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BOUNDED PROJECTIONS ONTO HOLOMORPHIC HARDY SPACES ON PLANAR DOMAINS

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1. Introduction. Throughout this paper, $D \subset C$ is a domain bounded by finitely many non-intersecting simple closed C^4 regular curves. We denote by m_0 the area Hausdorff measure on the boundary ∂D of the domain D, and by m_1 and m_2 two different harmonic measures relative to D. The holomorphic Hardy spaces $H^p(m_j)$ on ∂D are defined as the $L^p(m_j)$-norm closure of $A(\partial D)$ $1 \leqq p < \infty$, where $A(\partial D)$ is the class of continuous functions f on ∂D whose Poisson integral PI$[f]$ is analytic in D. This paper is concerned with projection operators of $L^p(m_j)$ onto $H^p(m_j)$.

As is well known, there are two bounded projection operators of $L^2(m_j)$ onto $H^2(m_j)$. One of them is the Cauchy projection H and the other is the orthogonal projection P_j. These operators are useful to study real or holomorphic Hardy spaces. In particular, H and P_0 also play important roles in the theory of partial differential equations and of conformal mappings. In addition, P_1 and P_2 are deeply related with uniform algebras.

In this paper, we show correlations between H, P_0, P_1 and P_2, and give some applications to holomorphic Hardy spaces. Our investigation is motivated by the following interesting theorem by Kerzman and Stein [10]:

THEOREM KS ([10]; see also [3]). *Let D be a bounded, simply connected C^∞ domain in the plane, and H^* be the adjoint of H on the Hilbert space $L^2(m_0)$. Then*:

(1) *$H^* - H$ is an integral operator with a smooth kernel. Hence it is compact on $L^2(m_0)$.*

(2) *Further, $I - (H^* - H)$ is an injective bounded operator of $L^2(m_0)$ onto $L^2(m_0)$, and*

$$P_0 = H(I - H^* + H)^{-1} .$$

This result tells us a relation between H and P_0. On the other hand,

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our main theorem stated later implies that P_1 also can be written in terms of H, and P_2 can be represented in terms of H and P_1. Moreover, it gurantees that each P_j is bounded on the Hardy space $H^1_{\max}$ in the sense of Fefferman and Stein.

Let the complement D^c of D have $n+1$ connected components. Denote by G_0 the unbounded component of D^c and by G_μ for $\mu=1, \cdots, n$ the bounded components of D^c. For every G_k ($k=0, \cdots, n$) let l_k be the length of ∂G_k and denote

$$\partial G_k = \left\{\alpha_k(s): \sum_{d=0}^{k} l_{d-1} \leqq s < \sum_{d=0}^{k+1} l_{d-1}\right\} ,$$

where $l_{-1}=0$ and α_k is a unit speed simple closed C^4 curve which surrounds G_k. We suppose that α_0 is positively oriented and $\alpha_1, \cdots, \alpha_n$ are negatively oriented. For simplicity we use the notation

$$\alpha(s) = \alpha_k(s) , \quad \text{for} \quad s \in I_k \equiv \left[\sum_{d=0}^{k} l_{d-1}, \sum_{d=0}^{k+1} l_{d-1}\right) ,$$

if there is no confusion.

Let $K(\cdot, \cdot)$ be the Cauchy kernel, that is,

$$K(s, t) = D_+\alpha(t)/[\alpha(t) - \alpha(s)] ,$$

for $(s, t) \in [0, L) \times [0, L) - \{\text{diagonal}\}$, where $L = \sum_{d=0}^{n} l_d$, and $D_+F(t) = \lim_{h\to+0} [F(t+h) - F(t)]/h$ for every right differentiable function F.

Then the operator H is given by the following singular integral operator:

$$Hf(x) = \frac{1}{2}h(x) + \frac{1}{2\pi i}\text{P.V.}\int_0^L K(\alpha^{-1}(x), t) f(\alpha(t))dt \quad (x \in \partial D) .$$

We now recall the definition of Hardy spaces introduced by Fefferman and Stein:

For a function $f \in L^1(m_j)$, let $N(f)$ be the non-tangential maximal function of f, that is,

$$N(f)(x) = \sup\{|\text{PI}[f](z)|: z \in \Gamma(x)\} , \quad x \in \partial D ,$$

where $\Gamma(x) = \{z \in D: |z - x| < 2\,\text{dist}(z, \partial D)\}$.

