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CERTAIN COMPACT COMPLEX MANIFOLDS WITH INFINITE CYCLIC FUNDAMENTAL GROUPS

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Introduction. In this paper, we give a method of constructing certain examples of compact complex manifolds U with $\pi_1(U) \simeq \boldsymbol{Z}$ and study the structure of U. Those manifolds are toroidal compactifications of the quotient spaces of open sets of algebraic tori $(\boldsymbol{C}^\times)^r$ by the groups $g^{\boldsymbol{Z}}$ generated by elements g in $GL(r, \boldsymbol{Z})$ satisfying certain conditions (Definition 1.1). As examples of such g, we can take integral matrices whose entries are all positive. Since the first Betti numbers of such manifolds are equal to one, they are not Kähler manifolds. In the two-dimensional case, those manifolds U are hyperbolic Inoue surfaces or half Inoue surfaces (see [3]). Hence we may regard our examples as higher-dimensional analogues of hyperbolic Inoue surfaces. On the other hand, one of them is bimeromorphic to that constructed by Kato [6]. Therefore, we call them Inoue-Kato manifolds. (The name was suggested by Ishida.) Sankaran [9] also constructs certain examples of compact complex manifolds M, which are in another sense higher-dimensional analogues of hyperbolic Inoue surfaces, and whose fundamental groups are free abelian groups of rank $\dim M - 1$.

This paper is organized as follows. In Section 1 and Section 2, we construct compact complex manifolds mentioned above and their degenerations, respectively. In Section 3, we show that a part of them contain global spherical shells. In Section 4 and Section 5, we calculate some of their analytic invariants. We show some examples in Section 6.

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1. The construction.

Let $N \simeq \boldsymbol{Z}^r$ be a free $\boldsymbol{Z}$-module of rank r and let $T = N \otimes \boldsymbol{C}^\times$ be an algebraic torus of rank r.

DEFINITION 1.1. Let $K(N)$ be the set of $\boldsymbol{Z}$-linear transformations g of N satisfying the following condition.

g has a simple real eigenvalue $\lambda = \lambda(g)$ such that $|\eta| < \lambda$ for all the

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other eigenvalues η of g.

Clearly, we have:

PROPOSITION 1.2. *If g is in $K(N)$, then ${}^t g$ is in $K(N^*)$, where the transpose ${}^t g$ is the linear transformation of $N^* := \mathrm{Hom}(N, \boldsymbol{Z})$ definined by $\langle m, gn\rangle = \langle {}^t gm, n\rangle$, for all $m \in N^*$ and for all $n \in N$.*

PROPOSITION 1.3. *Let g be in $K(N)$. Then there exists an open convex cone C such that the closure of gC is contained in $C \cup \{0\}$, that $H(g) := \bigcup_{l \in \boldsymbol{Z}} g^l C$ is a half-space of $N_{\boldsymbol{R}}$ and that $L(g) := \bigcap_{l \in \boldsymbol{Z}} g^l C$ is a half-line of $N_{\boldsymbol{R}}$, where we denote by the same letter g, the image of g under the natural map $GL(N) \to GL(N_{\boldsymbol{R}})$.*

