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ANALYTIC MAPPINGS BETWEEN TWO REGULARLY BRANCHED THREE-SHEETED ALGEBROID SURFACES

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

KIYOSHI NIINO*

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1. Introduction and results. Baker, Mutō and the author ([1], [4], [5], [6], [8], [9]) have discussed the family of analytic mappings between two ultrahyperelliptic surfaces. In this paper we investigate the structure of the family of analytic mappings between two regularly branched three-sheeted algebroid Riemann surfaces. Here we call a three-sheeted covering Riemann surface regularly branched if it has no branch point other than those of order two.

Let R (resp. S) be the three-sheeted covering algebroid Riemann surface formed by elements $p = (z, y)$ (resp. $q = (w, u)$) for each z, y (resp. w, u) satisfying the equation $y^3 = G(z)$ (resp. $u^3 = g(w)$), where G and g are entire functions, each of which has an infinite number of simple or double zeros and no other zeros. Then, since R and S have branch points of order two only, R and S are regularly branched. If the Nevanlinna counting function $N(r, 0, G)$ for the zeros of G is of finite order $\rho(G)$, then we may assume that G is the canonical product of order $\rho(G)$ over these zeros; a similar remark applies to g.

Let $\mathfrak{A}(R, S)$ denote the family of non-trivial analytic mappings of R into S. Mutō [3] proved:

THEOREM A. *To every $\phi \in \mathfrak{A}(R, S)$ there corresponds a non-constant entire function h such that one of the two functional equations*

$$f_1(z)^3 G(z) = g(h(z))$$

and

$$f_2(z)^3 G(z)^2 = g(h(z))$$

holds, where f_1 is entire and f_2 is a meromorphic function having at most simple poles only at the double zeros of G. The converse is also true.

We call such h the projection for the analytic mapping ϕ and say that

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a pair (f_1, f_2) of functions f_1 and f_2 satisfies the property (A) when f_1 and f_2 satisfy the property stated in Theorem A.

We denote by $\mathfrak{H}(R, S)$ the family of projections for the mappings in $\mathfrak{A}(R, S)$ and by $\mathfrak{H}_P(R, S)$ (resp. $\mathfrak{H}_T(R, S)$) the subfamily of $\mathfrak{H}(R, S)$ consisting of polynomials (resp. transcendental entire functions). It is clear that $\mathfrak{H}(R, S) = \mathfrak{H}_P(R, S) \cup \mathfrak{H}_T(R, S)$.

In this paper we shall obtain the following theorems:

THEOREM 1. *$\mathfrak{H}(R, S)$ is at most a countable set.*

THEOREM 2. *If $\mathfrak{H}_P(R, S)$ is not empty, then it consists of polynomials of the same degree.*

THEOREM 3. *If $\mathfrak{H}(R, S) \neq \varnothing$, then $\mathfrak{H}(R, S) = \mathfrak{H}_P(R, S)$ or $\mathfrak{H}(R, S) = \mathfrak{H}_T(R, S)$.*

THEOREM 4. *Assume that there exist two polynomials $h(z) = a_p z^p + \cdots + a_0$ $(a_p \neq 0)$ and $k(z) = b_p z^p + \cdots + b_0$ $(b_p \neq 0)$ belonging to $\mathfrak{H}(R, S)$. If $|a_p| < |b_p|$, then the following hold:*

(a) $\rho(g) = \rho(G) = 0$.

(b) *$k(z) = (b_p/a_p)h(z) + A$, where A is a constant.*

(c) *$\mathfrak{H}(R, S) = \mathfrak{H}_P(R, S)$ and $\mathfrak{H}(R, S)$ consists of just two elements h and k.*

(d) *g satisfies one of the following functional equations:*

(i) $g(\lambda w + A) = B(\lambda w + A - \alpha_1)g(w)$,

(ii) $g(\lambda w + A) = B(\lambda w + A - \alpha_1)^2 g(w)$,

(iii) $H(\lambda w + A)^3 g(\lambda w + A) = B(\lambda w + A - \alpha_1)g(w)^2$, $H(\alpha_1) \neq 0$,

(iv) $H(\lambda w + A)^3 g(\lambda w + A) = B(\lambda w + A - \alpha_1)^2 g(w)^2$, $H(\alpha_1) \neq 0$,

(v) $g(\lambda w + A)^2 = B(\lambda w + A - \alpha_1)H(\lambda w + A)^3 g(w)$, $H(\alpha_1) = 0$,

(vi) $g(\lambda w + A)^2 = B(\lambda w + A - \alpha_1)^2 H(\lambda w + A)^3 g(w)$, $H(\alpha_1) \neq 0$,

where $\lambda = b_p/a_p$, α_1 and B are constants such that $\alpha_1 \neq -A/(\lambda - 1)$ and $g((\alpha_1 - A)/\lambda) \neq 0$ and H is an entire function having only simple zeros.

(e) *p is a multiple of three and $k(z) = \alpha_1 + P(z)^3$, where P is a suitable polynomial of degree $p/3$.*

(f) *g has an infinite set of zeros only at the points α_j, $j = 1, 2, \cdots$, such that $\alpha_{j+1} = \lambda^j \alpha_1 + A(\lambda^j - 1)/(\lambda - 1)$. Moreover, if g satisfies the n-th equation in* (d), *then $\{\alpha_j\}$ satisfies the corresponding n-th condition below:*

(i) *$\{\alpha_j\}$ are all simple zeros of g,*

(ii) *$\{\alpha_j\}$ are all double zeros of g,*

(iii) *$\{\alpha_{2j-1}\}_{j=1}^{\infty}$ are simple zeros of g, $\{\alpha_{2j-1}\}_{j=3}^{\infty}$ are zeros of H and $\{\alpha_{2j}\}_{j=1}^{\infty}$ are double zeros of g,*

(iv) *$\{\alpha_{2j-1}\}$ are double zeros of g, while $\{\alpha_{2j}\}$ are simple zeros of g and H,*

(v) $\{\alpha_{2j-1}\}$ *are double zeros of* g *and are simple zeros of* H, *while* $\{\alpha_{2j}\}$ *are simple zeros of* g,

(vi) $\{\alpha_{2j-1}\}$ *are simple zeros of* g, *while* $\{\alpha_{2j}\}$ *are double zeros of* g *and are simple zeros of* H.