Fefferman and Stein's spaces are defined in terms of $N(\cdot)$ by:

$$H^p_{\max}(m_j) = \{f \in L^1(m_j): \|f\|_{p,j,\max} \equiv \|N(f)\|_{p,j} < \infty\} ,$$

where $\|\cdot\|_{p,j}$ is the $L^p(m_j)$-norm, $j = 0, 1, 2$.

It is well known that $H^1_{\max}(m_j)$ is a proper subspace of $L^1(m_j)$ and $H^p_{\max}(m_j) = L^p(m_j)$, $1 < p < \infty$ (see [5] and [8]).

In this paper we denote by $P(z, x)$ ($z \in D$, $x \in \partial D$) the Poisson kernel of D, and we put $W_j(t) = P(z_j, \alpha(t))$, where z_j is the point such that $m_j(F) = \int_F P(z_j, x)dm_0(x)$ for every Borel set F of ∂D, $j = 1, 2$.

Our main theorem is the following:

THEOREM 1. *Let D be a domain bounded by finitely many non-intersecting simple closed C^4 regular curves. Let*

$$a_j(s, t) = -\overline{K(t, s)} \cdot W_j(t) W_j(s)^{-1} - K(s, t) ,$$

for $(s, t) \in [0, L) \times [0, L) - \{diagonal\}$, $j = 0, 1, 2$, where $W_0(t) \equiv 1$, and the bar denotes the complex conjugation here and elsewhere. Then we have the following:

(1) *Each a_j can be extended to a function A_j on $[0, L) \times [0, L)$ in such a way that $A_j(\alpha^{-1}(\cdot), \alpha^{-1}(\cdot))$ is continuous on $\partial D \times \partial D$.*

(2) *Let*

$$A_j f(x) = \frac{1}{2\pi i}\int_0^L A_j(\alpha^{-1}(x), t) f(\alpha(t))dt , \quad f \in L^1(m_l) ,$$

$j, l = 0, 1, 2$. Then each mapping $I - A_j$ is a bijective bounded operator of $H^1_{\max}(m_l)$ to $H^1_{\max}(m_l)$, $j, l = 0, 1, 2$.

Hence $(I - A_j)^{-1}$ is also bounded on $H^1_{\max}(m_l)$, $j, l = 0, 1, 2$. Moreover, for every $f \in L^2(m_l)$, $l = 0, 1, 2$, we have

$$(3) \qquad P_j f = H(I - A_j)^{-1} f , \quad j = 0, 1, 2 ,$$

and

$$(4) \qquad P_j f = P_{j+1}(I - A_{j+1})(I - A_j)^{-1} f , \quad j = 0, 1 .$$

As a consequence of Theorem 1 we have the following:

COROLLARY 1. *Let D be as in Theorem 1. For every $j \in \{0, 1, 2\}$,* the *following are equivalent:*

(1) $\|Hf\|_{1,j} \leqq C_1 \|f\|_{1,j,\max}$ *for every $f \in L^2(m_j)$.*
(2) $\|P_0 f\|_{1,j} \leqq C_2 \|f\|_{1,j,\max}$ *for every $f \in L^2(m_j)$.*
(3) $\|P_1 f\|_{1,j} \leqq C_3 \|f\|_{1,j,\max}$ *for every $f \in L^2(m_j)$.*
(4) $\|P_2 f\|_{1,j} \leqq C_4 \|f\|_{1,j,\max}$ *for every $f \in L^2(m_j)$.*

Here C_1, C_2, C_3 and C_4 are constants independent of f.

From the atomic decomposition of $H^1_{\max}(m_j)$ (cf. [8]) and a result in Coifman and Weiss [4, p. 559] it follows that the map H can be extended to a bounded operator of $H^1_{\max}(m_j)$ to $L^1(m_j)$. Hence by Corollary 1 we have the following:

COROLLARY 2. *Let D be as in Theorem 1. Then the operator H and the orthogonal projections P_j can be extended to bounded projections of*

$H^1_{\max}(m_l)$ onto $H^1(m_l)$, $j, l = 0, 1, 2$.

When D is the open unit disc and $m_0 = m_1 = m_2$, then this corollary was obtained by Burkholder, Gundy and Silverstein [2].

The following result is an immediate consequence of Corollary 2 and the $H^1_{\max}$-BMO duality theorem. For the definition of BMO, see Section 2.

