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THE RICCI CURVATURE OF SYMPLECTIC QUOTIENTS OF FANO MANIFOLDS

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1. Introduction. Let M be a symplectic manifold with a symplectic form ω on which a compact connected Lie group K acts as symplectic diffeomorphisms. Let k^* be the dual of the Lie algebra k of K. A moment map for the action of K is a map $\mu: M \to k^*$ satisfying

$$d\langle \mu, X\rangle = i(X)\omega \quad \text{for all} \quad X \in k , \tag{1.1}$$

and

$$\mu \circ \sigma = \mathrm{Ad}(\sigma^{-1})^* \mu \quad \text{for all} \quad \sigma \in K . \tag{1.2}$$

It is convenient to put $\mu_X = \langle \mu, X\rangle$ which is a smooth function on M. Then (1.1) and (1.2) are equivalent to $d\mu_X = i(X)\omega$ and $\sigma^*\mu_X = \mu_{\mathrm{Ad}(\sigma^{-1})X}$. Obviously from (1.2), $\mu^{-1}(0)$ is K-invariant. When 0 is a regular value of μ and K acts on $\mu^{-1}(0)$ freely (which we assume throughout this paper), $M_K = \mu^{-1}(0)/K$ becomes a smooth manifold. Let $\iota: \mu^{-1}(0) \to M$ be the inclusion and $\pi: \mu^{-1}(0) \to M_K$ the projection. It is well known that there exists a unique symplectic form ω_K on M_K such that $\pi^*\omega_K = \iota^*\omega$. The symplectic manifold (M_K, ω_K) is called a *symplectic quotient* or a Marsden-Weinstein reduction [9] of (M, ω).

Assume further that M is a Kähler manifold with a Kähler form ω on which K acts as holomorphic isometries. Then it is also well known that M_K admits an integrable complex structure with respect to which ω_K is a Kähler form (see §2). The purpose of this paper is to compute the Ricci curvature of M_K in this situation. A formula we get is (3.12) in §3.

The most interesting case would be the case where M is a compact complex manifold of positive first Chern class, or simply a Fano manifold in algebraic geometers' terminology. Let ω be a Kähler form chosen in $c_1(M)$ and γ_ω the Ricci form of ω. Since both ω and γ_ω represent $c_1(M)$, there exists, uniquely up to a constant, a real valued smooth function F such that $\gamma_\omega - \omega = (i/2\pi)\partial\bar\partial F$. In this situation we have a natural moment map (see (4.2)) and obtain a simpler formula for the Ricci curvature of (M_K, ω_K). To write down the formula, first note that, since ω

and γ_ω are K-invariant, so is F. Therefore F descends to a smooth function $\check{F}$ on M_K. Let $\xi = \{X_1, \cdots, X_d\}$ be a basis of k and $\xi_i = (X_i - \sqrt{-1}\,JX_i)/2$. Let $\|\xi\|$ be the pointwise norm of $\xi_1 \wedge \cdots \wedge \xi_d$ considered as a section of $\wedge^d T^{1,0}M|_{\mu^{-1}(0)}$ and measured by the metric induced from the Kähler metric of M; thus $\|\xi\|$ is a smooth nowhere zero function on $\mu^{-1}(0)$. Furthermore, $\|\xi\|$ turns out to be K-invariant and thus projects to a function $\|\check{\xi}\|$ on M_K.

THEOREM 1. *In the above situation the Ricci form* γ_{ω_K} *of* (M_K, ω_K) *is expressed as*

$$\gamma_{\omega_K} = \omega_K + \frac{i}{2\pi}\partial\bar{\partial}(\check{F} + \log \|\check{\xi}\|^2)\,.$$

By the above theorem, γ_{ω_K} and ω_K are cohomologous. Since γ_{ω_K} represents $c_1(M_K)$ and ω_K is a positive form, we have:

COROLLARY 2. *If M is a Fano manifold, the symplectic quotient M_K is a Fano manifold again.*

The following corollary is also obvious.