(g) *Examples of these situations indeed occur.*

THEOREM 5. *Assume that* $\rho(g) < +\infty$, *there exist two polynomials* h *and* k *of degree* p *belonging to* $\mathfrak{H}(R, S)$ *and the leading coefficients of* h *and* k *are the same in modulus. Then one of the following three cases occurs*:

(i) $k(z) = Lh(z) + M$, *where* L *is a root of unity and* M *is a constant.*

(ii) p *is even and there is a polynomial* r *such that* $h(z) = r(z)^2 + A_0$ *and* $k(z) = \{r(z) + \beta\}^2 + D_0$, *where* A_0, D_0 *and* β *are constants.*

(iii) *The ratio of the leading coefficients of* h *and* k *is a primitive* s*-th root of unity, and the* (ps)*-th iterate* ψ_{ps} *of the expansion* ψ *of* $k^{-1}\circ h$ *about* ∞ *satisfies* $\psi_{ps}(z) \equiv z$. *Case* (iii) *can occur only if* $\rho(G) > 2$.

Further, examples of each of the cases exist.

REMARK 1. Hiromi-Mutō [2] obtained another interesting result that if $\rho(G) < +\infty$, $0 < \rho(g) < +\infty$ and $\mathfrak{A}(R, S) \neq \emptyset$, then $\rho(G) = p\rho(g)$ with a suitable positive integer p and $\mathfrak{H}(R, S)$ consists of polynomials of the same degree p.

REMARK 2. We assume that R and S have the maximal Picard constant, that is, $P(R) = P(S) = 6$. Then the following hold: (I) If $\mathfrak{H}_P(R, S) \neq \emptyset$, then either case (i) or case (ii) in our Theorem 5 occurs. (II) If $\mathfrak{H}_T(R, S) \neq \emptyset$, then $\mathfrak{H}_T(R, S)$ consists of transcendental entire functions of the same order, the same type and the same class ([10, Theorem 4]). We have no other information on $\mathfrak{H}_T(R, S)$. In general, is the above statement (II) true without the condition $P(R) = P(S) = 6$?

We can deduce our Theorems 1 and 3 from the argument of the proofs of Theorem 1 in Mutō [4] and Theorem in Mutō [5] combined with (II) and (III) of our Lemma 3.1. Hence their proofs are omitted here.

Proof of Theorem 5 is also omitted here, because by our Lemma 3.2 we can apply the argument of the proof of Theorem 1 in Baker [1] to prove our Theorem 5 and all of his examples satisfy the functional equations $G(z) = g(h(z)) = g(k(z))$ or the functional equations $G(z) = g(h(z))$ and $e^{\phi(z)}G(z) = g(k(z))$, which are desired for our cases.

So in this paper we shall give the proofs of Theorems 2 and 4.

We assume here that the reader is familiar with the Nevanlinna theory of meromorphic functions and usual notation such as $T(r, f)$,

$N(r, a, f)$, $\bar{N}(r, a, f)$, $m(r, f)$, $S(r, f)$ etc. (see e.g. [7]).

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2. Lemmas (I). In order to prove our theorems we need several lemmas.

The following lemma is clear.

LEMMA 2.1. *Let g be an entire function and h a polynomial such that $h(z) = a_p z^p + \cdots$ $(a_p \neq 0)$. For any $\varepsilon > 0$ there is $r_0 > 0$ such that*

$$h'(z) \neq 0 \quad \text{and} \quad |a_p| r^p(1-\varepsilon) < |h(z)| < |a_p| r^p(1+\varepsilon)$$

are valid for all z satisfying $r = |z| > r_0$ and so

$$p\bar{n}(|a_p| r^p(1-\varepsilon), 0, g) - (p-1) \leqq \bar{n}(r, 0, g\circ h) \leqq p\bar{n}(|a_p| r^p(1+\varepsilon), 0, g)$$

is true for all $r > r_0$.

We have the following:

LEMMA 2.2. *Let k be a polynomial of degree p and f a rational function whose zeros and poles are all of simple or double order. If the functional equation*

$$F(z)^3 = f(k(z)) \tag{2.1}$$

holds with a suitable rational function F, then f has only one zero or pole without counting its multiplicity and p is a multiple of three.

PROOF. Let α and β be zeros or poles of f. Since α and β are of order at most two, it follows from (2.1) that p is a multiple of three, and $k(z) - \alpha = q_1(z)^3$ and $k(z) - \beta = q_2(z)^3$ are valid with suitable polynomials q_1 and q_2 of degree $p/3$. Since $\alpha \neq \beta$, $k'(z)$ has at least $4p/3$ zeros, which is impossible. Hence f has only one zero or pole without counting its multiplicity, and p is clearly a multiple of three. q.e.d.

3. Lemmas (II). First, we study relations among the growths of G, g, $g\circ h$ and the counting functions for their zeros when h belongs to $\mathfrak{H}(R, S)$.

Let $N_2^*(r, 0, f)$ be the counting function for simple or double zeros of the function f and $N_2^c(r, 0, f)$ the counting function for the other zeros of f.

LEMMA 3.1. *If h belongs to $\mathfrak{H}(R, S)$, then we have the following:*

(I) $\bar{n}(r, 0, g\circ h) - \bar{n}(r, 0, h') \leqq \bar{n}(r, 0, G) \leqq \bar{n}(r, 0, g\circ h)$.

(I′) *Especially, if* h *is a polynomial of degree* p, *then for large* r

$$\bar{n}(r, 0, g\circ h) - (p-1) \leqq \bar{n}(r, 0, G) \leqq \bar{n}(r, 0, g\circ h)\ .$$

(II) $(1/2)N_2^*(r, 0, g\circ h) \leqq N(r, 0, G) \leqq 2N(r, 0, g\circ h)$.