COROLLARY 3. *Let D be as in Theorem 1. The dual of $H^1(m_j)$ is isomorphic to $BMOA(m_j)$, where $BMOA(m_j) = BMO(m_j) \cap H^2(m_j)$, $j = 0, 1, 2$.*

In connection with M. Riesz's inequality, Gamelin and Lumer [6] proved the following formula in an abstract setting:

$$L^p(m_1) = H^p(m_1) \oplus H^p_0(m_1)^- \oplus N\,, \quad 1 < p < \infty\,,$$

where $H^p_0(m_1) = \left\{f \in H^p(m_1)\colon \int f dm_1 = 0\right\}$ and N is a finite dimensional subspace of $L^\infty(m_1)$.

As an application of Corollary 2 we can extend the result to the case $p = 1$ as follows:

COROLLARY 4. *Let D be as in Theorem 1. Then*

$$H^1_{\max}(m_1) = H^1(m_1) \oplus H^1_0(m_1)^- \oplus N\,.$$

In Section 2, we obtain propositions which will be used for the proofs of these results, and in Section 3 we prove Theorem 1. Corollaries stated above are proved in Section 4.

2. Some preliminary results. Let D be as in Theorem 1. We will use $C_5, C_6, \cdots$ to denote positive constants depending only on D, m_1 and m_2.

PROPOSITION 1 (cf. [12]). *For every $j = 1, 2$ and every measurable set E, the following inequalities are valid:*

$$C_5 m_{j-1}(E) \leqq m_j(E) \leqq C_6 m_{j-1}(E)\,.$$

Denote by $BMO(m_j)$ the class of all integrable functions f such that

$$\|f\|_{bmo,j} = \sup\left\{(1/m_j(I))\int_I |f - f_I| dm_j\colon I \text{ is the intersection of } \partial D \text{ and a disc centered at a point in } \partial D\right\} < \infty. \quad \text{Here } f_I = m_j(I)^{-1}\int_I f dm_j\,.$$

PROPOSITION 2 ($H^1_{\max}$-BMO duality; see [8]). *The dual of $H^1_{\max}(m_j)$ is isomorphic to $BMO(m_j)$. Especially, for every $x^* \in (H^1_{\max}(m_j))^*$, there exists a unique element $b(x^*)$ of $BMO(m_j)$ such that*

$$x^*(f) = \int f \cdot b(x^*)^- dm_j\,, \quad \text{for all} \quad f \in L^2(m_j)\,.$$

To prove Theorem 1, we need the following:

PROPOSITION 3. (1) (cf. [9] and Proposition 1). *H is a bounded operator of $L^2(m_j)$ to $L^2(m_j)$, $j = 0, 1, 2$.*

(2) *Furthermore, H is a bounded projection of $L^2(m_j)$ onto $H^2(m_j)$, $j = 0, 1, 2$.*

PROOF OF (2). We note that the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{1}{\zeta - z} f(\zeta) d\zeta \quad (z \in D)$$

of a smooth function f is Hölder continuous near the boundary of D (see [11]).

To prove $HH = H$, we recall Mergelyan's theorem which asserts that every $f \in A(\partial D)$ is approximated uniformly by rational functions whose poles are off D (see [13]). For such rational functions g we have $Hg = g$ by the Cauchy integral formula. Hence by Propositions 3, (1) and the boundary property of the Cauchy integral stated above, we have

(i) $Hg = g$, for every $g \in H^2(m_j)$, and

(ii) $Hg \in A(\partial D)$, for every $g \in C^2(\partial D)$.

From (i) and (ii) it follows that H is a bounded projection of $L^2(m_j)$. By Proposition 3, (1), a density argument and the relation (i), we see immediately that the range of H coincides with $H^2(m_j)$.

In order to prove the corollaries we need the following:

PROPOSITION 4. *If $f \in H^1(m_j)$ then*

$$\|f\|_{1,j} \leqq \|f\|_{1,j,\max} \leqq C_7\|f\|_{1,j} .$$

PROOF. Since the first inequality is clear, we prove the second. Here we use the following lemma which extends a result of Burkholder, Gundy and Silverstein [2] to certain general domains:

LEMMA 1 (cf. [1]). *Let $\{B(t): 0 \leqq t < \infty\}$ be a complex Brownian motion starting at z_1. Then*

$$E[\sup_{0 \leqq t < T} |\mathrm{PI}[f](B(t))|] \leqq C_8\|f\|_{1,1,\max} ,$$

for every $f \in H^1_{\max}(m_1)$, and

$$C_9\|f\|_{1,1,\max} \leqq E[\sup_{0 \leqq t < T} |\mathrm{PI}[f](B(t))|] ,$$

for every $f \in L^1(m_1)$ with $E[\sup_{0 \leqq t < T}|\mathrm{PI}[f](B(t))|] < \infty$, where E is the expectation with respect to the Wiener measure which defines the Brownian motion B, and T is the first time at which B escapes from D.

