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A REMARK ON THE HYPERBOLIC COLLAR LEMMA

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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1. Preliminaries. Let G be a discrete subgroup of $PSL(2, \boldsymbol{C})$ acting on $H^3 = \{z + uj; z \in \boldsymbol{C}, u > 0\}$, the upper half space model of the hyperbolic space. If $X \in G - \{\text{id.}\}$ is not a parabolic element, then we denote by g_X the geodesic in H^3 joining the fixed points of X on the boundary of H^3. For a positive number k, we define a tubular neighborhood about g_X as the set

$$N_k(X) = \{x \in H^3; d(x, g_X) \leqq k\} ,$$

where d is the hyperbolic distance. Let G_X be the subgroup of G which leaves g_X invariant. We call $N_k(X)$ a collar for X in G, if $T(N_k(X)) \cap N_k(X) = \emptyset$ for all $T \in G - G_X$ and $T(N_k(X)) = N_k(X)$ for all $T \in G_X$. In this case, the number k is called the width of the collar $N_k(X)$.

The first purpose of this note is to prove the following theorem, the so-called collar lemma.

THEOREM. *Let $G \subset PSL(2, \boldsymbol{C})$ be a non-elementary discrete group.*

(i) *Suppose that $X \in G$ satisfies $0 < |\text{trace}^2 X - 4| = s < s_0 = 2(-1 + \sqrt{2})$. Then g_X has a collar $N_{k(s)}(X)$, where*

$$\sinh^2 k(s) = s^{-1}(1 - s)^{1/2} - 1/2 . \tag{1}$$

(ii) *Let X and Y be in G and suppose that X and Y generate a non-elementary group. If $0 < |\text{trace}^2 X - 4|$ and $|\text{trace}^2 Y - 4| < 2(-1 + \sqrt{2})$, then the collars $N_{k(s)}(X)$ for X and $N_{k(s')}(Y)$ for Y are disjoint, where $s = |\text{trace}^2 X - 4|$, $s' = |\text{trace}^2 Y - 4|$ and k is the function defined by* (1).

Brooks and Matelski [2] proved the above theorem for the constant $s_0 = 1/2$ and for the function k defined by $\sinh^2 k(s) = s^{-1} - 3/2$. Gallo [3] also obtained the theorem for the constant $s_0 = (\sqrt{41} - 5)/2$ and for k defined by $\sinh^2 k(s) = s^{-1} - (s + 3)/2$. The constant s_0 and the function k in the Theorem are better than those in [2] and [3].

Sections 2 through 4 are devoted to preliminary discussions for the

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proof of the Theorem. In Section 5, we give three examples: the first shows the asymptotic sharpness of our Theorem, the second gives the existence of a discrete group G and an element $X \in G$ where the geodesic g_X does not possess a collar for G and the third states a relation between the existence of an inscribed ball in the Dirichlet fundamental region and the existence of a collar.

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2. Complex distance and the cosine rule. Following [1], we introduce the notion of complex distance between two geodesics in H^3 and also state the cosine rule. Denote a directed geodesic L by the ordered pair of its endpoints; so $L = (a, b)$ for its endpoints $a, b \in \boldsymbol{C}$, $a \neq b$. The complex distance $t = \delta(L_1, L_2) \in \boldsymbol{C}$ between two directed geodesics $L_1 = (a_1, b_1)$ and $L_2 = (a_2, b_2)$ is defined as follows: $\operatorname{Re} t \geqq 0$ is the hyperbolic distance between the geodesics and $\operatorname{Im} t$ is the angle made by the geodesics along their common perpendicular and is determined modulo 2π unless $\operatorname{Re} t = 0$, in which case $\pm \operatorname{Im} t$ is determined modulo 2π. We can compute the complex distance by the formula

$$(2) \qquad \cosh^2 \frac{t}{2} = (a_1, a_2, b_2, b_1) .$$

The right hand side of this equality denotes the cross ratio of those four points. Therefore, for any $\gamma \in PSL(2, \boldsymbol{C})$, we see $\delta(L_1, L_2) = \delta(\gamma(L_1), \gamma(L_2))$.