(III) *For any positive constant* K *and any* ε *satisfying* $0 < 5/K < \varepsilon < 1$ *we have*

$$KT(r, h) < N_2^*(r, 0, g\circ h) \leqq N(r, 0, g\circ h)$$

and

$$(1/2)(1-\varepsilon)N(r, 0, g\circ h) \leqq N(r, 0, G)$$

for all large r *if* h *is of finite order, and for* r *outside a set* E *of* r *of finite measure otherwise.*

Proof. If h belongs to $\mathfrak{H}(R, S)$, then it follows from Theorem A that either

$$f_1(z)^3G(z) = g(h(z)) \tag{3.1}$$

or

$$f_2(z)^3G(z)^2 = g(h(z)) \tag{3.2}$$

is valid, where (f_1, f_2) satisfies the property (A).

If the functional equation (3.1) is valid, then we have

$$\bar{n}(r, 0, G) \leqq \bar{n}(r, 0, g\circ h)\ ,$$
$$\bar{n}(r, 0, g\circ h) \leqq \bar{n}(r, 0, G) + \bar{n}(r, 0, f_1) \leqq \bar{n}(r, 0, G) + \bar{n}(r, 0, h')\ ,$$
$$N(r, 0, g\circ h) = N(r, 0, G) + 3N(r, 0, f_1)$$

and

$$N_2^*(r, 0, g\circ h) \leqq N(r, 0, G)\ .$$

Hence we have (I) and (II). If the functional equation (3.2) is valid, then we have

$$\bar{n}(r, 0, G) \leqq \bar{n}(r, 0, g\circ h)\ ,$$
$$\bar{n}(r, 0, g\circ h) \leqq \bar{n}(r, 0, G) + \bar{n}(r, 0, f_2) \leqq \bar{n}(r, 0, G) + \bar{n}(r, 0, h')\ ,$$
$$N(r, 0, G) \leqq 2N(r, 0, g\circ h)$$

and

$$N_2^*(r, 0, g\circ h) \leqq 2N(r, 0, G)\ ,$$

because a double zero of G may be a simple zero of $g\circ h$ and a double zero of $g\circ h$ is a simple zero of G. Hence we have (I) and (II) in this case. Therefore (I) and (II) are valid in all cases.

It is clear from (I) that (I′) is true.

Since g has an infinite number of simple or double zeros only, we have

$$(3.3)\qquad \begin{aligned} &N(r, 0, g\circ h) = N_2^*(r, 0, g\circ h) + N_2^c(r, 0, g\circ h)\,, \\ &N_2^c(r, 0, g\circ h) \leqq 4N(r, 0, h') \leqq 4T(r, h) + S(r, h)\,. \end{aligned}$$

Let $\{w_j\}$ be the set of distinct zeros of g. For an arbitrary but fixed number q it follows from Nevanlinna's second fundamental theorem that

$$(3.4)\qquad N(r, 0, g\circ h) > \sum_{j=1}^{q} N(r, w_j, h) > (q-1)T(r, h) + S(r, h)\,.$$

So we deduce from (3.3), (3.4) and (II) in this lemma that

$$(3.5)\qquad (q-5)T(r, h) + S(r, h) < N_2^*(r, 0, g\circ h) \leqq N(r, 0, g\circ h)\,.$$

For any $K > 0$ choosing q such that $q > K + 6$, we obtain (III) from (3.3), (3.5), (II) and the property of $S(r, h)$. q.e.d.

Next we prove:

LEMMA 3.2. *Suppose that the polynomials* $h(z) = a_p z^p + \cdots$ *and* $k(z) = b_p z^p + \cdots$, $|a_p| = |b_p| \neq 0$, *belong to* $\mathfrak{H}(R, S)$. *Then there exists* $r_0 > 0$ *such that in* $|z| > r_0$ *each of the* p *branches of* $\psi = k^{-1}\circ h$ *is regular, except for the pole at* $z = \infty$. *Moreover,* $G(\psi(z_1)) = 0$ *holds for any* z_1 *such that* $G(z_1) = 0$ *and* $|z_1| > r_0$, *and for any branch of* ψ.

PROOF. Since h and k belongs to $\mathfrak{H}(R, S)$, it follows from Theorem A that h satisfies one of the following functional equations

$$f_{h1}(z)^3 G(z) = g(h(z)) \quad\text{and}\quad f_{h2}(z)^3 G(z)^2 = g(h(z))\,,$$

where (f_{h1}, f_{h2}) satisfies the property (A), and k satisfies one of the functional equations

$$f_{k1}(z)^3 G(z) = g(k(z)) \quad\text{and}\quad f_{k2}(z)^3 G(z)^2 = g(k(z))\,,$$

where (f_{k1}, f_{k2}) satisfies the property (A). Since h and k are also polynomials, there exists $r_1 > 1$ such that in $|z| > r_1$ we have $h'(z) \neq 0$, $k'(z) \neq 0$ and $(1/2)|a_p|r^p < |h(z)|$, $|k(z)| < 2|a_p|r^p$ $(r = |z|)$. Hence each branch of the inverse function k^{-1} of k is regular in $|z| > r_1$, except for the pole at ∞, while all roots of $h(z) = \alpha$ and $k(z) = \alpha$ for $|\alpha| > 2|a_p|r_1^p$ are of simple order. Hence the zeros of $g\circ h$ and $g\circ k$ in $|z| > 4r_1$ are all simple or double. Therefore, it follows from the above equations that they are also zeros of G. Conversely, the zeros of G in $|z| > 4r_1$ are also zeros of $g\circ h$ and $g\circ k$. q.e.d.