This lemma holds good if D is higher dimensional non-tangentially accessible domains (see [1]).

Now we proceed with the proof of Proposition 4. Let $Mf(t) = \mathrm{PI}[f](B(t))$ for $0 \leqq t < T$, and $Mf(t) = f(B(T))$ for $T \leqq t$. By Ito's formula, $Mf(t)$ is a continuous holomorphic martingale in the sense of Varopoulos [15]. Consequently, by [15], $|Mf(t)|^{1/2}$ is a submartingale. Hence Doob's inequality implies that

$$E[\sup_{0\leqq t<T}|\mathrm{PI}[f](B(t))|] = E[(\sup_{0\leqq t\leqq\infty}|Mf(t)|^{1/2})^2] \leqq 4E[|Mf(\infty)|] = 4\|f\|_{1,1} .$$

Here the last equality is guranteed by Kakutani's theorem. Thus by Proposition 1 we obtain Proposition 4.

3. Proof of Theorem 1. Proof of (1). Let

$$q(s, t) = -K(t, s)^{-} - K(s, t) \quad \text{and}$$
$$r_j(s, t) = W_j(s)^{-1}(W_j(s) - W_j(t))(K(t, s)^{-}) .$$

Then $a_j = q + r_j$.

By the proof of Theorem 1 in Kerzman and Stein [10], q is the restriction of a function which is continuous on each $I_k \times I_m$ ($k, m = 0, \cdots, n$). Hence it is sufficient to prove that r_j can be extended to a function which is continuous on each $I_k \times I_m$ ($k, m = 0, \cdots, n$). To prove this we show the following:

LEMMA 2. *W_j is twice continuously differentiable in each I_k ($k = 0, \cdots, n$), $j = 0, 1, 2$.*

PROOF. Let

$$n(t) = -iD_+\alpha(t) ,$$

For sufficiently small $\varepsilon > 0$, let $\alpha(\varepsilon, t) = \alpha(t) - \varepsilon n(t)$. It is easy to check that $\Gamma_{k,\varepsilon} \equiv \{\alpha(\varepsilon, t): t \in I_k\}$ is a simple closed curve and $\Gamma_{0,\varepsilon} + \cdots + \Gamma_{n,\varepsilon}$ is the boundary of a C^3 subdomain D_ε of D such that $D_\varepsilon \uparrow D$ as $\varepsilon \to 0+$.

Let $n(\varepsilon, t)$ be the outward normal field of ∂D_ε, that is,

$$n(\varepsilon, t) = (1/|D_+\alpha(\varepsilon, t)|)\left(\mathrm{Im}\, D_+\alpha(\varepsilon, t)\frac{\partial}{\partial x} - i\mathrm{Re}\, D_+\alpha(\varepsilon, t)\frac{\partial}{\partial y}\right) ,$$

where $D_+\alpha(\varepsilon, t) = \lim_{h\to 0+} [\alpha(\varepsilon, t + h) - \alpha(\varepsilon, t)]/h$.

By regularity properties of elliptic boundary value problems ([7]), the Green function $g(z_j, \cdot)$ of D possesses all derivatives of order $\leqq 3$ continuous in $D\setminus\{z_j\}$ and they have continuous extensions to $D\cup\partial D\setminus\{z_j\}$. Hence if we put

$$W_{j,\varepsilon}(t) = -n(\varepsilon, t)g(z_j, \alpha(\varepsilon, t)) ,$$

then $W_{j,\varepsilon} \in C^2(I_k)$, and $W_{j,\varepsilon}$ converges to the Poisson kernel W_j in the $C^2(I_k)$-topology as $\varepsilon \to 0+$ $(k = 0, \cdots, n)$. Consequently, $W_j \in C^2(I_k)$ $k = 0, \cdots, n$.

