finally, the last statement is a consequence of Hamilton [5].

REMARK. It is not difficult to see that

$$\sigma = -\frac{1}{2}(\delta\psi)\circ\varphi|B$$

and

$$\iota_{X_0}\delta\psi = 0 \ ,$$

where $\delta: S^2T^*M \to T^*M$ is the Berger-Ebin differential operator (cf. [1]) given by $(\delta\psi)X = \mathrm{trace}\, \nabla\psi(X, \cdot; \cdot)$, $X \in C^\infty(TM)$, and S^2 is the symmetric square. Clearly, $\sigma = 0$, if and only if $\delta\psi = 0$. This is the case for K-contact metric structures. In general, by the Berger-Ebin decomposition theorem, we have the orthogonal splitting

$$\psi = \psi_0 + L_Z g \ ,$$

where $Z \in C^\infty(TM)$ and $\delta\psi_0 = 0$. Thus, $\delta\psi = 0$ means that in the space $\mathscr{M}$ of all Riemannian metrics on M, the tangent vector $\psi \in T_g\mathscr{M}$ is perpendicular to the orbit of g under the group of diffeomorphisms of M.

References

[1] M. Berger and D. Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold, J. Diff. Geom. 3 (1969), 379–392.

[2] D. E. Blair, Contact manifolds in Riemannian Geometry, Lecture Notes in Math., 509, Springer-Verlag, Berlin and New York, 1976.

[3] S. S. Chern and R. S. Hamilton, On Riemannian metrics adapted to three-dimensional contact manifolds, Lecture Notes in Math., Springer-Verlag, Berlin and New York, Vol. 1111, 1985, 279–308.

[4] S. I. Goldberg, Nonnegatively curved contact manifolds, Proc. Amer. Math. Soc. 96 (1986), 651–656.

[5] R. S. Hamilton, Three-manifolds with positive Ricci curvature, J. Diff. Geom. 17 (1982), 255–306.

[6] J. Martinet, Formes de contact sur les variétés de dimension 3, Proc. Liverpool Singularities Sympos. II, Springer Lecture Notes in Math. 209 (1971), 142–163.

[7] H. Sato, Remarks concerning contact manifolds, Tohoku Math. J. 29 (1977), 577–584.

Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801
U.S.A.
and
Department of Mathematics
and Statistics
Queen's University
Kingston, Canada K7L 3N6
Canada

and

Department of Mathematics
Rutgers University
Camden, N. J. 08102
U.S.A.