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A CANONICAL DECOMPOSITION OF AUTOMORPHIC FORMS WHICH VANISH ON AN INVARIANT MEASURABLE SUBSET

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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Introduction. Let Γ be a discrete subgroup of the real Möbius group $PSL(2; \boldsymbol{R})$. We denote by $\Omega(\Gamma)$ the region of discontinuity of Γ. Let σ be a Γ-invariant closed subset of the extended real line $\hat{\boldsymbol{R}}$ such that $\sharp\sigma \geqq 3$ and $\sigma \ni \infty$, and let D be the component of $\Omega(\Gamma) - \sigma$ containing the upper half-plane U. Then $D = U$ or $D = \Omega(\Gamma) - \sigma$ according as $\sigma = \hat{\boldsymbol{R}}$ or not. Let E be a Γ-invariant measurable subset of D, and put $V = D - E$, where if $D \neq U$, then E is assumed to be symmetric with respect to $\boldsymbol{R}$ in the sense that $\bar{z} \in E$ whenever $z \in E$. Furthermore, for an integer $q \geqq 2$, let L^p, $1 \leqq p < \infty$, (resp. L^∞) be the Banach space consisting of all the p-integrable (resp. bounded) measurable automorphic forms of weight $-2q$ on D for Γ, which are symmetric if D is symmetric (see Section 1 for the precise definition). We denote by A^p, $1 \leqq p \leqq \infty$, the closed subspace consisting of all the holomorphic elements in L^p, and set $L^p(V) = \{\mu \in L^p; \mu|_E = 0\}$ and $A^p|_V = \{\chi_V\phi; \phi \in A^p\}$, where χ_V is the characteristic function of V. For $1 \leqq p < \infty$ and p' satisfying $1/p + 1/p' = 1$, $L^{p'}$ is isomorphic to the dual space of L^p. We denote by $(A^p)^\perp$ $(\subset L^{p'})$ the annihilator of A^p.

In the present paper, we investigate conditions for E under which $(A^p)^\perp \cap L^{p'}(V)$ and $A^{p'}|_V$ are closed and complementary to each other in $L^{p'}(V)$, and give two kinds of answers to this question (see Theorems 1 and 3 below). This problem occured in studying extremal quasiconformal mappings with dilatation bound (see, for example, Sakan [10]). Our results can be applied to the study of quasiconformal mappings and Teichmüller spaces. These applications will be discussed in Ohtake [9].

Throughout this paper, as natural assumptions for the problem, we require that V *has positive measure and* $A^p \neq \{0\}$. We note that if E has (2-dimensional Lebesgue) measure zero, then the spaces $(A^p)^\perp \cap$

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$L^{p'}(V)(=(A^p)^{\perp})$ and $A^{p'}|_V$ $(=A^{p'})$ are closed and complementary to each other; this is classical and well-known.

In Section 1, we give some definitions and recall known results. In Section 2, we state our main results on the problem mentioned above. The proofs will be given in Sections 3 and 4.

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1. Preliminaries. Let Γ, σ, D, E and V be as in Introduction and let $\lambda = \lambda_D$ be the hyperbolic metric for D with constant negative curvature -4. We fix once and for all an integer $q \geqq 2$. A measurable automorphic form of weight $-2q$ on D for Γ is a measurable function μ on D which satisfies

$$(\mu\circ\gamma)(\gamma')^q = \mu \quad \text{for all} \quad \gamma\in\Gamma .$$

Such an automorphic form μ is said to be p-integrable for p, $1 \leqq p < \infty$, (resp. bounded), if

$$\|\mu\|_p = \left(\iint_{D/\Gamma} \lambda(z)^{2-qp}|\mu(z)|^p|dz\wedge d\bar{z}|\right)^{1/p} < \infty$$

$$(\text{resp. } \|\mu\|_\infty = \operatorname*{ess\,sup}_{z\in D} \lambda(z)^{-q}|\mu(z)| < \infty) .$$

We then denote by $L_q^p(D, \Gamma)$ (resp. $L_q^\infty(D, \Gamma)$) the complex Banach space consisting of all the p-integrable (resp. bounded) automorphic forms of weight $-2q$ on D for Γ. For p, $1 \leqq p \leqq \infty$, $A_q^p(D, \Gamma)$ denotes the closed subspace of all the holomorphic elements in $L_q^p(D, \Gamma)$. Furthermore, if D is symmetric with respect to $\boldsymbol{R}$, then we define the real Banach spaces of all the symmetric functions in $L_q^p(D, \Gamma)$ and $A_q^p(D, \Gamma)$ by

$$L_q^p(D, \Gamma)_{\mathrm{sym}} = \{\mu\in L_q^p(D, \Gamma);\ \mu(\bar{z}) = \bar{\mu}(z) \text{ for a.e. } z\in D\}$$

and

$$A_q^p(D, \Gamma)_{\mathrm{sym}} = A_q^p(D, \Gamma)\cap L_q^p(D, \Gamma)_{\mathrm{sym}} ,$$

respectively.