PROOF. Let v and v^* be eigenvectors of g and ${}^t g$, respectively, associated with the real eigenvalue $\lambda(g)$. Then clearly, $\langle v^*, v\rangle \neq 0$. Hence we may assume that $\langle v^*, v\rangle > 0$. Then the half-space $H := \{y \in N_{\boldsymbol{R}} \mid \langle v^*, y\rangle > 0\}$ contains the half-line $L := \boldsymbol{R}_{>0} v$. Take an open polygonal cone $C_0 = \boldsymbol{R}_{>0} n_1 + \boldsymbol{R}_{>0} n_2 + \cdots + \boldsymbol{R}_{>0} n_s$ containing L and contained in H. Then $\bigcap_{l \in \boldsymbol{Z}} g^l C_0 = L$. Hence there exists a positive integer l_0 such that the closure of $g^{l_0} C_0$ is contained in $C_0 \cup \{0\}$. Let $C_j = \boldsymbol{R}_{>0} n_1(j) + \boldsymbol{R}_{>0} n_2(j) + \cdots + \boldsymbol{R}_{>0} n_s(j)$, where $n_k(j) = n_k + (j\varepsilon / l_0) v$. Then the closure of $g^{l_0} C_0$ is contained also in $C_{l_0} \cup \{0\}$ for a positive real number ε small enough. Moreover, the closure of C_{j+1} is contained in $C_j \cup \{0\}$. Let $C = C_{l_0 - 1} \cap g C_{l_0 - 2} \cap \cdots \cap g^{l_0 - 1} C_0$. Then the closure of $gC = g^{l_0} C_0 \cap g C_{l_0 - 1} \cap \cdots \cap g^{l_0 - 1} C_1$ is contained in $C \cup \{0\}$. Since also C contains L and contained in H, we have $\bigcap_{l \in \boldsymbol{Z}} g^l C = L$ and $\bigcup_{l \in \boldsymbol{Z}} g^l C = H$. q.e.d.

Let g be in $K(N)$. Then we see by the above proposition that the cyclic group $g^{\boldsymbol{Z}}$ generated by g acts on $D(g) := (H(g) \setminus L(g)) / \boldsymbol{R}_{>0}$ properly discontinuously and without fixed points and that the quotient $D(g)/g^{\boldsymbol{Z}}$ is compact. Moreover, $D(g)/g^{\boldsymbol{Z}}$ is homeomorphic to $S^{r-2} \times S^1$, if g is in $SL(N)$.

PROPOSITION 1.4. *Let g be in the group $GL(N)$ of $\boldsymbol{Z}$-automorphisms of N and let C be the interior of a non-singular rational cone of dimension r in $N_{\boldsymbol{R}}$. Assume that gC is contained in C and that the closure of $g^l C$ is contained in $C \cup \{0\}$, for a positive integer l. Then g is in $K(N)$.*

PROOF. By assumption, there exists a $\boldsymbol{Z}$-basis $\{n_1, n_2, \cdots, n_r\}$ of N with $C = \boldsymbol{R}_{>0} n_1 + \boldsymbol{R}_{>0} n_2 + \cdots + \boldsymbol{R}_{>0} n_r$. Then g (resp. g^l) is represented with respect to the basis $\{n_1, n_2, \cdots, n_r\}$ by a matrix whose entries are all non-negative (resp. positive) integers. Hence by the Perron-Frobenius theorem (see [10, Ex. 37]), g (resp. g^l) has a real eigenvalue λ (resp. a simple real eigenvalue λ') such that $\lambda \geqq |\eta|$ (resp. $\lambda' > |\eta'|$) for the other eigenvalues η of g (resp. η' of g^l). Here clearly, $\lambda' = \lambda^l$. Therefore, λ is a simple real eigenvalue of g and $\lambda > |\eta|$ for the other eigenvalues η of g. q.e.d.

PROPOSITION 1.5. *Let g be in $K(N)$. Then g^Z acts on $H(g)$ properly discontinuously and without fixed points.*

PROOF. Let v and v^* be the same as in the proof of Proposition 1.3. By Proposition 1.3, we have a cone C such that the closure of gC is contained in $C \cup \{0\}$. Let $F = \{y \in C \mid \langle v^*, y\rangle > 1\}$. Then $\bigcup_{l \in Z} g^l F = H(g)$, $\bigcap_{l \in Z} g^l F = \emptyset$ and the closure of gF is contained in F, because $\langle v^*, gy\rangle = \langle {}^t g v^*, y\rangle = \lambda(g)\langle v^*, y\rangle > \lambda(g) > 1$ for any y in F. Hence the action of g^Z on $H(g)$ is properly discontinuous and fixed point free. q.e.d.