4. **Proof of Theorem 2.** Assume that $h(z) = a_p z^p + \cdots + a_0$ $(a_p \neq 0)$

and $k(z) = b_q z^q + \cdots + b_0$ $(b_q \neq 0)$ belong to $\mathfrak{H}_P(R, S)$. Then it follows from Theorem A that h satisfies one of the functional equations

$$f_{h1}(z)^3 G(z) = g(h(z)) \quad \text{and} \quad f_{h2}(z)^3 G(z)^2 = g(h(z)) , \tag{4.1}$$

where (f_{h1}, f_{h2}) satisfies the property (A), and k satisfies one of the functional equations

$$f_{k1}(z)^3 G(z) = g(k(z)) \quad \text{and} \quad f_{k2}(z)^3 G(z)^2 = g(k(z)) , \tag{4.2}$$

where (f_{k1}, f_{k2}) satisfies the property (A).

We contrarily suppose that $q > p$. For any fixed ε $(0 < \varepsilon < 1)$ there exists $r_0 > 0$ such that

$$\begin{gathered} |a_p| r^p(1 - \varepsilon) < |h(z)| < |a_p| r^p(1 + \varepsilon) , \\ |b_q| r^q(1 - \varepsilon) < |k(z)| < |b_q| r^q(1 + \varepsilon) , \end{gathered} \tag{4.3}$$

$$h'(z) \neq 0 , \qquad k'(z) \neq 0 , \tag{4.4}$$

$$(|a_p|/|b_q|)\{(1 + \varepsilon)/(1 - \varepsilon)\}^{\max\{3, q/p+1\}} < r^{q-p} \tag{4.5}$$

are valid for all $r > r_0$, $r = |z|$. Then it follows from Lemma 2.1 and (I′) of Lemma 3.1 that

$$q\bar{n}(|b_q| r^q(1 - \varepsilon), 0, g) - 2(q - 1) \leqq \bar{n}(r, 0, G) \leqq p\bar{n}(|a_p| r^p(1 + \varepsilon), 0, g)$$

and so

$$\bar{n}(|a_p| r^p(1 + \varepsilon), 0, g) - \bar{n}(|b_q| r^q(1 - \varepsilon), 0, g) + 2 \geqq (2/q) .$$

Hence, since $|a_p| r^p(1 + \varepsilon) < |b_q| r^q(1 - \varepsilon)$ from (4.5), we obtain

$$\bar{n}(|b_q| r^q(1 - \varepsilon), 0, g) - \bar{n}(|a_p| r^p(1 + \varepsilon), 0, g) = 0 \text{ or } 1 \tag{4.6}$$

for all $r > r_0$.

Let $\{w_j\}_{j=1}^{\infty}$ be the set of zeros of g satisfying $|w_j| > |b_q| r_0^q(1 + \varepsilon)$ without considering their multiplicities and assume that $|w_1| \leqq |w_2| \leqq \cdots$. We set

$$K(r) = \{z; |a_p| r^p(1 + \varepsilon) \leqq |z| < |b_q| r^q(1 - \varepsilon)\} .$$

The equation (4.6) means that the number of elements of $\{w_j\}$ belonging to $K(r)$ is at most one for all $r > r_0$.

We take r_l so that $|a_p| r_l^p(1 + \varepsilon) = |w_l|$ for some w_l. Then (4.6) implies

$$|w_{l+1}| > |b_q| r_l^q(1 - \varepsilon) . \tag{4.7}$$

Let z_{lj} $(j = 1, \cdots, p)$ be p roots of the equation $h(z) = w_l$. It follows from (4.3) that

$$|b_q| r_l^q(1 - \varepsilon) < |k(z_{lj})| < |b_q| r_l^q\{(1 + \varepsilon)/(1 - \varepsilon)\}^{q/p}(1 + \varepsilon) \tag{4.8}$$

and so from (4.5) that all $k(z_{lj})$ $(j = 1, \cdots, p)$ belong to $K(r'_l)$, where r'_l is a number satisfying $|a_p| r'^p_l(1 + \varepsilon) = |b_q| r^q_l(1 - \varepsilon)$. Therefore (4.6) implies that all $k(z_{lj})$ take the same value, say, $k(z_{lj}) = w'_l$ $(j = 1, \cdots, p)$. Since z_{lj} are zeros of G by (4.1) and (4.4), (4.2) implies that w'_l is a zero of g. If $w'_l \neq w_{l+1}$, then it follows from (4.7) and (4.8) that $|w'_l| \geqq |w_{l+1}|$, and consequently that w'_l and w_{l+1} belong to $K(r'_l)$, which contradicts (4.6). Hence we have $w'_l = w_{l+1}$. Since k is a polynomial of degree q $(>p)$, there is a root z' of the equation $k(z) = w_{l+1}$ different from z_{lj} $(j = 1, \cdots, p)$. It follows from (4.4) that z' is a simple root and so from (4.2) that z' is a zero of G. Hence $w' = h(z')$ is a zero of $g(w)$ and $w' \neq w_l$ because of $z' \neq z_{lj}$ $(j = 1, \cdots, p)$. On the other hand, since w_{l+1} belongs to the ring (4.8), we deduce from (4.3) that

$$|a_p| r^p_l\{(1 - \varepsilon)/(1 + \varepsilon)\}^{p/q}(1 - \varepsilon) < |w'| < |a_p| r^p_l\{(1 + \varepsilon)/(1 - \varepsilon)\}^{p/q+1}(1 + \varepsilon) .$$

Hence it follows from (4.5) that two elements w' and w_l of $\{w_j\}$ belong to the ring $K(r')$, where r' is a number satisfying $|a_p| r'^p(1 + \varepsilon) = |a_p| r^p_l\{(1 - \varepsilon)/(1 + \varepsilon)\}^{p/q}(1 - \varepsilon)$. This contradicts (4.6). Hence we obtain $q \leqq p$. Similarly, we also have $p \leqq q$. Therefore we obtain $p = q$, that is, all the degrees of polynomials belonging to $\mathfrak{H}_P(R, S)$ are the same p.

q.e.d.