We use the following result:

PROPOSITION A. *There exists a unique function $F = F_{D,\Gamma}$ on $D\times D$ with the following properties, where $c_q = (2q-1)/(q-1)$:*

$$F(z, \zeta) = -\bar{F}(\zeta, z) , \tag{1.1}$$

$$F(\cdot, \zeta)\in A_q^p(D, \Gamma) \tag{1.2}$$

for every fixed $\zeta \in D$ *and every* p, $1 \leqq p \leqq \infty$,

$$\iint_{D/\Gamma} \lambda(\zeta)^{2-q}|F(z,\zeta)||d\zeta \wedge d\bar{\zeta}| \leqq c_q \lambda(z)^q\,, \quad \textit{and} \tag{1.3}$$

$$\phi(z) = \iint_{D/\Gamma} \lambda(\zeta)^{2-2q}F(z,\zeta)\phi(\zeta)d\zeta \wedge d\bar{\zeta} \tag{1.4}$$

for every $\phi \in A_q^p(D, \Gamma)$, $1 \leqq p \leqq \infty$, *and every* $z \in D$.

The uniqueness of $F_{D,\Gamma}$ above follows from (1.1), (1.2) and (1.4). In fact, let F_1 and F_2 have these three properties. Then we have

$$\begin{aligned} F_1(z,\zeta) &= \iint_{D/\Gamma} \lambda(w)^{2-2q}F_2(z,w)F_1(w,\zeta)dw \wedge d\bar{w} \\ &= \iint_{D/\Gamma} \lambda(w)^{2-2q}\bar{F}_2(w,z)\bar{F}_1(\zeta,w)dw \wedge d\bar{w} \\ &= \left(-\iint_{D/\Gamma} \lambda(w)^{2-2q}F_1(\zeta,w)F_2(w,z)dw \wedge d\bar{w}\right)^- \\ &= -\bar{F}_2(\zeta,z) = F_2(z,\zeta)\,. \end{aligned}$$

For a proof of the assertion except the uniquess of $F_{D,\Gamma}$, see Kra [5, p. 89 and p. 101]. In [5, p. 101] D is assumed to be conformally equivalent to the unit disk, but we can easily check that the argument is applicable to our case.

For $\mu \in L_q^p(D, \Gamma)$, $1 \leqq p \leqq \infty$, define

$$\beta[\mu](z) = \iint_{D/\Gamma} \lambda(\zeta)^{2-2q}F(z,\zeta)\mu(\zeta)d\zeta \wedge d\bar{\zeta}\,, \quad z \in D\,.$$

Then β is a bounded projection of $L_q^p(D, \Gamma)$ onto $A_q^p(D, \Gamma)$, of norm $\leqq c_q$ (see [5, p. 90 and p. 101]). When D is symmetric with respect to $\boldsymbol{R}$, (1.1), (1.2) and (1.4) imply

$$\bar{F}(\bar{z}, \bar{\zeta}) = -F(z,\zeta)\,,$$

since

$$\begin{aligned} \bar{F}(\bar{z},\bar{\zeta}) &= \iint_{D/\Gamma} \lambda(w)^{2-2q}F(z,w)\bar{F}(\bar{w},\bar{\zeta})dw \wedge d\bar{w} \\ &= \iint_{D/\Gamma} \lambda(\bar{w})^{2-2q}F(z,\bar{w})\bar{F}(w,\bar{\zeta})dw \wedge d\bar{w} \\ &= \iint_{D/\Gamma} \lambda(w)^{2-2q}\bar{F}(\bar{w},z)F(\bar{\zeta},w)dw \wedge d\bar{w} \\ &= \bar{F}(\zeta,z) = -F(z,\zeta)\,. \end{aligned}$$

Hence we see that $\beta[\mu] \in A_q^p(D, \Gamma)_{\mathrm{sym}}$ whenever $\mu \in L_q^p(D, \Gamma)_{\mathrm{sym}}$, since we have

$$\overline{\beta[\mu]}(\bar{z}) = \left(\iint_{D/\Gamma} \lambda(\zeta)^{2-2q} F(\bar{z}, \zeta)\mu(\zeta) d\zeta \wedge d\bar{\zeta}\right)^{-}$$
$$= \iint_{D/\Gamma} \lambda(\bar{\zeta})^{2-2q} F(z, \bar{\zeta})\mu(\bar{\zeta}) d\zeta \wedge d\bar{\zeta} = \beta[\mu](z) .$$

This implies that the integral operator β above is also a bounded projection of $L_q^p(D, \Gamma)_{\mathrm{sym}}$ onto $A_q^p(D, \Gamma)_{\mathrm{sym}}$ of norm $\leqq c_q$.