COROLLARY 3. *Let M be a compact Kähler-Einstein manifold of positive Ricci curvature. Then the symplectic quotient (M_K, ω_K) is a Kähler-Einstein manifold if and only if $\|\xi\|$ is constant on $\mu^{-1}(0)$.*

This work was motivated by the problem of finding Kähler-Einstein manifolds of positive Ricci curvature. Corollary 3 suggests that one may find new examples of Kähler-Einstein manifolds out of well-known ones. The simplest manifolds, on which it is unknown whether a Kähler-Einstein metric of positive Ricci curvature exists, are three and four point blow-ups of $\boldsymbol{P}^2(\boldsymbol{C})$, see [1]. In §5 we give examples where these two manifolds appear as symplectic quotients of $(\boldsymbol{P}^1(\boldsymbol{C}))^3$ and $(\boldsymbol{P}^1(\boldsymbol{C}))^5$. Unfortunately however, $\|\xi\|$ is not constant in these examples and the problem remains open. We remark that the only known non-homogeneous examples of Kähler-Einstein manifolds of positive Ricci curvature are Sakane's examples [10].

This work was also motivated by Kobayashi's work [6] in which he computed the holomorphic sectional curvature of M_K in terms of the holomorphic sectional curvature of M and the second fundamental form of $\mu^{-1}(0)$ in M. His set-up is in a situation where M and K may be infinite dimensional, so that his computation applies to the moduli spaces of Hermitian-Einstein vector bundles, which have been studied by Itoh [3] (see also [7]). Our formula does not apply to this infinite dimensional situation, since $||\xi||$ does not make sense.

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2. Symplectic Quotients of Kähler Manifolds. Let M be a Kähler manifold, g its Kähler metric, and J its complex structure. The Kähler form ω is defined by

$$\omega(X, Y) = \frac{1}{2\pi} g(JX, Y)$$

for any real or complex vector fields X and Y of M. By the Kähler condition ω is closed, and since g is positive definite, ω is nondegenerate; thus ω is considered as a symplectic form. Let K be a compact connected Lie group which acts on M as holomorphic isometries and μ: $M \to k^*$ a moment map for the action of K. Any element X of k defines a vector field of M, which we denote by the same letter X. For each point p of M, k_p denotes the vector subspace of the tangent space T_pM spanned by X_p, $X \in k$. If $p \in \mu^{-1}(0)$ and $Y \in T_p\mu^{-1}(0)$, then $g(JX, Y) = \omega(X, Y) = Y\mu_X = 0$. It follows from this and $\operatorname{codim} \mu^{-1}(0) = \dim K$ we have an orthogonal decomposition

$$T_pM = T_p\mu^{-1}(0) \oplus Jk_p \tag{2.1}$$

at any $p \in \mu^{-1}(0)$. Letting E_p be the orthogonal complement of k_p in $T_p\mu^{-1}(0)$, we have from (2.1) an orthogonal decomposition

$$T_pM = E_p \oplus k_p \oplus Jk_p\ . \tag{2.2}$$

Clearly E_p is J-invariant and the distribution $E = \{E_p\}_{p \in \mu^{-1}(0)}$ is K-invariant. Since E is J-invariant we have a decomposition $E \otimes \boldsymbol{C} = E^{1,0} \oplus E^{0,1}$ into $\pm i$ eigenspaces. It is obvious that

$$E^{1,0} = T^{1,0}M|_{\mu^{-1}(0)} \cap (T\mu^{-1}(0) \otimes \boldsymbol{C})\ . \tag{2.3}$$

It follows from (2.3) that $E^{1,0}$ is integrable (but E may not be).

Let π: $\mu^{-1}(0) \to M_K = \mu^{-1}(0)/K$ be the projection. Then $d\pi|_{E_p}$ induces an isomorphism from E_p onto $T_{\pi(p)}M_K$. We define an almost complex structure J_K of M_K so that $d\pi|_E \circ J = J_K \circ d\pi|_E$.

LEMMA 2.4. *J_K is integrable.*

PROOF. Let s_1 and s_2 be sections of $T^{1,0}M_K$ and s_1' and s_2' the unique K-invariant sections of $E^{1,0}$ such that $d\pi(s_i') = s_i$, $i = 1, 2$. Since $E^{1,0}$ is integrable, $[s_1', s_2']$ is a K-invariant section of $E^{1,0}$. Thus $d\pi[s_1', s_2'] = [s_1, s_2]$ is a section of $T^{1,0}M_K$. q.e.d.