Let $X \in PSL(2, \boldsymbol{C})$ be non-parabolic and let g_X be the directed geodesic in the hyperbolic space joining the fixed points of X. If L is a perpendicular to g_X, then the complex distance t between L and $X(L)$ is called the complex translation length of X. In this case, we have

$$(3) \qquad \operatorname{trace}^2 X = 4 \cosh^2 \frac{t}{2} ,$$

which makes sense even if X is not loxodromic.

For the geodesics L_0, L_1, L_2, put $\omega = \delta(L_1, L_2)$, $t_1 = \delta(L_0, L_1)$ and $t_2 = \delta(L_0, L_2)$ and denote by α the complex distance from the perpendicular between L_0 and L_1 to the perpendicular between L_0 and L_2. Brooks-Matelski [1] proved the so-called cosine rule:

$$(4) \qquad \cosh \omega = \cosh t_1 \cosh t_2 - \cosh \alpha \sinh t_1 \sinh t_2 .$$

3. An application of Jørgensen's inequality. If X and Y generate a discrete non-elementary subgroup of $PSL(2, \boldsymbol{C})$, then the inequality

$$(5) \qquad |\mathrm{trace}^2 X - 4| + |\mathrm{trace}\, XYX^{-1}Y^{-1} - 2| \geqq 1$$

holds. This inequality is called Jørgensen's inequality. Using the cosine rule (4) and Jørgensen's inequality (5), we have the following lemma.

LEMMA 1. *Let X and Y be non-parabolic elements of $PSL(2, \boldsymbol{C})$ whose complex translation lengths are t and t', respectively, and let β be the complex distance between g_X and g_Y. If X and Y generate a non-elementary discrete group, then*

$$(6) \qquad |1 - \cosh t|(2 + |\sinh^2 \beta||1 - \cosh t'|) \geqq 1 ,$$

$$(7) \qquad |1 - \cosh t'|(2 + |\sinh^2 \beta||1 - \cosh t|) \geqq 1 .$$

PROOF. Let ω be the complex distance between g_X and $g_{YXY^{-1}}$. Then we can normalize X and YXY^{-1} as follows:

$$X = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \quad YXY^{-1} = \begin{pmatrix} \cosh\frac{t}{2} & e^{\omega}\sinh\frac{t}{2} \\ e^{-\omega}\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}.$$

We have trace $XYX^{-1}Y^{-1} - 2 = -(1 - \cosh t)(1 - \cosh \omega)$. Thus (3) and (5) yield

$$(8) \qquad |(1 - \cosh t)(1 - \cosh \omega)| + 2|1 - \cosh t| \geqq 1 .$$

Recall the cosine rule (4) and take $L_2 = g_X$, $L_0 = g_Y$ and $L_1 = g_{YXY^{-1}} = Y(g_X)$. It is easy to show that $\beta = \delta(g_Y, g_X) = \delta(g_Y, g_{YXY^{-1}})$. Thus we have $\cosh \omega = \cosh^2 \beta - \cosh t' \sinh^2 \beta$. From (8) and this equality, we have the inequality (6) and similarly (7).

There exist various discrete groups which satisfy the equality in (6) or (7) and we shall show these examples in Section 5.

4. **Proof of the Theorem.** First we prove (i) of the Theorem stated in Section 1. From the assumption of the Theorem, $X \in G$ is not parabolic. Let t be the complex translation length of X. We may assume trace $X = 2\cosh(t/2) \neq 2$. Put $Y = TXT^{-1}$ for any $T \in G - G_X$. Let β be the complex distance between g_X and g_Y. Considering the conjugate of G, we may assume that X and $Y \in G$ are of the form

$$X = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \cosh\frac{t}{2} & e^{\beta}\sinh\frac{t}{2} \\ e^{-\beta}\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}.$$

Let X be elliptic. If $\exp(\sqrt{-1}\theta)$ is the multiplier of X, then we have $2(1-\cos\theta)=|\mathrm{trace}^2 X-4|$. Hence, if $|\mathrm{trace}^2 X-4|<1$, then the order of X is not less than 7. Hence the order of Y is also not less than 7. Therefore X and Y generate a non-elementary discrete subgroup $\langle X,Y\rangle$ of G.