For simplicity we often write L^p (resp. A^p) instead of $L_q^p(D, \Gamma)$ (resp. $A_q^p(D, \Gamma)$) when $D = U$, and $L_q^p(D, \Gamma)_{\mathrm{sym}}$ (resp. $A_q^p(D, \Gamma)_{\mathrm{sym}}$) when $D \neq U$. We set

$$L^p(V) = \{\mu \in L^p;\ \mu|_E = 0\} ,$$

and

$$A^p|_V = \{\chi_V \phi;\ \phi \in A^p\} ,$$

where χ_X stands for the characteristic function of a measurable subset X of D. In what follows, we assume that the numbers p and p' satisfy $1 \leqq p < \infty$ and $1/p + 1/p' = 1$ $(1/\infty = 0)$.

For $\mu \in L^p$ and $\nu \in L^{p'}$, we define the Petersson scalar product (μ, ν) of μ and ν by

$$(1.5) \qquad (\mu, \nu) = \iint_{D/\Gamma} \lambda(z)^{2-2q} \mu(z)\bar{\nu}(z) |dz \wedge d\bar{z}| .$$

Obviously we have

$$(1.6) \qquad |(\mu, \nu)| \leqq \|\mu\|_p \|\nu\|_{p'} .$$

We note that (μ, ν) above is i times (μ, ν) in [5, p. 88]. We adopt (1.5), however, because for symmetric μ and ν, we have

$$(\mu, \nu) = 2 \operatorname{Re} \iint_{U/\Gamma} \lambda_D(z)^{2-2q} \mu(z)\bar{\nu}(z) |dz \wedge d\bar{z}| \in \boldsymbol{R} .$$

This scalar product establishes isometric isomorphisms between $L^{p'}$ and $(L^p)^*$, and between $L^{p'}(V)$ and $L^p(V)^*$, where X^* stands for the dual space of a normed vector space X. These isomorphisms are anti-linear when $D = U$. By (1.1) and Fubini's theorem, we have

$$(1.7) \qquad (\beta[\mu], \nu) = (\mu, \beta[\nu]) \quad \text{for} \quad \mu \in L^p \quad \text{and} \quad \nu \in L^{p'} .$$

For a subset S of L^p, we set

$$S^\perp = \{\nu \in L^{p'}; (\mu, \nu) = 0 \text{ for all } \mu \in S\} .$$

Since β is a projection satisfying (1.7), we see

$$(1.8) \qquad (\ker \beta) \cap L^{p'} = \{\nu - \beta[\nu]; \nu \in L^{p'}\} = (A^p)^\perp .$$

2. Statements of the main results. In this section we state our results on the problem in Introduction.

A closed subspace X_1 of a Banach space X is said to *split in* X if there exists a closed subspace X_2 of X, complementary to X_1, that is, $X_1 + X_2 = X$ and $X_1 \cap X_2 = \{0\}$.

THEOREM 1. *Let* $1 \leqq p < \infty$ *and* p' *satisfying* $1/p + 1/p' = 1$, *and set*

$$b = \sup_{\phi \in A^p} \|\chi_V \phi\|_p / \|\beta[\chi_V \phi]\|_p ,$$

here and in what follows, we conform to the convention:

$$0/0 = 0 , \quad \textit{and} \quad a/0 = +\infty \quad \textit{if} \quad a > 0 .$$

(I) *Then the following four conditions are equivalent to each other.*

(a) *The subspaces* $(A^p)^\perp \cap L^{p'}(V)$ *and* $A^{p'}|_V$ *of the Banach space* $L^{p'}(V)$ *are closed and complementary to each other. In particular,* $(A^p)^\perp \cap L^{p'}(V)$ *splits in* $L^{p'}(V)$.

(b) *There exists a bounded linear mapping* β_V *of* $L^{p'}(V)$ *onto* $A^{p'}$ *such that*

$$\ker \beta_V = (A^p)^\perp \cap L^{p'}(V) = \{\nu - \chi_V \beta_V[\nu]; \nu \in L^{p'}(V)\} . \tag{2.1}$$

(c) *The number* b *is finite and*

$$A^{p'}|_V \cap (A^p)^\perp = \{0\} . \tag{2.2}$$

(d) *The number* b *is finite and*

$$\beta[A^p|_V] = \{\beta[\chi_V \phi]; \phi \in A^p\} \quad \textit{is dense in} \quad A^p . \tag{2.3}$$

(II) *In* (I) *we have the inequality*

$$b \leqq \|\beta_V\| \leqq c_q b . \tag{2.4}$$

REMARK. It follows from Taylor [12, §4.8] that the condition (a) of Theorem 1 is equivalent to the following:

(a′) There exists a bounded projection of $L^{p'}(V)$ onto $A^{p'}|_V$ with kernel $(A^p)^\perp \cap L^{p'}(V)$.

We can easily see that, for β_V in the condition (b), $\chi_V \beta_V$ is a bounded projection with the property in (a′) above. A bounded projection in (a′) is unique ([12, §4.8]), and χ_V: $A^{p'} \to A^{p'}|_V$ is bijective. Hence, when (b) holds, a bounded linear mapping $\beta_V = \chi_V^{-1}(\chi_V \beta_V)$ is uniquely determined, and satisfies

$$\beta_V \chi_V = \text{id. on } A^{p'} . \tag{2.5}$$

In particular, β_V is none other than β whenever E is a null set.