Finally we define a Riemannian metric g_K of M_K so that

$$g(X_p, Y_p) = g_K(d\pi(X_p), d\pi(Y_p)) \tag{2.5}$$

for all $X_p, Y_p \in E_p$. Then g_K is Hermitian with respect to J_K, namely g_K is J_K-invariant. Moreover, we have:

LEMMA 2.6. *g_K is a Kähler metric and the Kähler form ω_K for g_K satisfies $\pi^*\omega_K = \iota^*\omega$ where $\iota: \mu^{-1}(0) \to M$ is the inclusion.*

PROOF. We first prove the last equality. Then we have $\pi^*d\omega_K = \iota^*d\omega = 0$, since ω is closed. Since π is surjective, $d\omega_K = 0$. This proves that g_K is a Kähler metric.

The Kähler form ω_K for g_K is by definition

$$\omega_K(Z, W) = \frac{1}{2\pi} g_K(J_K Z, W)$$

for any vector fields Z and W. If Z' and W' are the unique K-invariant section of E such that $d\pi(Z') = Z$ and $d\pi(W') = W$, then

$$\begin{aligned} \pi^*\omega_K(Z', W') &= \frac{1}{2\pi} g_K(J_K d\pi(Z'), d\pi(W')) \circ \pi \\ &= \frac{1}{2\pi} g_K(d\pi(JZ'), d\pi(W')) \circ \pi \\ &= \frac{1}{2\pi} g(JZ', W') = \iota^*\omega(Z', W') . \end{aligned}$$

If $Z' \in T_p(Kp)$, then $\pi^*\omega_K(Z', W') = 0$ for any W'. On the other hand, for the same Z' we have $\iota^*\omega(Z', W') = (1/2\pi)g(JZ', W') = 0$ since JZ' is perpendicular to $\mu^{-1}(0)$ by (2.1). Thus we have proved $\pi^*\omega_K = \iota^*\omega$. q.e.d.

REMARK 2.7. Let ∇ and ∇_K be the Levi-Civita connections of (M, g) and (M_K, g_K). Let $p_1: \iota^*TM \to E$ be the orthogonal projection. Then we have

$$(\nabla_K)_X Y = d\pi \circ p_1(\nabla_{X'} Y') , \tag{2.8}$$

where X and Y are arbitrary local vector fields of M_K and X' and Y' are the unique K-invariant sections of E such that $d\pi(X') = X$ and $d\pi(Y') = Y$. We can see (2.8) by proving that, defining ∇_K by (2.8), it is compatible with g_K and is torsion-free.

REMARK 2.9. If $\dim_C M = n$ and $\dim_R K = d$, then $\dim_C M_K = \dim_C E^{1,0} = n - d$.

3. The Ricci Curvature of M_K. Let $X_1, \cdots, X_d$ be a basis of k. Then $\xi_i = (X_i - \sqrt{-1}\, JX_i)/2$, $1 \leqq i \leqq d$, are holomorphic vector fields and the real parts X_i are Killing vector fields.

LEMMA 3.1. *$\xi_1 \wedge \cdots \wedge \xi_d$ and its norm are K-invariant.*

PROOF. The tangent vector X_p at p corresponding to $X \in k$ is defined by $X_p = (d/dt)|_{t=0} \exp(tX)p$. Thus if $\sigma \in K$ then $X_{\sigma p} = \sigma_*(\mathrm{Ad}(\sigma^{-1})X)_p$ and

$$(\xi_1 \wedge \cdots \wedge \xi_d)_{\sigma p} = \det(\mathrm{Ad}(\sigma^{-1})|_k)\sigma_*(\xi_1 \wedge \cdots \wedge \xi_d)_p = \sigma_*(\xi_1 \wedge \cdots \wedge \xi_d)_p ,$$

since $\det(\mathrm{Ad}(\sigma^{-1})|_k) = 1$ by the compactness of K. Since σ is an isometry we have $\|\xi\|_{\sigma p} = \|\xi\|_p$. q.e.d.

Let F be the distribution $\{k_p \oplus Jk_p\}_{p \in M}$. Then we have decompositions $F \otimes \boldsymbol{C} = F^{1,0} \oplus F^{0,1}$ and $\iota^* T^{1,0}M = E^{1,0} \oplus F^{1,0}$, the latter being an orthogonal decomposition. Let ∇^h and ∇^v be the connections of $E^{1,0}$ and $F^{1,0}$ induced from $\iota^*\nabla$ of $\iota^* T^{1,0}M$. The connections $\iota^*\nabla$, ∇^h and ∇^v induce connections of $\det \iota^* T^{1,0}M$, $\det E^{1,0}$, and $\det F^{1,0}$, which we shall denote by the same letters. Let $Z_1, \cdots, Z_d$ be a local orthonormal K-invariant frame of $E^{1,0}$. Let θ, θ^h and θ^v be the connection forms of $\iota^*\nabla$, ∇^h and ∇^v with respect to the frames $Z_1 \wedge \cdots \wedge Z_d \wedge \xi_1 \wedge \cdots \wedge \xi_d$, $Z_1 \wedge \cdots \wedge Z_d$ and $\xi_1 \wedge \cdots \wedge \xi_d$, respectively. Then we have $\theta = \theta^h + \theta^v$; this is a merit of having taken wedge product. We further define θ^h_h, θ^h_v, θ^v_h and θ^v_v by