5. Proof of Theorem 4. Since h and k belong to $\mathfrak{H}(R, S)$, h satisfies one of the following functional equations:

$$f_{h1}(z)^3 G(z) = g(h(z)) , \tag{5.1}$$

$$f_{h2}(z)^3 G(z)^2 = g(h(z)) , \tag{5.2}$$

where (f_{h1}, f_{h2}) satisfies the property (A), and k satisfies one of the following functional equations:

$$f_{k1}(z)^3 G(z) = g(k(z)) , \tag{5.3}$$

$$f_{k2}(z)^3 G(z)^2 = g(k(z)) , \tag{5.4}$$

where (f_{k1}, f_{k2}) satisfies the property (A). For $\varepsilon > 0$ satisfying $|a_p|(1 + \varepsilon)^3 < |b_p|(1 - \varepsilon)^3$, there exists a large number r_0 such that

$$|a_p| r^p(1 - \varepsilon) < |h(z)| < |a_p| r^p(1 + \varepsilon) , \tag{5.5}$$

$$|b_p| r^p(1 - \varepsilon) < |k(z)| < |b_p| r^p(1 + \varepsilon) , \tag{5.6}$$

$$h'(z) \neq 0 \quad \text{and} \quad k'(z) \neq 0 \tag{5.7}$$

are valid for all $r > r_0$, $r = |z|$.

Let $\{w_j\}_{j=1}^{\infty}$ be the set of zeros of g satisfying $|w_j| > |b_p| r^p_0(1 + \varepsilon)$ without considering their multiplicities and assume that $|w_1| \leqq |w_2| \leqq \cdots$.

Then from an argument similar to that in the proof of our Theorem 2 we deduce that

$$(5.8) \qquad \bar{n}(|b_p|r^p(1-\varepsilon), 0, g) - \bar{n}(|a_p|r^p(1+\varepsilon), 0, g) = 0 \text{ or } 1$$

for all $r > r_0$ and consequently

$$0 < \left|\frac{w_j}{w_{j+1}}\right| \leqq \frac{|a_p|(1+\varepsilon)}{|b_p|(1-\varepsilon)} < 1 \qquad \text{for all } j .$$

Hence the exponent of convergence of the sequence $\{w_j\}$ is zero. Since the zeros of g are of order at most two, we have $\rho(N(r, 0, g)) = 0$, that is, $\rho(g) = 0$. From (II) of Lemma 3.1 we also have $\rho(G) = 0$ because of $\rho(N(r, 0, g \circ h)) = 0$ for a polynomial h. Thus we obtain (a).

Now we shall prove (b). From a discussion similar to that in the proof of Theorem 2, we can deduce that the images under k of all roots of the equation $h(z) = w_j$ must be w_{j+1}. Hence we have

$$(5.9) \qquad k(z) = (b_p/a_p)h(z) + w_{j+1} - (b_p/a_p)w_j \qquad \text{for all } j \geqq 2 .$$

By setting $A = w_{j+1} - (b_p/a_p)w_j$ we obtain (b).

Next we shall prove (c). Let $h_1(z)$ be an arbitrary element belonging to $\mathfrak{H}(R, S)$. Then it follows from Theorems 2 and 3 that $\mathfrak{H}(R, S) = \mathfrak{H}_P(R, S)$ and $h_1(z)$ is a polynomial of degree p. We put $h_1(z) = c_p z^p + \cdots + c_0$ $(c_p \neq 0)$. If $|c_p| < |b_p|$, then, by the above argument, we deduce that

$$k(z) = (b_p/c_p)h_1(z) + w_{j+1} - (b_p/c_p)w_j \qquad \text{for all } j > j_1 \geqq 2$$

and so using (5.9) we have

$$(1/c_p - 1/a_p)w_j = h_1(z)/c_p - h(z)/a_p \qquad \text{for all } j > j_1 .$$

Therefore we have $c_p = a_p$ and consequently $h_1(z) \equiv h(z)$. If $|c_p| \geqq |b_p|$, then we similarly deduce that

$$h_1(z) = (c_p/a_p)h(z) + w_{j+1} - (c_p/a_p)w_j \qquad \text{for all } j > j_2 \geqq 2$$

and so

$$(1/c_p - 1/b_p)w_{j+1} = h_1(z)/c_p - k(z)/b_p \qquad \text{for all } j > j_2 .$$

Hence we have $c_p = b_p$ and so $h_1(z) \equiv k(z)$. Thus we have proved (c).

Next we shall prove (d) and (e). We now note that $k(z) = \lambda h(z) + A$, where $\lambda = b_p/a_p$ and $A = w_{j+1} - \lambda w_j$ for all $j \geqq 2$, and that all roots of the equations $h(z) = w_j$ and $k(z) = w_j$ are simple. We consider the following two cases (A) and (B):

(A) The case where w_j is a simple zero of g. Suppose that the functional equation (5.1) is valid. Then the roots z_{jl} $(l = 1, \cdots, p)$ of the equation $h(z) = w_j$ are simple zeros of G and $k(z_{jl}) = \lambda w_j + A = w_{j+1}$.

If w_{j+1} is a simple zero of g, then, since z_{jl} are simple zeros of $g \circ k$, the functional equation (5.3) must be valid. Since the roots $z_{j+1,l}$ of the equation $h(z) = w_{j+1}$ are also simple zeros of G, it follows from (5.3) that $k(z_{j+1,l}) = w_{j+2}$ is a simple zero of g.

If w_{j+1} is a double zero of g, then z_{jl} are double zeros of $g \circ k$. Hence the functional equation (5.4) must be true. Since the roots $z_{j+1,l}$ of the equation $h(z) = w_{j+1}$ are double zeros of $G(z)$, (5.4) implies that $k(z_{j+1,l}) = w_{j+2}$ is a simple zero of g.

Next suppose that the functional equation (5.2) is valid. Then the roots z_{jl} $(l = 1, \cdots, p)$ of the equation $h(z) = w_j$ are double zeros of G.