We note that an operator similar to β_V has been studied from a different point of view, for example, in Schiffer-Spencer [11] and Komatsu-Ozawa [4].

THEOREM 2. *Suppose that one of the four conditions of Theorem* 1 *holds for* $1 \leqq p < \infty$ *and* p' *satisfying* $1/p + 1/p' = 1$. *If* $D = U$ (*resp.* $D \neq U$), *then an anti-linear* (*resp. linear*) *isomorphism between* $A^{p'}|_V$ *and* $(A^p|_V)^*$ *is established by the Petersson scalar product. Furthermore, if* $l \in (A^p|_V)^*$ *corresponds to* $\chi_V\psi \in A^{p'}|_V$ *under this isomorphism, then*

$$\|l\| \leqq \|\chi_V\psi\|_{p'} \leqq \|\chi_V\beta_V\| \, \|l\| \, .$$

Finally we give a sufficient condition for E under which (c) of Theorem 1 holds. To simplify the statements, we use the following notation:

$$W(z, \zeta) = \lambda(z)^{-q}\lambda(\zeta)^{-q}|F(z, \zeta)| \, , \quad z, \zeta \in D \, , \tag{2.6}$$

$$M(\zeta) = \sup_{z \in D} W(z, \zeta) \, , \tag{2.7}$$

and

$$dA(z) = \lambda(z)^2|dz \wedge d\bar{z}| \, .$$

THEOREM 3. *When* $p = 1$ *and* $p' = \infty$, *suppose that*

$$\int_{E/\Gamma} M^2 dA < \infty \, , \tag{2.8}$$

and

$$\text{Area}(E/\Gamma) = \int_{E/\Gamma} dA < \infty \, . \tag{2.9}$$

When $1 < p < 2 < p' < \infty$ *or* $1 < p' < 2 < p < \infty$, *suppose that*

$$\int_{E/\Gamma} W(z, z)^t dA(z) < \infty \quad \text{for} \quad t = p/2 \quad \text{and} \quad p'/2 \, , \tag{2.10}$$

and

$$\int_{E/\Gamma} M dA < \infty \, . \tag{2.11}$$

When $p = p' = 2$, *suppose that*

$$\int_{E/\Gamma} W(z, z) dA(z) < \infty \, .$$

Then we have (2.2) *and*

$$\sup_{\phi \in A^p} \|\phi\|_p / \|\beta[\chi_V\phi]\|_p < \infty \, . \tag{2.12}$$

In particular, (*c*) *of Theorem* 1 *holds.*

Here we note that (2.8) and (2.9) imply (2.11).

It is obvious that $W(\cdot, \cdot)$ is continuous and M is lower semi-continuous. Moreover, from results due to Bers [1], Earle [2], Lehner [6, 7], and Metzger and Rajeswara Rao [8], we can derive an estimate for M and a condition under which M is bounded. Namely, we have the following:

PROPOSITION 1. *For each real* $t > 1$ *and a fixed* (*holomorphic*) *universal covering* $\rho: \Delta = \{|w| < 1\} \to D$, *we have*

$$M(z) \leqq C \inf\{(1 - |w|^2)^{-t}; w \in \rho^{-1}(z)\} ,$$

where the constant C *depends on* q, t, ρ *and* Γ.

PROPOSITION 2. *If* $A^1 \subset A^\infty$, *then* M *is bounded. In particular, if a Fuchsian model* G *of* Γ *satisfies the condition*:

$$\inf\{|\mathrm{trace}\, g|;\ g \text{ is hyperbolic and in } G\} > 2 , \tag{2.13}$$

then M *is bounded.*

We regard the condition (2.13) above to hold, when G contains no hyperbolic elements. Note that the left hand-side of (2.13) is independent of the choice of G. By Theorem 3 and Proposition 2, we easily obtain:

THEOREM 4. *Suppose that* $\mathrm{Area}(E/\Gamma) < \infty$ *and* $A^1 \subset A^\infty$. *Then, for* $1 \leqq p < \infty$ *and* p' *satisfying* $1/p + 1/p' = 1$, (2.2) *and* (2.12) *hold.*

3. Proofs of Theorems 1 and 2. We use the following result due to Bers [1]:

PROPOSITION B. *For* $1 \leqq p < \infty$ *with* $1/p + 1/p' = 1$, *the Petersson scalar product induces an isomorphism between* $A^{p'}$ *and* $(A^p)^*$, *and this isomorphism is anti-linear if* $D = U$. *Furthermore, for* $\psi \in A^{p'}$ *and* $l \in (A^p)^*$ *corresponding to each other under this isomorphism, we have*

$$\|l\| \leqq \|\psi\|_{p'} \leqq c_q \|l\| . \tag{3.1}$$

Proposition B follows fromL emma 1 below.