If X is not elliptic, then we can conclude that $\beta\neq 0$ and $\langle X,Y\rangle$ is a non-elementary discrete subgroup of G. Thus $\langle X,Y\rangle$ is non-elementary for any X in G. Lemma 1 shows that $|\sinh^2\beta|\geqq(1-2|1-\cosh t|)|1-\cosh t|^{-2}$. Clearly $\cosh\mathrm{Re}\,\beta\geqq|\sinh\beta|$ and the equality holds for $\mathrm{Im}\,\beta=\pm\pi/2$. Therefore we have

$$(9)\qquad 2\sinh^2(\mathrm{Re}\,\beta/2)+1=\cosh\mathrm{Re}\,\beta \geqq\left(1-4\left|\sinh^2\frac{t}{2}\right|\right)^{1/2}\left(2\left|\sinh^2\frac{t}{2}\right|\right)^{-1}.$$

Setting $|\mathrm{trace}^2 X-4|=|4\sinh^2(t/2)|=s<s_0=2(\sqrt{2}-1)$ and defining $k(s)$ by $\sinh^2 k(s)=s^{-1}(1-s)^{1/2}-1/2$, we see that $N_{k(s)}(X)$ is a collar for X.

Next we prove (ii) of the Theorem. Let t and t' be the complex translation lengths of X and Y, respectively, and let β be the complex distance between g_X and g_Y. We may assume that $s=|\mathrm{trace}^2 X-4|\leqq|\mathrm{trace}^2 Y-4|=s'$, or equivalently, that $|\sinh(t/2)|\leqq|\sinh(t'/2)|$. Then (6) and (9) imply

$$\begin{aligned}|\sinh^2\beta| &\geqq(1-2|1-\cosh t|)|1-\cosh t|^{-1}|1-\cosh t'|^{-1}\\ &\geqq(2\sinh^2 k(s)+1)(2\sinh^2 k(s')+1)\\ &=(\cosh^2 k(s)+\sinh^2 k(s))(\cosh^2 k(s')+\sinh^2 k(s'))\\ &\geqq\{\cosh k(s)\cosh k(s')+\sinh k(s)\sinh k(s')\}^2\\ &=\cosh^2(k(s)+k(s')),\end{aligned}$$

where k is the function defined by (1). From $\cosh^2\mathrm{Re}\,\beta\geqq|\sinh^2\beta|$, we have $\mathrm{Re}\,\beta\geqq k(s)+k(s')$, which proves (ii) of the Theorem.

5. Examples. Here we give an example of a non-elementary discrete group with two elliptic generators of order n ($\geqq 7$) which satisfy the equality in Jørgensen's inequality. This example shows the asymptotic sharpness of our Theorem. First we state a lemma due to Jørgensen and Kiikka [4].

LEMMA 2. *Suppose that X and Y generate a non-elementary discrete group and that*

$$(10)\qquad |\mathrm{trace}^2 X-4|+|\mathrm{trace}\,XYX^{-1}Y^{-1}-2|=1.$$

Then X is elliptic and is of order at least 7 or X is parabolic. Furthermore, X and $Y_1=YXY^{-1}$ generate a non-elementary discrete group *and the equality*

$$(11) \qquad |\mathrm{trace}^2 X - 4| + |\mathrm{trace}\, XY_1X^{-1}Y_1^{-1} - 2| = 1$$

holds.

Now, take

$$X = \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \varepsilon_n^{-1} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\varepsilon_n = \exp(\sqrt{-1}\pi/n)$ $(n \geqq 7)$, $a, b, c, d \in \boldsymbol{C}$ and $ad - bc = 1$. Put $X' = WXW^{-1}$ and

$$Y' = WYW^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad \text{for} \quad W = \sqrt{2}^{-1}\begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}.$$

The condition (10) for X and Y is equivalent to

$$(12) \qquad 4|bc| = (1 - 4\sin^2(\pi/n))(\sin(\pi/n))^{-2}.$$

For two real numbers p and $q (\neq 0)$, we put $a' = d' = \sqrt{-1}pq^{-1}$. $b' = -(p^2q^{-1} + q)$ and $c' = q^{-1}$. Then we have

$$(13) \qquad 2a = \sqrt{-1}(2pq^{-1} - p^2q^{-1} - q - q^{-1}),$$

$$2d = \sqrt{-1}(2pq^{-1} + p^2q^{-1} + q + q^{-1}) \text{ and } 2b = 2c = -(p^2q^{-1} + q - q^{-1}).$$