If w_{j+1} is a simple zero of g, then z_{jl} are simple zeros of $g \circ k$. Hence the functional equation (5.4) must be true. Since the roots $z_{j+1,l}$ of $h(z) = w_{j+1}$ are double zeros of G, (5.4) implies that $k(z_{j+1,l}) = w_{j+2}$ is a simple zero of g.

If w_{j+1} is a double zero, then z_{jl} are double zeros of $g \circ k$. Hence the functional equation (5.3) must be valid. Similarly we deduce that w_{j+2} is a simple zero of g.

(B) The case where w_j is a double zero of g. From a discussion similar to that for (A) we can deduce that if (5.1) is valid and w_{j+1} is simple, then (5.4) is valid and w_{j+2} is double; if (5.1) is valid and w_{j+1} is double, then (5.3) is valid and w_{j+2} is double; if (5.2) is valid and w_{j+1} is simple, then (5.3) is valid and w_{j+2} is double; finally if (5.2) is valid and w_{j+1} is double, then (5.4) is valid and w_{j+2} is double.

Therefore, from the arguments for (A) and (B) we have the following three cases (I), (II) and (III):

(I) The case where $\{w_j\}$ are all simple zeros. It follows from the reasoning for (A) that the functional equations (5.1) and (5.3) are valid or the functional equations (5.2) and (5.4) are valid. In either case we have the functional equation

$$F(z)^3 g(h(z)) = g(k(z)) , \tag{5.10}$$

where F is a rational function of degree at most $p - 1$ because $\rho(g \circ h) = \rho(g \circ k) = 0$ and F has at most zeros and poles at the zeros of k' and h'. We put

$$g(w) = q(w) \prod_{j=2}^{\infty} (1 - w/w_j) ,$$

where q is a polynomial having only simple or double zeros which are distinct from w_j $(j \geqq 2)$. Then noting that $k(z) = \lambda h(z) + A$, $\lambda = b_p/a_p$ and $A = w_{j+1} - \lambda w_j$ for all $j \geqq 2$, we obtain

$$g(k(z)) = q(k(z))\Big(1 - \frac{k(z)}{w_2}\Big)\prod_{j=2}^{\infty}\Big(1 - \frac{\lambda h(z) + w_{j+1} - \lambda w_j}{w_{j+1}}\Big)$$
$$= q(k(z))\Big(1 - \frac{k(z)}{w_2}\Big)C\prod_{j=2}^{\infty}\Big(1 - \frac{h(z)}{w_j}\Big),$$

where $C = \prod_{j=2}^{\infty}(\lambda w_j/w_{j+1})$. Hence it follows from (5.10) that

$$F(z)^3 = q(k(z))(1 - k(z)/w_2)/q(h(z)) ,$$

that is,

$$F(k^{-1}(w))^3 = q(w)(1 - w/w_2)/q((w - A)/\lambda) . \tag{5.11}$$

We deduce from (5.11) that $f(w) \equiv F(k^{-1}(w))^3$ is a single-valued rational function of w of the form $f(w) = p_1(w)/p_2(w)$, where p_1 and p_2 are mutually prime polynomials with only simple or double zeros such that $\deg p_1 = \deg p_2 + 1$. Since $F(z)^3 = f(k(z))$, it follows from Lemma 2.2 that $f(w) = B(w - \alpha_1)$, where B and α_1 are constants. Hence (5.10) implies

$$B(w - \alpha_1)g((w - A)/\lambda) = g(w) , \tag{5.12}$$

that is,

$$g(\lambda w + A) = B(\lambda w + A - \alpha_1)g(w) .$$

Further, taking the multiplicity of zeros of both sides of (5.12) into account and noting that $g(w)$ has only simple zeros in $|w| > r_0$, we deduce that

$$(\alpha_1 - A)/\lambda \neq \alpha_1, \text{ that is, } \alpha_1 \neq -A/(\lambda - 1) \quad \text{and} \quad g((\alpha_1 - A)/\lambda) \neq 0 .$$

Thus we obtain (i) in (d). Moreover in this case, we have $F(z)^3 = f(k(z)) = B(k(z) - \alpha_1)$, and consequently (e).

(II) The case where $\{w_j\}$ are all double zeros. In this case, either (5.1) and (5.3) are valid or (5.2) and (5.4) are valid. Hence in either case we have the functional equation

$$F(z)^3 g(h(z)) = g(k(z)) , \tag{5.13}$$

where F is a rational function of degree at most $p - 1$. We put

$$g(w) = q(w)\prod_{j=2}^{\infty}(1 - w/w_j)^2 ,$$

where q is a polynomial having only simple or double zeros which are distinct from w_j $(j \geqq 2)$. By the same procedure as in the case (I) we deduce that

$$f(w) \equiv F(k^{-1}(w))^3 = q(w)(1 - w/w_2)^2/q((w - A)/\lambda) . \tag{5.14}$$

Since f is a single-valued function of w, (5.14) and Lemma 2.2 imply $f(w) =$

$B(w-\alpha_1)^2$, where B and α_1 are constants. Hence it follows from (5.13) that

$$B(w-\alpha_1)^2 g((w-A)/\lambda) = g(w) ,$$

that is,

$$g(\lambda w + A) = B(\lambda w + A - \alpha_1)^2 g(w)$$

is valid. Similarly, we also have

$$\alpha_1 \neq -A/(\lambda - 1) \quad \text{and} \quad g((\alpha_1 - A)/\lambda) \neq 0 .$$

Thus we obtain (ii) in (d). Further, in this case we have $F(z)^3 = f(k(z)) = B(k(z)-\alpha_1)^2$ and consequently $k(z) = \alpha_1 + P(z)^3$, where $P(z)$ is a polynomial, that is, we have (e).

(III) The case where $\{w_j\}$ are alternately simple and double zeros. We put

$$g(w) = q(w) \prod_{j=2}^{\infty} (1 - w/w_j)^{(3+(-1)^j)/2} ,$$

where q is a polynomial having only simple or double zeros which are distinct from w_j ($j \geqq 2$). In this case, it follows from the discussions in (A) and (B) that either the functional equations (5.1) and (5.4) are valid or the functional equations (5.2) and (5.3) are valid.