LEMMA 1. *Let* X *be a Banach space*, A *a subspace of* X, *and* ι *the inclusion map of* A *into* X. *Let* ρ *be a bounded projection of a Banach space* Y *onto a closed subspace* B *of* Y, *and let* τ *be an isometric isomorphism of* Y *onto* X^*. *Suppose that*

$$\tau(\ker \rho) = \{l \in X^*; l(a) = 0 \text{ for all } a \in A\} . \tag{3.2}$$

Then there is an isomorphism $\tilde{\tau}$ *of* B *onto* A^* *such that* $\iota^*\tau = \tilde{\tau}\rho$, *where*

$\iota^*: X^* \to A^*$ is the conjugate mapping of ι, and

$$\|\tilde{\tau}(y)\| \leqq \|y\| \leqq \|\rho\| \, \|\tilde{\tau}(y)\| \quad \textit{for all } y \in B\,.$$

PROOF. Since $\iota^*(l) \in A^*$ is the restriction of $l \in X^*$ to A, (3.2) implies $\ker \rho = \ker(\iota^* \tau)$. Hence the existence of $\tilde{\tau}$ is trivial. Note that ι^* is surjective by the Hahn-Banach theorem. Since $\rho(y) = y$ for every $y \in B$, we have $\|\tilde{\tau}(y)\| = \|\iota^* \tau(y)\| \leqq \|y\|$ for $y \in B$. Let $l' \in X^*$ be one of the norm-preserving extensions of $l = \tilde{\tau}(y) \in A^*$, $y \in B$, by the Hahn-Banach theorem. Then $\|y\| = \|\rho \tau^{-1}(l')\| \leqq \|\rho\| \, \|l'\| = \|\rho\| \, \|l\|$. □

Let $X = L^p$, $A = A^p$, $\rho = \beta$, $Y = L^{p'}$ and $B = A^{p'}$, and let τ be the isomorphism induced by the Petersson scalar product. Since (1.8) implies (3.2), we obtain Proposition B.

PROOF OF THEOREM 1. (a) $\Leftrightarrow$ (b): By Remark following Theorem 1, it suffices to show that (a′) implies (b). Suppose that (a′) holds. Then, since (a′) is equivalent to (a), the subspace $A^{p'}|_V$ is closed in $L^{p'}(V)$, thus $A^{p'}|_V$ is a Banach space. Then, by Taylor [12, Theorem 4.2–H], χ_V is an isomorphism of $A^{p'}$ onto $A^{p'}|_V$. Hence we can take $\chi_V^{-1}\pi$ to be β_V in (b), where π is the bounded projection in (a′).

(2.2) $\Leftrightarrow$ (2.3) (hence (c) $\Leftrightarrow$ (d)): Suppose that (2.3) does not hold. Then there is a non-zero $l \in (A^p)^*$ such that $\ker l \supset \beta[A^p|_V]$. It follows from Proposition B that there is a non-zero $\psi \in A^{p'}$ for which $l(\cdot) = (\cdot, \psi)$. Thus by (1.7) we see that for all $\phi \in A^p$, $0 = (\beta[\chi_V \phi], \psi) = (\chi_V \phi, \beta[\psi]) = (\chi_V \phi, \psi) = (\phi, \chi_V \psi)$. Hence $A^{p'}|_V \cap (A^p)^\perp \neq \{0\}$. Conversely, let $\chi_V \psi \in A^{p'}|_V \cap (A^p)^\perp$. Then we see that $0 = (\phi, \chi_V \psi) = (\beta[\chi_V \phi], \psi)$ for all $\phi \in A^p$. By (2.3) and Proposition B, we have $\psi = 0$.

(d) $\Rightarrow$ (b): The condition (d) implies that the bounded linear operator $\beta: A^p|_V \to \beta[A^p|_V] \subset A^p$ has a bounded inverse β^{-1} which is defined on the dense subspace $\beta[A^p|_V]$ of A^p and maps $\beta[A^p|_V]$ into $L^p(V)$. Then the conjugate operator $(\beta^{-1})^*$ of β^{-1} is defined on $L^p(V)^*$, which maps $L^p(V)^*$ onto $(A^p)^*$ ([12, Theorem 4.7–A]); $(\beta^{-1})^*$ is bounded, in fact,

$$\|(\beta^{-1})^*\| = \|\beta^{-1}\| = b \tag{3.3}$$

([12, p. 214]), and $\ker(\beta^{-1})^* = (A^p|_V)^\perp (\subset L^p(V)^*)$ ([12, Theorem 4.6–C]). We define β_V as the mapping of $L^{p'}(V)$ to $A^{p'}$ induced by $(\beta^{-1})^*$ by means of the isomorphism of Proposition B and the isometric isomorphism between $L^p(V)^*$ and $L^{p'}(V)$. It is obvious that β_V is a bounded surjective linear mapping whose kernel is $(A^p)^\perp \cap L^{p'}(V) = (A^p|_V)^\perp$ $(\subset L^{p'}(V))$. The estimate (2.4) follows from (3.1) and (3.3). By the definition of β_V, we have

(3.4) $\quad (\chi_V\phi, \nu) = (\beta[\chi_V\phi], \beta_V[\nu])$ for all $\phi \in A^p$ and $\nu \in L^{p'}(V)$.