$$\begin{aligned} &\theta^h_h(Z) = \theta^h(Z) , &&\theta^h_h(X) = 0 ,\\ &\theta^h_v(Z) = 0 , &&\theta^h_v(X) = \theta^h(X) ,\\ &\theta^v_h(Z) = \theta^v(Z) , &&\theta^v_h(X) = 0 ,\\ &\theta^v_v(Z) = 0 , &&\theta^v_v(X) = \theta^v(X) \end{aligned} \tag{3.2}$$

for any $Z \in E$ and $X \in k_p$, $p \in \mu^{-1}(0)$. Then naturally, we have $\theta = \theta^h_h + \theta^h_v + \theta^v_h + \theta^v_v$. Let θ_K be the connection form of $\det T^{1,0}M_K$ with respect to the local frame $d\pi(Z_1) \wedge \cdots \wedge d\pi(Z_{n-d})$. Then by Remark 2.7 we have $\pi^*\theta_K = \theta^h_h$. This is proved as follows:

$$\begin{aligned} &\theta_K(X)d\pi(Z_1) \wedge \cdots \wedge d\pi(Z_{n-d})\\ &= \sum_{i=1}^{n-d} d\pi(Z_1) \wedge \cdots \wedge d\pi(p\nabla_{X'}Z_i) \wedge \cdots \wedge d\pi(Z_{n-d})\\ &= d\pi(\nabla_{X'}(Z_1 \wedge \cdots \wedge Z_{n-d}))\\ &= \theta^h(X')d\pi(Z_1) \cdots \wedge d\pi(Z_{n-d}) , \end{aligned}$$

where X and X' are as in Remark 2.7. Thus we get

$$\begin{aligned}\pi^*\gamma_{\omega_K} &= \frac{i}{2\pi} d\pi^*\theta_K = \frac{i}{2\pi} d\theta_h^h \\ &= \frac{i}{2\pi}(d\theta - d\theta_v^h - d\theta_h^v - d\theta_v^v) \\ &= \iota^*\gamma_\omega - \frac{i}{2\pi}(d\theta_v^h + d\theta_h^v + d\theta_v^v) .\end{aligned} \tag{3.3}$$

LEMMA 3.4. $d\theta_h^v = d\pi^*(\partial \log \|\check{\xi}\|^2) = \pi^*(\bar{\partial}\partial \log \|\check{\xi}\|^2)$.

PROOF. If $Y \in E^{1,0}$, then since ξ_i are holomorphic,

$$\nabla_{\bar{Y}}^v \xi = 0$$

and

$$\nabla_Y^v \xi = \frac{\langle \nabla_Y \xi, \bar{\xi} \rangle}{\|\xi\|^2} \xi = (Y \log \|\xi\|^2)\xi .$$

Thus $\theta_h^v(\bar{Y}) = 0$ and $\theta_h^v(Y) = Y \log \|\xi\|^2$. This implies $\theta_h^v = \pi^*(\partial \log \|\check{\xi}\|^2)$.
q.e.d.

Let ∇' be the Levi-Civita connection of $T\mu^{-1}(0)$. For any vector field X on $\mu^{-1}(0)$, we denote by X^v the $(\ker d\pi)$-component of the decomposition $T\mu^{-1}(0) = E \oplus \ker d\pi$. We define $C: E \times E \to F$ by

$$C(Y, W) = (\nabla_Y' W)^v . \tag{3.5}$$

Then C is a skew-symmetric bilinear form and satisfies

$$2C(Y, W) = [Y, W]^v \tag{3.6}$$

(see [6]).