Now, we return to the proof of Theorem 1. By Lemma 2 it suffices to show that $(W_j(s) - W_j(t))K(t, s)$ is the restriction of a function which is continuous on each $I_k \times I_m$ $(k, m = 0, \cdots, n)$. Let

$$F_j(s, t) = \begin{cases} [W_j(s) - W_j(t)]/(s - t) & \text{if } s \neq t \\ D_+ W_j(t) & \text{if } s = t \end{cases}$$

and

$$G(s, t) = \begin{cases} [\alpha(s) - \alpha(t)]/(s - t) & \text{if } s \neq t \\ D_+\alpha(t) & \text{if } s = t\,. \end{cases}$$

Then $(W_j(s) - W_j(t))K(t, s) = D_+\alpha(s)\cdot F_j(s, t)/G(s, t)$ for every $(s, t) \in [0, L)\times[0, L) - \{\text{diagonal}\}$. By the Taylor expansion of $W_j(s) - W_j(t)$ we see that

$$F_j(t + u, t + v) = D_+ W_j(t) + o(1)$$

if $u \neq v$ and if t, $t + u$ and $t + v$ belong to a same interval I_k. Moreover, F_j is continuous on each $(I_k \times I_m) \cap \{\text{diagonal}\}$ by definition. Hence F_j is continuous on each $I_k \times I_m$. A similar proof yields that $G \in C(I_k \times I_m)$ and $G \neq 0$, $k, m = 0, \cdots, n$. This complete the proof of (1).

Proof of Theorem 1, (2). By Proposition 1 it is sufficient to show (2) when $j = l$.

For a compact operator T on a Banach space the mapping $I - T$ is a Fredholm operator of index zero ([14, p. 301]). Hence (2) is valid if the following assertions hold true:

Assertion 1. A_j is compact on $H^1_{\max}(m_j)$.

Assertion 2. The range of $I - A_j$ is equal to $H^1_{\max}(m_j)$.

We begin by proving Assertion 1. By Theorem 1, (1) and the generalized Stone-Weierstrass theorem ([13, Corollary 12.5]) there exist $p_{j,\nu} \in \{\sum_{m=1}^N b_m(x)c_m(y): b_m \in C(\partial D),\ c_m \in C(\partial D),\ N = 1, 2, \cdots.\}$ such that

$$\lim_{\nu\to\infty} \|A_j(\alpha^{-1}(\cdot), \alpha^{-1}(\cdot)) - p_{j,\nu}\|_\infty = 0\,.$$

Since the integral operators

$$P_{j,\nu}f(x) = \int p_{j,\nu}(x, \alpha(t))f(\alpha(t))dt$$

are of finite rank, we obtain Assertion 1 by Hölder's inequality.

Before verifying Assertion 2, we introduce some notation:

For a Banach space X, the algebra of all bounded linear operators from X to itself will be denoted by $\mathrm{BL}(X)$. If $T \in \mathrm{BL}(H^1_{\max}(m_j))$, then

$(T)_m$ denotes the conjugate operator of T as an operator on $H^1_{\max}(m_j)$. If $T \in \mathrm{BL}(L^2(m_j))$, then $(T)_L$ represents the adjoint as an element of $\mathrm{BL}(L^2(m_j))$. For every $T \in \mathrm{BL}(H^1_{\max}(m_j))$, we put $R(T) = \{Tf: f \in H^1_{\max}(m_j)\}$.

By the closed range theorem and [14, V. Theorem 7.8] we have

$$R(I - A_j) = \{g \in H^1_{\max}(m_j): g^*(g) = 0 \text{ for every element } g^* \text{ of the kernel of } (I - A_j)_m\} .$$

Hence, to prove Assertion 2 we need only to show that the kernel K of $(I - A_j)_m$ consists of zero.

LEMMA 3. *For every* $g^* \in (H^1_{\max}(m_j))^*$ *and every* $g \in L^2(m_j)$,

$$[(A_j)_m(g^*)](g) = \int g \cdot H(b(g^*))^- dm_j - \int g \cdot (H)_L(b(g^*))^- dm_j ,$$

where $b(g^*)$ *is defined as in Proposition* 2.

PROOF OF LEMMA 3. It is easy to check that

$$(H)_L(h)(x) = \frac{1}{2}h(x) - \frac{1}{2\pi i}\mathrm{P.V.}\int K(t, \alpha^{-1}(x))^- h(\alpha(t)) W_j(t) W_j(\alpha^{-1}(x))^{-1} dt$$

and $A_j(h) = (H)_L(h) - H(h)$, for every $h \in L^2(m_j)$. Hence by Proposition 2 we have

$$\begin{aligned}[(A_j)_m(g^*)](g) &= \int (H)_L(g) \cdot b(g^*)^- dm_j - \int H(g) \cdot b(g^*)^- dm_j \\ &= \int g \cdot H(b(g^*))^- dm_j - \int g \cdot (H)_L(b(g^*))^- dm_j ,\end{aligned}$$

which prove Lemma 3.