Since $(\chi_V\phi, \nu) = (\phi, \nu)$ and $(\beta[\chi_V\phi], \beta_V[\nu]) = (\phi, \chi_V\beta_V[\nu])$, we have

$$\nu - \chi_V\beta_V[\nu] \in (A^p)^\perp \cap L^{p'}(V) \quad \text{for all} \quad \nu \in L^{p'}(V) .$$

Since $(A^p)^\perp \cap L^{p'}(V) \subset \{\nu - \chi_V\beta_V[\nu]; \nu \in L^{p'}(V)\}$ is obvious, we obtain (2.1).

(b) $\Rightarrow$ (c): From (2.1) we obtain (3.4). This and (1.6) imply

$$\|\chi_V\phi\|_p = \sup_{\nu \in L^{p'}(V)} |(\chi_V\phi, \nu)|/\|\nu\|_{p'} \leqq \|\beta[\chi_V\phi]\|_p \|\beta_V\| ,$$

hence $b \leqq \|\beta_V\| < \infty$. Next, let $\chi_V\psi \in (A^p)^\perp \cap A^{p'}|_V$. From (2.5) and (2.1), we see $\psi = \beta_V[\chi_V\psi] = 0$. Hence we have (2.2). □

Theorem 2 follows easily from Theorem 1 and Lemma 1.

4. **Proofs of Theorem 3 and Propositions 1 and 2.** Again we begin by presenting some preliminary lemmas.

LEMMA 2. *For $1 \leqq p < \infty$ and p' satisfying $1/p + 1/p' = 1$, we have*

(4.1) $$\lambda(z)^{-q}\|F(\cdot, z)\|_{p'} \leqq c_q^{1/p'} M(z)^{1/p} \quad (c_q^{1/\infty} = 1) ,$$

(4.2) $$\lambda(z)^{-q}\|F(\cdot, z)\|_2 = W(z, z)^{1/2} ,$$

(4.3) $$\lambda(z)^{-q}|\phi(z)| \leqq c_q^{1/p'}\|\phi\|_p M(z)^{1/p} \quad \textit{for} \quad \phi \in A^p ,$$

and

(4.4) $$\lambda(z)^{-q}|\phi(z)| \leqq \|\phi\|_2 W(z, z)^{1/2} \quad \textit{for} \quad \phi \in A^2 .$$

PROOF. By Hölder's inequality we have

$$\|F(\cdot, z)\|_{p'} \leqq \|F(\cdot, z)\|_1^{1/p'} \|F(\cdot, z)\|_\infty^{1/p} .$$

Since $M(z) = \lambda(z)^{-q}\|F(\cdot, z)\|_\infty$, (4.1) follows from (1.1) and (1.3). Next, we have

$$\begin{aligned} \|F(\cdot, z)\|_2^2 &= \int_{D/\Gamma} \lambda(\zeta)^{-2q}\bar{F}(\zeta, z)F(\zeta, z)dA(\zeta) \\ &= -i\iint_{D/\Gamma} \lambda(\zeta)^{2-2q}F(z, \zeta)F(\zeta, z)d\zeta \wedge d\bar{\zeta} \\ &= -iF(z, z) . \end{aligned}$$

Hence we get (4.2) by (2.6). Finally, by (1.4), (1.1) and Hölder's inequality, we have

$$|\phi(z)| \leqq \|\phi\|_p \|F(\cdot, z)\|_{p'} .$$

Thus (4.3) and (4.4) follow from (4.1) and (4.2), respectively. □

By (4.3), (4.4) and Lebesgue's convergence theorem, we have the following:

LEMMA 3. *Let* $\{\phi_n\}_{n=1}^{\infty}$ *be a sequence in* A^p, $1 \leqq p < \infty$, *such that* $\{\|\phi_n\|_p\}_{n=1}^{\infty}$ *is bounded and* $\lim_{n\to\infty} \phi_n = 0$. *Suppose that* $\int_{E/\Gamma} W(z, z)dA(z) < \infty$ *if* $p = 2$, *and that* $\int_{E/\Gamma} MdA < \infty$ *if* $p \neq 2$. *Then* $\lim_{n\to\infty} \|\chi_E \phi_n\|_p = 0$.

LEMMA 4. *If* $\phi \in A^2$ *satisfies*

$$\beta[\chi_E\phi] = \phi\ , \quad \text{i.e.,} \quad \beta[\chi_V\phi] = 0\ , \tag{4.5}$$

then $\phi = 0$.

PROOF. $\int_{V/\Gamma} \lambda^{-2q}|\phi|^2 dA = (\chi_V\phi, \phi) = (\chi_V\phi, \beta[\chi_E\phi]) = (\beta[\chi_V\phi], \chi_E\phi) = 0$.

Hence $\chi_V\phi = 0$ and the assertion follows from $\mathrm{Area}(V/\Gamma) > 0$.