Choose $2q = v + (v^2 + 4)^{1/2} > 0$, where $v = -(\sin(\pi/n))^{-1} + (1 - 4\sin^2(\pi/n))^{1/2} \times (2\sin(\pi/n))^{-1}$. Since $q^2 - qv = 1$, we see that $q^2 + 2q(\sin(\pi/n))^{-1} = 1 + q \times [v + 2(\sin(\pi/n))^{-1}] > 1$. Hence there is a positive p such that $p^2 + 1 = q^2 + 2q(\sin(\pi/n))^{-1}$. Moreover, we see $q - q^{-1} = v$, so we have $(p^2q^{-1} + q - q^{-1})^2 = 4[v + (\sin(\pi/n))^{-1}]^2$. Thus we obtain positive numbers p and q satisfying

$$(14) \qquad (p^2q^{-1} + q - q^{-1})^2 = (1 - 4\sin^2(\pi/n))(\sin(\pi/n))^{-2},$$

$$(15) \qquad p^2 + 1 = q^2 + 2q(\sin(\pi/n))^{-1}.$$

By considering the isometric circles of X', X'^{-1}, Y', Y'^{-1} and noting that X' is elliptic and is of order not less than 7, we see from (15) that X' and Y' generate a non-elementary discrete group. Hence X and Y also generate a non-elementary discrete group and (14) implies that (12) and (10) hold. By Lemma 2, we see that two elliptic elements X and $Y_1 = YXY^{-1}$ of order $n(\geqq 7)$ generate a non-elementary discrete group and that the equality (11) holds.

Let $\beta = \delta(g_X, g_{Y_1})$ be the complex distance between the two geodesics g_X and g_{Y_1} in H^3. We see from (13) that the fixed points $Y(0)$, $Y(\infty)$ of Y_1 on $\hat{\boldsymbol{C}}$ are both purely imaginary and that $\mathrm{Im}\,\beta = 0$. Furthermore, we see by (2) that $\cosh^2(\beta/2) = ad$, which gives

$$\sinh^2 \beta = 4bc(bc+1) = (1 - 4\sin^2(\pi/n))(4\sin^4(\pi/n))^{-1}.$$

Putting $|\text{trace}^2 X - 4| = 4\sin^2(\pi/n) = s$, we have

$$\sinh^2(\beta/2) = (1 - 4\sin^2(\pi/n))(4\sin^2(\pi/n))^{-1} = s^{-1} - 1 \cdot$$

Summing up the above, we have the following:

EXAMPLE 1. Let

$$X = \begin{pmatrix} \varepsilon_n & 0 \\ 0 & \varepsilon_n^{-1} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\varepsilon_n = \exp(\sqrt{-1}\pi/n)$ $(n \geqq 7)$ and a, b, c and d are determined by (13) for positive numbers p and q satisfying (14) and (15). Put $Y_1 = YXY^{-1}$. Then $\langle X, Y_1 \rangle$ is a non-elementary discrete group and $|\text{trace}^2 X - 4| + |\text{trace}\, XY_1X^{-1}Y_1^{-1} - 2| = 1$, Furthermore, $\sinh^2 k(s) = (1 - 4\sin^2(\pi/n)) \times (4\sin^2(\pi/n))^{-1} = s^{-1} - 1$ where k is the width of the collar of $N_{k(s)}(X)$ for X in $\langle X, Y_1 \rangle$ and $s = |\text{trace}^2 X - 4|$.

EXAMPLE 2. If the assumption of the Theorem is dropped, then we cannot assert the conclusion stated in the Theorem. The following example shows this. Let G be a group generated by

$$A = \begin{pmatrix} 2 & -1 - 4\sqrt{-1} \\ 1 & -2\sqrt{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2\sqrt{-1} & -1 - 4\sqrt{-1} \\ 1 & 2 \end{pmatrix}.$$