LEMMA 3.7. *Let Y be a section of $E^{1,0}$ and let $2\,\mathrm{Re}\, Y = u$ and $2\,\mathrm{Re}\, Z_i = v_i$. Then*

$$\begin{aligned} d\theta_v^h(Y, \bar{Y}) &= 2 \sum_{i=1}^{n-d} \langle C(Y, \bar{Y}), C(Z_i, \bar{Z}_i) \rangle \\ &= -\frac{1}{2} \sum_{i=1}^{n-d} \langle C(u, Ju), C(v_i, Jv_i) \rangle .\end{aligned}$$

PROOF. This follows from the next three equalities (3.8)–(3.10).

$$d\theta_v^h(Y, \bar{Y}) = -\theta_v^h([Y, \bar{Y}]^v) , \tag{3.8}$$

$$\begin{aligned}&\nabla_{[Y,\bar{Y}]^v}^h (Z_1 \wedge \cdots \wedge Z_{n-d}) \\ &\quad = \sum Z_1 \wedge \cdots \wedge \langle \nabla_{[Y,\bar{Y}]^v} Z_i, \bar{Z}_i \rangle Z_i \wedge \cdots \wedge Z_{n-d} .\end{aligned} \tag{3.9}$$

If $[Y, \bar{Y}]^v = \sum_{i=1}^d f_i X_i$, where $\{X_i\}$ is the basis of k and f_i are complex valued functions, then, since Z_i are K-invariant and $[X_i, Z_j] = 0$, we have

$$\begin{aligned}(3.10)\qquad \langle \nabla_{[Y,\bar{Y}]^v} Z_i, \bar{Z}_i\rangle &= \langle \nabla_{Z_i}[Y, \bar{Y}]^v, \bar{Z}_i\rangle - \left\langle \sum_{j=1}^{d} (Z_i f_j) X_j, \bar{Z}_i\right\rangle \\ &= \langle \nabla_{Z_i}[Y, \bar{Y}]^v, \bar{Z}_i\rangle = -\langle [Y, \bar{Y}]^v, \nabla_{Z_i}\bar{Z}_i\rangle \\ &= -2\langle C(Y, \bar{Y}), C(Z_i, \bar{Z}_i)\rangle .\end{aligned}$$

q.e.d.

LEMMA 3.11. *Let Y and u be as in Lemma 3.7. Then*

$$d\theta_v^v(Y, \bar{Y}) = -\frac{1}{2}(JC(u, Ju))\log \|\xi\|^2 .$$

PROOF. Clearly one has

$$d\theta_v^v(Y, \bar{Y}) = -\theta_v^v([Y, \bar{Y}]^v) .$$

If we put $X = ([Y, \bar{Y}]^v - iJ[Y, \bar{Y}]^v)/2$, then, since $[Y, \bar{Y}]^v$ is purely imaginary, we get $[Y, \bar{Y}]^v = X - \bar{X}$. Thus

$$\begin{aligned}\theta_v^v([Y, \bar{Y}]^v)\xi_1 \wedge \cdots \wedge \xi_d &= \nabla^v_{[Y,\bar{Y}]^v}(\xi_1 \wedge \cdots \wedge \xi_d) \\ &= \nabla^v_{X-\bar{X}}(\xi_1 \wedge \cdots \wedge \xi_d) = \nabla_X(\xi_1 \wedge \cdots \wedge \xi_d) \\ &= (X \log \|\xi\|^2)\xi \\ &= -\frac{i}{2}(J[Y, \bar{Y}]^v \log \|\xi\|^2)\xi .\end{aligned}$$

The last equality holds, since $\|\xi\|$ is K-invariant. We get the lemma from $[Y, \bar{Y}]^v = iC(u, Ju)$. q.e.d.

Combining (3.3), (3.4), (3.7) and (3.11), we obtain:

PROPOSITION 3.12. *Let Ric_{M_K} and Ric_M be the curvature of M_K and M, respectively. Let Y be a vector in $E^{1,0}$ and $2\,\mathrm{Re}\, Y = u$. Then*

$$\begin{aligned}\mathrm{Ric}_{M_K}(d\pi(Y), d\pi(\bar{Y})) &= \mathrm{Ric}_M(Y, \bar{Y}) + (\pi^*\partial\bar{\partial} \log \|\check{\xi}\|^2)(Y, \bar{Y}) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n-d} \langle C(u, Ju), C(v_i, Jv_i)\rangle \\ &\quad + \frac{1}{2} JC(u, Ju)\log \|\xi\|^2 .\end{aligned}$$

where $\{v_1, \cdots, v_{n-d}, Jv_1, \cdots, Jv_{n-d}\}$ is an orthonormal basis of E.