In the following, we use the notation in [8]. Let g be in $K(N)$ and let $\tilde{W} = \mathrm{ord}^{-1}(H(g))$ be the inverse image of $H(g)$ under the $GL(N)$-equivariant map $\mathrm{ord} = -\log|\ |: T \to N_R$. Then the quotient $W := \tilde{W}/g^Z$ of $\tilde{W}$ with respect to the action of g^Z is a complex manifold by the above proposition. In the following, we construct a toroidal compactification of W. First, we show that there exists a g^Z-invariant r.p.p. decomposition Σ in N with $|\Sigma|$ $(:= \bigcup_{\sigma \in \Sigma} \sigma) = (H(g) \setminus L(g)) \cup \{0\}$. We can take a strongly convex rational polyhedral cone C such that gC is contained in $\mathrm{Int}(C) \cup \{0\}$, that $\bigcup_{l \in Z} g^l C = H(g) \cup \{0\}$ and that $\bigcap_{l \in Z} g^l C = L(g) \cup \{0\}$, by Proposition 1.3. Let $\Lambda = \{\text{faces of } C\} \setminus \{C\}$. Then since $\Lambda \cup g\Lambda$ is an r.p.p. decomposition in N, we have a complete r.p.p. decomposition Λ' containing $\Lambda \cup g\Lambda$, by [11, Theorem 3] and [8, Theorem 4.1]. Let $\Sigma_0 = \{\sigma \in \Lambda' \mid \sigma \subset C \setminus \mathrm{Int}(gC)\}$. Then $|\Sigma_0| = C \setminus \mathrm{Int}(gC)$, because $C \setminus \mathrm{Int}(gC)$ is the closure of a connected component of $N_R \setminus |\Lambda \cup g\Lambda|$. Hence $\Sigma = \{g^l \sigma \mid \sigma \in \Sigma_0, l \in Z\}$ is a g^Z-invariant r.p.p. decomposition in N and $|\Sigma| = \bigcup_{l \in Z}(g^l C \setminus \mathrm{Int}(g^{l+1}C)) = (\bigcup_{l \in Z} g^l C) \setminus (\bigcap_{l \in Z} \mathrm{Int}(g^{l+1}C)) = (H(g) \setminus L(g)) \cup \{0\}$. Let $\tilde{X} = T\mathrm{emb}(\Sigma) \setminus T$ and let $\tilde{U} = \tilde{W} \cup \tilde{X}$. Then $\tilde{U}$ is an open set of $T\mathrm{emb}(\Sigma)$ and is invariant under the action of g^Z.

PROPOSITION 1.6. *$\tilde{U}$ is simply connected.*

PROOF. Note that the inclusion map $\tilde{W} \hookrightarrow T$ induces an isomorphism $\pi_1(\tilde{W}) \simeq \pi_1(T)$ of the fundamental groups. Hence we get the assertion of the proposition in the same way as in the proof of [8, Proposition 10.2]. q.e.d.

We obtain from Σ, a g^Z-invariant polygonal decomposition $\Delta := \{(\sigma \setminus \{0\})/R_{>0} \mid \sigma \in \Sigma \setminus \{\{0\}\}\}$ on $D(g)$, which coincides with the dual graph of $\tilde{X}$. Since g^Z has no fixed points on $D(g)$, neither does it on $\tilde{X}$. Let

$U = \tilde{U}/g^{Z}$ and let $X = \tilde{X}/g^{Z}$. Then X is a divisor on U and the dual graph of X is the graph on $D(g)/g^{Z}$ which is the image of Δ under the projection $D(g) \to D(g)/g^{Z}$.