The sets $\{z; |z - 2\sqrt{-1}| = 1\}$, $\{z; |z - 2| = 1\}$, $\{z; |z + 2| = 1\}$ and $\{z; |z + 2\sqrt{-1}| = 1\}$ are isometric circles of A, A^{-1}, B and B^{-1}, respectively and these four circles are mutually disjoint in $\boldsymbol{C}$. Therefore the group G is a Schottky group. By a simple calculation, we have

$$AB = \begin{pmatrix} -1 - 8\sqrt{-1} & -4 - 16\sqrt{-1} \\ -4\sqrt{-1} & -1 - 8\sqrt{-1} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} -1 - 8\sqrt{-1} & -16 + 4\sqrt{-1} \\ 4 & -1 - 8\sqrt{-1} \end{pmatrix}.$$

It is easily seen that the geodesic g_{AB} has endpoints w, $-w$ in $\boldsymbol{C}$, where $w = -(2 + (17/4)^{1/2})^{1/2} + (-2 + (17/4)^{1/2})^{1/2}\sqrt{-1}$. The geodesic g_{AB} has endpoints $-w\sqrt{-1}$, $w\sqrt{-1}$. Hence we see $g_{AB} \cap g_{BA} = \{17^{1/4}j\}$. Thus we have $d(g_{AB}, g_{BA}) = d(g_{AB}, g_{BABB^{-1}}) = d(g_{AB}, B(g_{AB})) = 0$. Therefore we cannot have a collar for g_{AB}, because B is not an element of a purely loxodromic cyclic group $\langle AB \rangle$.

EXAMPLE 3. First we give the definition of the Dirichlet fundamental region of a discrete group.

A region D in H^3 is called a fundamental region of G, if any two points of the interior of D are not equivalent and any point x has its equivalent in D. Let G be a discrete group acting on H^3 and let w be any point of H^3 which is not fixed by any element of G. The Dirichlet fundamental region $D(w)$ with centre w is defined by

$$D(w) = \{x \in H^3; d(x, w) \leqq d(x, g(w)) \quad \text{for all} \quad g \in G\} ,$$
$$= \bigcap_{g \in G} \{x \in H^3; d(x, w) \leqq d(x, g(w))\} .$$

The Dirichlet fundamental region $D(w)$ is a convex fundamental region for G.

Let G_n be a group generated by $X: z \to (nz + n^2 - 1)/(z + n)$ and $Y: z \to (n + 1)z/(n - 1)$ $(n > 1)$. Then the group G_n acts discontinuously on H^3 and the Dirichlet fundamental region D_n of G_n centered at $(n^2 - 1)^{1/2}j$ is the set $D_n = H^3 \cap \{z + uj; |z^2| + u^2 \leqq (n + 1)^2\} \cap \{z + uj; |z^2| + u^2 \geqq n^2\} \cap \{z + uj; |z \pm n|^2 + u^2 \geqq 1\}$, where $\{z + uj; |z \pm n|^2 + u^2 = 1\}$ are isometric spheres of $X^{\pm 1}$. Considering the tessellation of D_n by G_n, we can easily check that the set D_n satisfies the conditions for the fundamental region of G_n. We have $d(g_X, Y^{\pm 1}(g_X)) = \log((n + 1)(n - 1)^{-1})$ for $n > 1$. It is easily seen that $d(g_X, X^p Y^q X^r(g_X)) = d(g_X, Y^q(g_X)) = |q| \log((n + 1)(n - 1)^{-1})$, where p, q and r are integers. Furthermore, we have $d(g_X, Y^p X^q Y^r(g_X)) \geqq (|p|/2)\log((n + 1)(n - 1)^{-1})$, where $p, q, r \in \boldsymbol{Z} - \{0\}$. Therefore we have $(1/2)\log((n + 1)(n - 1)^{-1}) \leqq d(g_X, \gamma(g_X))$ for all $\gamma \in G_n - G_X$. Thus we have a collar $N_k(X)$ for g_X, where $k = (1/4)\log((n + 1)(n - 1)^{-1})$. But we cannot have a collar independent of n, because k tends to 0 as n tends to ∞. Nevertheless, we have an inscribed ball tangent to $\boldsymbol{C}$ in D_n, because $D_n \cap \boldsymbol{C}$ contains an inscribed disc with Euclidean radius 1/2.

Waterman [5] showed the existence of a ball with the hyperbolic radius 1/300 inscribed in the Dirichlet fundamental region for any discrete group. It is easily seen that the existence of a collar implies the existence of an inscribed ball, but the converse is not true.

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