4. Fano Manifolds. In this section we assume that M is a Fano manifold, i.e., a compact complex manifold of positive first Chern class. We choose a Kähler form ω in $c_1(M)$. Since both ω and the Ricci form γ_ω of ω represent $c_1(M)$ there exists a real smooth function F such that $\gamma_\omega - \omega = (i/2\pi)\,\partial\bar{\partial}F$. We define a second order elliptic differential operator Δ_F by

$$\Delta_F u = \Delta u + u^\alpha F_\alpha \,, \quad u^\alpha F_\alpha = g^{\alpha\bar\beta}\frac{\partial u}{\partial \bar z^\beta}\frac{\partial F}{\partial z^\alpha} \,.$$

Then Δ_F is self-adjoint with respect to the volume form $e^F\omega^m$ and its eigenvalues are nonnegative, i.e., if $\Delta_F u + \lambda u = 0$ for some $u \neq 0$, then $\lambda \geqq 0$. We let Λ_λ be the eigenspace belonging to an eigenvalue λ. Let $i(M)$ (resp. $h(M)$) be the real (resp. complex) Lie algebra of all Killing (resp. holomorphic) vector fields of M. Then $i(M)$ is imbedded in $h(M)$ by $i(M) \ni X \to \xi_X = (X - iJX)/2 \in h(M)$. We identify $i(M)$ with its image by this imbedding. The following is a generalization of Matsushima-Lichnerowicz's theorem and can be proved quite analogously if we replace the canonical volume form ω^m by $e^F\omega^m$; for this reason we shall omit the proof (see [8]).

PROPOSITION 4.1. *Let the situation be as above.*

(1) *The first non-zero eigenvalue* λ_1 *of* Δ_F *satisfies* $\lambda_1 \geqq 1$.

(2) $\lambda_1 = 1$ *if and only if* $h(M) \neq 0$. *When this is the case,* Λ_1 *is isomorphic to* $h(M)$ *through the correspondence* $u \mapsto \bar\partial u^\sharp := g^{\alpha\bar\beta}(\partial u/\partial \bar z^\beta)(\partial/\partial z^\alpha)$ *and* $\partial u^\sharp$ *is a Killing vector field if and only if* u *is purely imaginary.*

Let K be a connected closed subgroup of the group of isometries and k its Lie algebra. By Proposition 4.1 for any $X \in k$ there exists a unique $u_X \in \Lambda_1$ such that $\xi_X = \bar\partial u_X^\sharp$. We put $\mu_X = (i/2\pi)u_X$, which is a real function by Proposition 4.1, and define $\mu\colon M \to k^*$ by $\langle \mu(p), X\rangle = \mu_X(p)$.

LEMMA 4.2. $\mu\colon M \to k^*$ *is a moment map for the action of* K.

PROOF. Since $\omega = (i/2\pi)g_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$ *we have*

$$i(\xi_X)\omega = i(\bar\partial u_X^\sharp)\omega = \bar\partial \mu_X$$

and

$$i(X)\omega = i(\xi_X + \bar\xi_X)\omega = i(\xi_X)\omega + \overline{i(\xi_X)\omega} = d\mu_X \,.$$

This proves (1.1).

If σ is an isometry, then σ^* commutes with Δ, $\sigma^* F = F$, and thus σ^* commutes with Δ_F. Therefore if $u_X \in \Lambda_1$, then $\sigma^* u_X \in \Lambda_1$. For any vector field Y of type (0, 1),

$$\begin{aligned}\omega(\bar\partial\sigma^* u_X^\sharp, Y) &= Y\sigma^*\mu_X = (\sigma_* Y)\mu_X = \omega(\bar\partial u_X^\sharp, \sigma_* Y)\\ &= \omega(\sigma_*^{-1}\xi_X, Y) = \omega(\xi_{\mathrm{Ad}(\sigma^{-1})X}, Y) = \omega(\bar\partial u^\sharp_{\mathrm{Ad}(\sigma^{-1})X}, Y) \,.\end{aligned}$$

This shows $\sigma^* u_X = u_{\mathrm{Ad}(\sigma^{-1})X}$ and thus $\sigma^*\mu_X = \mu_{\mathrm{Ad}(\sigma^{-1})X}$, proving (1.2).

q.e.d.