LEMMA 5. *On the same assumption as in Theorem* 3, *if* $\phi \in A^p \cup A^{p'}$ *satisfies* (4.5) *then* $\phi = 0$.

PROOF. It suffices to show $\phi \in A^2$.

The case $p = 1$, $p' = \infty$: Let $\phi \in A^\infty$. Then $\chi_E\phi \in L^2$ by (2.9), hence $\phi = \beta[\chi_E\phi] \in A^2$. On the other hand, if $\phi \in A^1$, then by (4.3) and (2.8) we have

$$\|\chi_E\phi\|_2^2 = \int_{E/\Gamma} \lambda^{-2q}|\phi|^2 dA \leqq \int_{E/\Gamma} (\|\phi\|_1 M)^2 dA < \infty\ .$$

This implies $\phi \in A^2$.

The case $1 < p < \infty$, $p \neq 2$: Let $\phi \in A^p$. By (4.5), Minkowski's inequality (Hardy, Littlewood and Pólya [3, Theorem 202]), (4.2) and Hölder's inequality, we get

$$\begin{aligned}
&\left(\int_{D/\Gamma} \lambda^{-2q}|\phi|^2 dA\right)^{1/2} \\
&\quad = \left(\int_{D/\Gamma} \lambda(z)^{-2q}\left|\int_{E/\Gamma} \lambda(\zeta)^{-2q}F(z, \zeta)\phi(\zeta)dA(\zeta)\right|^2 dA(z)\right)^{1/2} \\
&\quad \leqq \int_{E/\Gamma} \lambda(\zeta)^{-2q}|\phi(\zeta)|\left(\int_{D/\Gamma} \lambda(z)^{-2q}|F(z, \zeta)|^2 dA(z)\right)^{1/2} dA(\zeta) \\
&\quad = \int_{E/\Gamma} \lambda(\zeta)^{-q}|\phi(\zeta)|W(\zeta, \zeta)^{1/2} dA(\zeta) \\
&\quad \leqq \|\phi\|_p\left(\int_{E/\Gamma} W(\zeta, \zeta)^{p'/2} dA(\zeta)\right)^{1/p'} .
\end{aligned}$$

Hence by (2.10) we see $\phi \in A^2$. The same holds for $\phi \in A^{p'}$, because the assumption is symmetric for p and p'.

PROOF OF THEOREM 3. First, we show (2.2). Suppose that $\psi \in A^{p'}$ satisfies $\chi_V\psi \in (A^p)^\perp$. Then by (1.8) we have $\beta[\chi_V\psi] = 0$. Thus (2.2) follows from Lemmas 4 and 5. Next, we show (2.12). Suppose that (2.12) does not hold. Then there is a sequence $\{\phi_n\}_{n=1}^{\infty}$ in A^p such that $\|\phi_n\|_p = 1$ for each n and

$$\|\beta[\chi_V \phi_n]\|_p \to 0 . \tag{4.6}$$

Since $\{\phi_n\}$ is a normal family, by taking a subsequence if necessary, we may assume that ϕ_n converges to some ϕ in A^p, $\|\phi\|_p \leqq 1$, uniformly on compact subsets of D. Let Δ' be a relatively compact disk in D such that $\Delta' \cap \gamma(\Delta') = \varnothing$ for every $\gamma \in \Gamma - \{\mathrm{id}\}$, and let χ be the characteristic function of $\Gamma(\Delta') = \bigcup_{\gamma \in \Gamma} \gamma(\Delta')$. Then we have $\|(\phi - \phi_n)\chi\|_p \to 0$ and $\|(\phi_n - \beta[\chi_E \phi_n])\chi\|_p \leqq \|\beta[\chi_V \phi_n]\|_p \to 0$. Since $\|\phi - \phi_n\|_p \leqq 2$, by Lemma 3 we get

$$\|\beta[\chi_E(\phi - \phi_n)]\|_p \leqq c_q \|\chi_E(\phi - \phi_n)\|_p \to 0 . \tag{4.7}$$

Thus we obtain $\|(\phi - \beta[\chi_E \phi])\chi\|_p \leqq \|(\phi - \phi_n)\chi\|_p + \|(\phi_n - \beta[\chi_E \phi_n])\chi\|_p + \|\chi\beta[\chi_E(\phi - \phi_n)]\|_p \to 0$, that is, $\phi = \beta[\chi_E \phi]$ on $\Gamma(\Delta')$, and hence on D. By Lemmas 4 and 5 we have $\phi = 0$ and hence

$$1 = \|\phi_n\|_p \leqq \|\beta[\chi_V \phi_n]\|_p + \|\beta[\chi_E(\phi_n - \phi)]\|_p ,$$

a contradiction to (4.6) and (4.7). □

For a Fuchsian group G acting on the unit disk Δ, we denote by $A_q^p(\Delta, G)$, $1 \leqq p < \infty$, (resp. $A_q^\infty(\Delta, G)$) the Banach space of all the p-integrable (resp. bounded) holomorphic automorphic forms of weight $-2q$ on Δ for G. When G is the trivial group $1 = \{\mathrm{id}\}$, the spaces $A_t^p(\Delta, 1)$, $1 \leqq p \leqq \infty$, can be defined for all real $t > 0$.