Now we are ready to prove that $K = \{0\}$. Fix any $g^* \in K$. Then for every $g \in L^2(m_j)$ we obtain

$$\begin{aligned}0 &= [(I - A_j)_m(g^*)](g) \\ &= g^*(g) - \int g \cdot H(b(g^*))^- dm_j + \int g \cdot (H)_L(b(g^*))^- dm_j \\ &= \int g \cdot [I - H + (H)_L](b(g^*))^- dm_j .\end{aligned}$$

Hence $b(g^*) = [H - (H)_L](b(g^*))$. The last relation implies that

$$\begin{aligned}\|b(g^*)\|^2_{2,j} &= \int H(b(g^*)) \cdot b(g^*)^- dm_j - \int (H)_L(b(g^*)) \cdot b(g^*)^- dm_j \\ &= 2i \,\mathrm{Im}\left[\int H(b(g^*)) \cdot b(g^*)^- dm_j\right] .\end{aligned}$$

Consequently, $\|b(g^*)\|_{2,j} = 0$. Thus $K = \{0\}$.

Proof of Theorem 1, (3) and (4). We have $P_jH = H$ and $P_j(H)_L = P_j$, because $(P_j)_L = P_j$. Hence $P_j(I + H - (H)_L) = H$. By Theorem 1, (2), $I - A_j$ is an injective bounded operator of $L^2(m_j)$ to $L^2(m_j)$. Furthermore $I - A_j$ is a Fredholm operator on $L^2(m_j)$ of index zero. Therefore by the proof of Lemma 3 we obtain (3). From (3) follows

$$P_j = H(I - A_j)^{-1} = P_{j+1}(I - A_{j+1})(I - A_j)^{-1} ,$$

which completes our proof.

4. **Proofs of corollaries.** Corollary 1 is proved directly by Theorem 1.

Corollary 2 is an immediate consequence of Corollary 1, Proposition 3, (2) and Proposition 4, because the operator H is a bounded map of $H^1_{\max}(m_j)$ to $L^1(m_j)$ as mentioned in Section 1. Corollary 3 is proved easily by Corollary 2 and a usual argument.

PROOF OF COROLLARY 4. Corollary 2 implies that

$$H^1_{\max}(m_1) = H^1(m_1) \oplus [I - P_1](H^1_{\max}(m_1)) .$$

Let $Z = H^1_0(m_1)^- \oplus N$, where N is a finite dimensional subspace of $L^\infty(m_1)$ defined by Gamelin and Lumer [6]. We show that $[I - P_1](H^1_{\max}(m_1)) = Z$.

The space Z is closed in $H^1_{\max}(m_1)$, because N is finite dimensional. Hence applying the open mapping theorem to the operator $T(g, h) = g + h$ $((g, h) \in H^1_0(m_1)^- \times N)$, we have

$$\text{(i)} \qquad C(\|g\|_{1,1,\max} + \|h\|_{1,1,\max}) \leqq \|g + h\|_{1,1,\max} \leqq \|g\|_{1,1,\max} + \|h\|_{1,1,\max} ,$$

where $(g, h) \in H^1_0(m_1)^- \times N$ and C is a constant independent of g and h. From Corollary 2 follows

$$\langle [I - P_1](L^2(m_1)) \rangle = [I - P_1](H^1_{\max}(m_1)) ,$$

where $\langle U \rangle$ denotes the $H^1_{\max}(m_1)$-norm closure of U. Furthermore from Proposition 4 follow

$$\langle H^2_0(m_1)^- \rangle = H^1_0(m_1)^- \quad \text{and} \quad \langle N \rangle = N .$$

Since

$$L^2(m_1) = H^2(m_1) \oplus H^2_0(m_1)^- \oplus N ,$$

(cf. [6]), we have by (i)

$$[I - P_1](H^1_{\max}(m_1)) = \langle H^2_0(m_1)^- \oplus N \rangle = \langle H^2_0(m_1)^- \rangle \oplus \langle N \rangle = H^1_0(m_1)^- \oplus N ,$$

from which Corollary 4 follows.

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