Assuming that 0 is a regular value of μ and that K acts on $\mu^{-1}(0)$

freely, we have a quotient Kähler manifold (M_K, ω_K) by Lemmas (2.8) and (2.10). By (3.3), (3.4), (2.10) and $\gamma_\omega = \omega + (i/2\pi)\partial\bar{\partial}F$, we have

$$\pi^*\gamma_{\omega_K} = \iota^*\gamma_\omega + \frac{i}{2\pi}\pi^*\partial\bar{\partial}\log\|\check{\xi}\|^2 - \frac{i}{2\pi}d(\theta_v^h + \theta_v^v)$$
$$= \pi^*\Big(\omega_K + \frac{i}{2\pi}\partial\bar{\partial}\log\|\check{\xi}\|^2\Big) + \frac{i}{2\pi}\iota^*\partial\bar{\partial}F - \frac{i}{2\pi}d(\theta_v^h + \theta_v^v)\,. \tag{4.3}$$

We now compute the last two terms of the right-hand side of (4.3).

LEMMA 4.4. *Let s be any section of* $\det T^{1,0}M$ *and $L_X s$ the Lie derivative of s with respect to $X \in k$. Then*

$$L_X s = \nabla_X s + (2\pi i \Delta\mu_X)s\,.$$

PROOF. Since $L_X - \nabla_X$ is $C^\infty(M)\otimes \boldsymbol{C}$-linear, it is sufficient to prove it for an appropriate s. Let $Z_1, \cdots, Z_n$ be an orthonormal frame of $T^{1,0}M$ and take s to be $Z_1 \wedge \cdots \wedge Z_n$. Then

$$L_X s = \sum_{i=1}^n Z_1 \wedge \cdots \wedge (\nabla_X Z_i - \nabla_{Z_i} X) \wedge \cdots \wedge Z_n$$
$$= \nabla_X s - \sum_{i=1}^n g(\nabla_{Z_i} X, \bar{Z}_i)s\,.$$

Thus it is sufficient to show $\sum_{i=1}^n g(\nabla_{Z_i}X, \bar{Z}_i) = -2\pi i \Delta\mu_X$. Note that $i(X)\omega = d\mu_X$ implies

$$\frac{1}{2\pi}g(JX, Y) = Y\mu_X$$

and thus if Y is of type (0, 1) then

$$g(X, Y) = -ig(JX, Y) = -2\pi i Y\mu_X\,.$$

From this we have

$$\sum_{i=1}^n g(\nabla_{Z_i}X, \bar{Z}_i) = \sum_{i=1}^n Z_i g(X, \bar{Z}_i) - g(X, \nabla_{Z_i}\bar{Z}_i)$$
$$= -2\pi i \sum_{i=1}^n Z_i(\bar{Z}_i\mu_X) - (\nabla_{Z_i}\bar{Z}_i)\mu_X$$
$$= -2\pi i \sum_{i=1}^n (\partial\bar{\partial}\mu_X)(Z_i, \bar{Z}_i)$$
$$= -2\pi i \Delta\mu_X\,.$$

q.e.d.

Now we restrict our attention to $\mu^{-1}(0)$. Since $Z_1 \wedge \cdots \wedge Z_{n-d}$ and $\xi_1 \wedge \cdots \wedge \xi_d$ are K-invariant by our choice of Z_i and Lemma 3.1, if we put $s = Z_1 \wedge \cdots \wedge Z_{n-d} \wedge \xi_1 \wedge \cdots \wedge \xi_d$, we have along $\mu^{-1}(0) = \{u_X = 0$ for all $X \in k\}$,

$$\nabla_X s = L_X s - (2\pi i \Delta\mu_X)s = (\Delta u_X)s = -(u_X^\alpha F_\alpha)s = -(\xi_X F)s$$

If we put $\theta_v = \theta_v^h + \theta_v^v$, this shows $\theta_v(X) = -\xi_X F$. Since $\theta_v(Z) = 0$ for any $Z \in E$, we have

$$\theta_v = -\iota^*\partial F + \pi^*\partial\check{F} ,$$

$$d\theta_v = \iota^*\partial\bar{\partial} F - \pi^*\partial\bar{\partial}\check{F} .$$

Putting this into (4.3) we get

$$\pi^*\gamma_{\omega_K} = \pi^*\left(\omega_K + \frac{i}{2\pi}\partial\bar{\partial}\log\|\check{\xi}\|^2 + \frac{i}{2\pi}\partial\bar{\partial}\check{F}\right) .$$

Since π is surjective, we get Theorem 1.