Bers [1, p. 199] has shown that $A_t^1(\Delta, 1) \subset A_t^\infty(\Delta, 1)$ for all real $t \geqq 2$, and the inclusion map is continuous. Earle [2] has shown that for all real $t > 1$, $A_q^1(\Delta, G) \subset A_{q+t}^1(\Delta, 1)$ with a continuous inclusion map.

PROOF OF PROPOSITION 1. Let G be the Fuchsian model of Γ induced by a universal covering $\rho: \Delta \to D$. The map: $\phi \mapsto (\phi \circ \rho)(\rho')^q$ is an isometric isomorphism of $A_q^p(D, \Gamma)$ onto $A_q^p(\Delta, G)$, $1 \leqq p \leqq \infty$. By the above results due to Bers and Earle, we may regard this map to be a continuous mapping of $A_q^1(D, \Gamma)$ into $A_{q+t}^\infty(\Delta, 1)$ for $t > 1$. In particular, we have

$$\sup_{w \in \Delta} \lambda_\Delta(w)^{-(q+t)} |F(\rho w, \zeta)| |\rho'(w)|^q \leqq C' \|F(\cdot, \zeta)\|_1 , \quad \zeta \in D ,$$

where $\lambda_\Delta(w) = (1 - |w|^2)^{-1}$ is the hyperbolic metric for Δ with constant negative curvature -4, and C' is a constant depending only on q, t, ρ and Γ. Hence by (2.6), (1.1) and (1.3) we see that

$$W(\zeta, z) \leqq c_q C' \lambda_\Delta(w)^t , \quad w \in \Delta , \quad z = \rho(w) \in D \quad \text{and} \quad \zeta \in D .$$

This implies the assertion. □

For w and ξ in Δ, we set

$$K_\Delta(w, \xi) = (2q-1)i/\{2\pi(1-w\bar{\xi})^{2q}\} .$$

For a Fuchsian group G acting on Δ, define

$$\alpha_\Delta(w, \xi) = \sum_{g \in G} K_\Delta(gw, \xi) g'(w)^q .$$

Metzger and Rajeswara Rao [8] has proved that $A_q^1(\Delta, G) \subset A_q^\infty(\Delta, G)$ if and only if $\sup_{w \in \Delta} \lambda_\Delta(w)^{-2q} |\alpha_\Delta(w, w)| < \infty$, for an arbitrary Fuchsian group G. Lehner [6, 7] has proved that if a Fuchsian group G satisfies the condition (2.13), then $A_q^1(\Delta, G) \subset A_q^\infty(\Delta, G)$.

Proof of Proposition 2. Let $\rho: \Delta \to D$ be a universal covering which induces the Fuchsian model G of Γ. As in the proof of Proposition 1, ρ induces an isometric isomorphism of $A_q^p(D, \Gamma)$ onto $A_q^p(\Delta, G)$, $1 \leqq p \leqq \infty$. Obviously, $A^1 \subset A^\infty$ if and only if $A_q^1(\Delta, G) \subset A_q^\infty(\Delta, G)$. Hence it suffices to show that

$$\alpha_\Delta(w, w) = F_{D,\Gamma}(\rho w, \rho w) |\rho'(w)|^{2q} , \quad w \in \Delta , \tag{4.8}$$

and

$$\sup_{z \in D} M(z) \leqq \sup_{z \in D} W(z, z) . \tag{4.9}$$

By [5, p. 101] we see that $\alpha_\Delta(\cdot, \xi) \in \bigcap_{1 \leqq p \leqq \infty} A_q^p(\Delta, G)$ and α_Δ possesses the properties corresponding to (1.1) and (1.4), that is,

$$\alpha_\Delta(w, \xi) = -\bar{\alpha}_\Delta(\xi, w)$$

and

$$\phi(w) = \iint_{\Delta/G} \lambda_\Delta(\xi)^{2-2q} \alpha_\Delta(w, \xi) \phi(\xi) d\xi \wedge d\bar{\xi}$$

for every $\phi \in A_q^p(\Delta, G)$, $1 \leqq p \leqq \infty$, respectively. Define $\alpha_D(z, \zeta)$, z and $\zeta \in D$, via

$$\alpha_D(\rho w, \rho \xi) \rho'(w)^q \bar{\rho}'(\xi)^q = \alpha_\Delta(w, \xi) .$$

Then α_D is well-defined and satisfies (1.1), (1.2) and (1.4). Since such a function is unique, we see $\alpha_D = F_{D,\Gamma}$. Hence we obtain (4.8).

Next, we have

$$F(z, \zeta) = i(F(\cdot, \zeta), F(\cdot, z)) .$$

Thus it follows from (1.6) and (4.2) that

$$W(z, \zeta)^2 \leqq W(z, z) W(\zeta, \zeta) .$$

This inequality yields (4.9). □

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