Suppose that (5.1) and (5.4) are valid. Then we have

$$F(z)^3 g(h(z))^2 = g(k(z)) , \tag{5.15}$$

where F is a meromorphic function. By the same reasoning as in the case (I) we obtain

$$F(z)^3 = Cq(k(z))(1 - k(z)/w_2)^2 \Big/ \left[q(h(z))^2 \left\{ \prod_{j=2}^{\infty} (1 - h(z)/w_j)^{(1+(-1)^j)/2} \right\}^3 \right] ,$$

that is,

$$\begin{aligned} f(w) &\equiv F(k^{-1}(w))^3 \\ &= Cq(w)(1 - w/w_2)^2 \Big/ \left[q((w-A)/\lambda)^2 \right. \\ &\quad \times \left. \left\{ \prod_{j=2}^{\infty} (1 - (w-A)/(\lambda w_j))^{(1+(-1)^j)/2} \right\}^3 \right] . \end{aligned} \tag{5.16}$$

Now we can put $f(w) = Q(w)/H(w)^3$, where Q is a rational function whose zeros and poles are simple or double and H is an entire function having only simple zeros which are different from the zeros of Q. Since we can write $Q(k(z)) = F_1(z)^3$, where F_1 is a suitable rational function, (5.16) and Lemma 2.2 imply $Q(w) = B(w-\alpha_1)$ or $Q(w) = B(w-\alpha_1)^2$ and $H(\alpha_1) \neq 0$, where B and α_1 are constants. Hence it follows from (5.15) that

$$(5.17) \qquad B(w-\alpha_1)g((w-A)/\lambda)^2 = H(w)^3 g(w)\,, \qquad H(\alpha_1) \neq 0$$

or

$$(5.18) \qquad B(w-\alpha_1)^2 g((w-A)/\lambda)^2 = H(w)^3 g(w)\,, \qquad H(\alpha_1) \neq 0\,.$$

Further, taking the multiplicity of the zeros of both sides of (5.17) and (5.18) and $H(\alpha_1) \neq 0$ into account and noting that $g(w)$ has only zeros of order at most two, we deduce that $\alpha_1 \neq -A/(1-\lambda)$ and $g((\alpha_1 - A)/\lambda) \neq 0$. Thus we obtain either (iii) or (iv) in (d). In this case we have $F_1(z)^3 = Q(k(z)) = B(k(z)-\alpha_1)$ or $= B(k(z)-\alpha_1)^2$ and consequently (e).

Next suppose that (5.2) and (5.3) are valid. Then we have

$$(5.19) \qquad F(z)^3 g(h(z)) = g(k(z))^2\,,$$

where F is a meromorphic function. So we can deduce that

$$(5.20) \qquad \begin{aligned} f(w) &\equiv F(k^{-1}(w))^3 \\ &= C^2 q(w)^2 (1 - w/w_2) \\ &\quad \times \left\{(1-w/w_2) \prod_{j=2}^{\infty} (1-(w-A)/(\lambda w_j))^{(1-(-1)^j)/2}\right\}^3 \Big/ q((w-A)/\lambda) \\ &\equiv Q(w)H(w)^3\,, \end{aligned}$$

where Q is a rational function whose zeros and poles are of order at most two and H is an entire function having only simple zeros such that the simple zeros of Q are also zeros of H and the double zeros and poles of Q are not zeros of H. Since we can write $Q(k(z)) = F_1(z)^3$ with a suitable rational function F_1, (5.20) and Lemma 2.2 imply that $Q(w) = B(w-\alpha_1)$, $H(\alpha_1) = 0$ or $Q(w) = B(w-\alpha_1)^2$, $H(\alpha_1) \neq 0$, where B and α_1 are constants. Hence from (5.19) and (5.20) we have

$$B(w-\alpha_1)H(w)^3 g((w-A)/\lambda) = g(w)^2\,, \qquad H(\alpha_1) = 0$$

or

$$B(w-\alpha_1)^2 H(w)^3 g((w-A)/\lambda) = g(w)^2\,, \qquad H(\alpha_1) \neq 0\,.$$

Similarly, we also have $\alpha_1 \neq -A/(1-\lambda)$ and $g((\alpha_1 - A)/\lambda) \neq 0$. Thus we obtain either (v) or (vi) in (d). In this case we have $F_1(z)^3 = Q(k(z)) = B(k(z)-\alpha_1)$ or $= B(k(z)-\alpha_1)^2$ and consequently (e). Therefore the proofs of (d) and (e) are complete.

Now, from the equations of (i)–(vi) in (d), $\alpha_1 \neq -A/(\lambda-1)$, $g((\alpha_1 - A)/\lambda) \neq 0$ and the property of H, we can deduce that in every case $\{\alpha_j\}_{j=1}^{\infty}$ defined by $\alpha_{j+1} = \lambda\alpha_j + A$ $(j \geqq 1)$, that is, $\lambda_{j+1} = \lambda^j \alpha_1 + (\lambda^j - 1)A/(\lambda - 1)$ $(j = 0, 1, \cdots)$ are zeros of g, and moreover with respect to their multiplicities and zeros of H, the corresponding one of (i)–(vi) in (f) is valid.

Let β_1 be a zero of g distinct from α_j $(j \geqq 1)$. Then we deduce from the equations (i)-(vi) in (d) that in every case $\{\beta_j\}_{j=1}^{\infty}$ defined by $\beta_{j+1} = (\beta_j - A)/\lambda$ $(j \geqq 1)$, that is, $\beta_{j+1} = \lambda^{-j}\beta_1 - \lambda(1 - \lambda^{-j-1})A/(\lambda - 1)$ are zeros of g without counting their multiplicities, and the sequence $\{\beta_j\}$ converges to $-\lambda A/(\lambda - 1)$, which is a contradiction. Therefore g has no zero other than $\{\alpha_j\}_{j=1}^{\infty}$. Thus we obtain (f).