5. Examples. Compact complex surfaces of positive first Chern class are classically known as del Pezzo surfaces which are either $\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$, $\boldsymbol{P}^2(\boldsymbol{C})$ or a surface obtained by blowing up $\boldsymbol{P}^2(\boldsymbol{C})$ at $k \leqq 8$ points in general position (see, e.g., [11]). We shall denote by $\boldsymbol{P}_{\hat{k}}^2$ the surface obtained by blowing up at k points. Note that if $k \leqq 4$ the complex structure of $\boldsymbol{P}_{\hat{k}}^2$ does not depend on the points where the blowing up is carried out, but that if $k \geqq 5$ it does. Note also that the second Betti number $b_2(\boldsymbol{P}_{\hat{k}}^2)$ of $\boldsymbol{P}_{\hat{k}}^2$ is equal to $k+1$.

EXAMPLE 5.1. Let M be $(\boldsymbol{P}^1(\boldsymbol{C}))^3 = \boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})\times\boldsymbol{P}^1(\boldsymbol{C})$ and K be $S^1 = \{e^{2\pi i\theta} | \theta \in \boldsymbol{R}\}$. S^1 acts on $\boldsymbol{P}^1(\boldsymbol{C})$ by $[z_0 : z_1] \mapsto [z_0 : e^{2\pi i\theta}z_1]$ and on $(\boldsymbol{P}^1(\boldsymbol{C}))^3$ by the diagonal action. The moment map $\mu\colon (\boldsymbol{P}^1(\boldsymbol{C}))^3 \to k = \boldsymbol{R}$ for this action is

$$\mu([z_0, z_1], [w_0 : w_1], [u_0 : u_1]) = \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2} + \frac{|w_0|^2 - |w_1|^2}{|w_0|^2 + |w_1|^2} + \frac{|u_0|^2 - |u_1|^2}{|u_0|^2 + |u_1|^2} .$$

This can be interpreted as follows: $(\boldsymbol{P}^1(\boldsymbol{C}))^3$ can be regarded as the set of ordered three points of $\boldsymbol{P}^1(\boldsymbol{C}) \cong S^2 \subset \boldsymbol{R}^3 = \{(x, y, z)\}$ and then μ is nothing more than the sum of z-coordinates of the three points. It is easy to see that 0 is a regular value of μ and S^1 acts on $\mu^{-1}(0)$ freely.

EXAMPLE 5.2. Let M be $(\boldsymbol{P}^1(\boldsymbol{C}))^5$ and K be $SO(3)$. K is the identity component of the group of isometries of $\boldsymbol{P}^1(\boldsymbol{C}) \cong S^2$ and acts on $(\boldsymbol{P}^1(\boldsymbol{C}))^5$ diagonally. The moment map $\mu\colon (\boldsymbol{P}^1(\boldsymbol{C}))^5 \to k$ is interpreted as follows. Identifying $\boldsymbol{P}^1(\boldsymbol{C})$ with the unit sphere S^2 in $k \cong \boldsymbol{R}^3$ and regarding $(\boldsymbol{P}^1(\boldsymbol{C}))^5$ as the set of ordered five points in S^2, μ is nothing but the sum of the position vectors of the five points. In this case again, 0 is a regular value of μ and K acts on $\mu^{-1}(0)$ freely.

In both Examples 5.1 and 5.2 it is not an easy task to see what the symplectic quotient M_K looks like. But we can invoke a result of Kirwan [4], who derived a formula for the Poincaré series $P_t(M_K)$ of M in terms of the Poincaré series of M and the classifying spaces of K and certain stabilizer groups. In fact, Example 5.2 is nothing but her Example 5.18 in [4]. Applying her formula we can easily get $P_t(M_K) = 1 + 4t^2 + t^4$ for Example 5.1 and $P_t(M_K) = 1 + 5t^2 + t^4$ for Example 5.2. Since $(\boldsymbol{P}^1(\boldsymbol{C}))^3$ and $(\boldsymbol{P}^1(\boldsymbol{C}))^5$ are Fano manifolds so are the symplectic quotients M_K by Corollary 2. But $b_2(M_K) = 4$ and 5 for 5.1 and 5.2, respectively. By the classification of the first paragraph of this section, M_K must be biholomorphic to $\boldsymbol{P}^2_3$ and $\boldsymbol{P}^2_4$ respectively.

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