Finally we shall prove (g), that is, we shall give examples of realization of our six cases.

Let g be the following:

$$g(w) = \prod_{j=1}^{\infty}(1 - w/\alpha_j) \qquad \text{in the case (i) in (d),}$$

$$g(w) = \prod_{j=1}^{\infty}(1 - w/\alpha_j)^2 \qquad \text{in the case (ii),}$$

$$g(w) = \prod_{j=1}^{\infty}(1 - w/\alpha_j)^{(3+(-1)^j)/2} \qquad \text{in the cases (iii) and (vi)}$$

and

$$g(w) = \prod_{j=1}^{\infty}(1 - w/\alpha_j)^{(3-(-1)^j)/2} \qquad \text{in the cases (iv) and (v).}$$

Here since the exponent of convergence of the sequence is zero, all the above products converge. Let $P(z)$ be a polynomial such that $P'(z) \neq 0$ on the set $\{z; z = P^{-1}((\alpha_j - \alpha_1)^{1/3}),\ j = 2, 3, \cdots\}$. Put $k(z) = \alpha_1 + P(z)^3$ and $h(z) = (k(z) - A)/\lambda$. Then all roots of the equations $k(z) = \alpha_j$ $(j = 2, 3, \cdots)$ and $h(z) = \alpha_j$ $(j = 1, 2, \cdots)$ are simple. Hence the zeros of $G_1(z) := g(h(z))$ are simple or double according to the order of zeros of g. Let R_1, R_2 and S be regularly branched three-sheeted algebroid Riemann surfaces defined by $y^3 = G_1(z)$, $y^3 = G_2(z)$ and $u^3 = g(w)$, respectively, where G_2 is an entire function defined later.

First of all, we have $h \in \mathfrak{H}(R_1, S)$ by $G_1(z) = g(h(z))$ and Theorem A.

Case (i). It follows from the equation (i) in (d) that

$$\{B_1P(z)\}^3G_1(z) = g(k(z)) ,$$

where $B_1 = B^{1/3}$ and consequently $k \in \mathfrak{H}(R_1, S)$ by Theorem A. Here the zeros of G_1 are all simple. On the other hand, we put $G_2(z) = g(h(z))^2$. Then the zeros of $G_2(z)$ are all double and from the equation (i) in (d) we have

$$\{1/g(h(z))\}^3G_2(z)^2 = g(h(z)) , \qquad \{B_1P(z)/g(h(z))\}^3G_2(z)^2 = g(k(z))$$

and so $h, k \in \mathfrak{H}(R_2, S)$ by Theorem A.

Case (ii). From the equation (ii) in (d) we have

$$\{B_1P(z)^2\}^3G_1(z) = g(k(z))$$

and so $k \in \mathfrak{H}(R_1, S)$. Here the zeros of $G_1(z) = g(h(z))$ are all double. Hence we can define $G_2(z)$ by $g(h(z))^{1/2}$ with a fixed branch. Then we have

$$G_2(z)^2 = g(h(z)) \quad \text{and} \quad \{B_1P(z)^2\}^3 G_2(z)^2 = g(k(z))$$

and consequently $h,\ k \in \mathfrak{H}(R_2, S)$.

Case (iii). From the equation (iii) in (d) we have

$$\{B_1P(z)/H(k(z))\}^3 G_1(z)^2 = g(k(z))$$

and so $k \in \mathfrak{H}(R_1, S)$. Next we put $G_2(z) = g(h(z))^2/H(k(z))^3$. Then it follows from (iii) in (f) that the zeros of G_2 are all simple or double and from the definition of G_2 and the equation (iii) in (d) that

$$\{H(k(z))^2/g(h(z))\}^3 G_2(z)^2 = g(h(z)) \quad \text{and} \quad \{B_1P(z)\}^3 G_2(z) = g(k(z))\ ,$$

and so $h,\ k \in \mathfrak{H}(R_2, S)$.

Case (iv). We have

$$\{B_1P(z)^2/H(k(z))\}^3 G_1(z)^2 = g(k(z))$$

and so $k \in \mathfrak{H}(R_1, S)$. We put $G_2(z) = g(h(z))^2/H(k(z))^3$. Then we have

$$\{H(k(z))^2/g(h(z))\}^3 G_2(z)^2 = g(h(z)) \quad \text{and} \quad \{B_1P(z)^2\}^3 G_2(z) = g(k(z))\ ,$$

and so $h,\ k \in \mathfrak{H}(R_2, S)$.

Case (v). We have

$$\{B_1^2P(z)^2H(k(z))^2/g(k(z))\}^3 G_1(z)^2 = g(k(z))$$

and so $k \in \mathfrak{H}(R_1, S)$. We put $G_2(z) = g(h(z))^2/H(h(z))^3$. Then we have

$$\{H(h(z))^2/g(h(z))\}^3 G_2(z)^2 = g(h(z))$$

and

$$\{B_1P(z)^2H(k(z))^2H(h(z))/g(k(z))\}^3 G_2(z) = g(k(z))\ ,$$

and so $h,\ k \in \mathfrak{H}(R_2, S)$.

Case (vi). We have

$$\{B_1^2P(z)^4H(k(z))^2/g(k(z))\}^3 G_1(z)^2 = g(k(z))$$

and so $k \in \mathfrak{H}(R_1, S)$. We put $G_2(z) = g(h(z))^2/H(h(z))^3$. Then we have

$$\{H(h(z))^2/g(h(z))\}^3 G_2(z)^2 = g(h(z))$$

and

$$\{B_1^2P(z)^4H(k(z))^2H(h(z))/g(k(z))\}^3 G_2(z) = g(k(z))\ ,$$

and so $h,\ k \in \mathfrak{H}(R_2, S)$.

Thus the proof of (g) is complete and consequently so is the proof of Theorem 4. q.e.d.

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FACULTY OF TECHNOLOGY
KANAZAWA UNIVERSITY
40-20, KODATSUNO 2-CHOME
KANAZAWA 920
JAPAN