PROPOSITION 1.7. *U is an r-dimensional compact complex variety with the fundamental group $\pi_1(U) \simeq \boldsymbol{Z}$.*

PROOF. Since g^{Z} has no fixed points on $\tilde{X}$ and on $\tilde{W}$, neither does it on $\tilde{U} = \tilde{W} \cup \tilde{X}$. Let F be the same as in the proof of Proposition 1.5. Then the closure G of $F \setminus gF$ in $\tilde{U}/CT$ is compact, where CT is the compact real torus $N \otimes U(1)$ in T. Hence the inverse image of G under the map ord: $\tilde{U} \to \tilde{U}/CT$ is also compact and is a fundamental domain with respect to the action of g^{Z}. Therefore, U is an r-dimensional compact complex variety. Moreover, $\pi_1(U) \simeq g^{Z} \simeq \boldsymbol{Z}$ by Proposition 1.6. q.e.d.

Assume that Σ consists of non-singular cones. Then $\tilde{U}$ and U are complex manifolds. Moreover, the dual graph Δ of $\tilde{X}$ is a triangulation.

REMARK. When $r = 2$, U is a hyperbolic Inoue surface or is a half Inoue surface, according as g belongs to $SL(N)$ or not.

2. **Degenerations.** Since λ in Definition 1.1 is greater than one, we have:

PROPOSITION 2.1. *If g is in $K(N)$, then $\hat{g}_{\pm}$ is in $K(N \oplus \boldsymbol{Z})$ where $\hat{g}_{\pm}$ is the linear transformation of $N \oplus \boldsymbol{Z}$ sending (n, l) to $(gn, \pm l)$.*

Let g be in $K(N)$ and assume that there exists an r.p.p. decomposition Σ with $|\Sigma| = (H(g) \setminus L(g)) \cup \{0\}$. Let $\Lambda = \{\boldsymbol{R}_{\geqq 0}1, \{0\}, \boldsymbol{R}_{\geqq 0}(-1)\}$. Then Λ is an r.p.p. decomposition in $\boldsymbol{Z}$ and $B := T_{\boldsymbol{Z}}\,\mathrm{emb}(\Lambda)$ is a non-singular rational curve. Assume that there exists an r.p.p. decomposition $\hat{\Sigma}$ in $N \oplus \boldsymbol{Z}$ satisfying the following condition.

(D) $\hat{\Sigma}$ is $(\hat{g}_+)^{Z}$-invariant, $|\hat{\Sigma}| = ((H(g) \times \boldsymbol{R}) \setminus (L(g)) \times \{0\})) \cup \{0\}$, the subcomplex $\{\sigma \in \hat{\Sigma} \mid \sigma \subset N_{\boldsymbol{R}}\}$ of $\hat{\Sigma}$ is equal to Σ and the natural projection $N \oplus \boldsymbol{Z} \to \boldsymbol{Z}$ induces a morphism $(N \oplus \boldsymbol{Z}, \hat{\Sigma}) \to (\boldsymbol{Z}, \Lambda)$ of r.p.p. decompositions.

Then we have an $(r+1)$-dimensional compact complex variety $\mathscr{U} := (\mathrm{ord}^{-1}(H(g) \times \boldsymbol{R}) \cup (T_{N \oplus Z}\mathrm{emb}(\hat{\Sigma}) \setminus T_{N \oplus Z}))/(\hat{g}_+)^{Z}$, a divisor $\mathscr{X} := (T_{N \oplus Z}\mathrm{emb}(\hat{\Sigma}) \setminus T_{N \oplus Z})/(\hat{g}_+)^{Z}$ on $\mathscr{U}$ and a holomorphic map $\varphi: \mathscr{U} \to B$ with $\varphi^{-1}(T_Z) \simeq U \times T_Z$ $(\varphi^{-1}(T_Z) \cap \mathscr{X} \simeq X \times T_Z)$

THEOREM 2.2. *Assume that there exist a convex rational cone C and a set Σ_r^0 of r-dimensional cones in Σ such that $gC \subset \mathrm{Int}(C) \cup \{0\}$ and that $|\Sigma_r^0| = C \setminus \mathrm{Int}(gC)$. Then there exists an r.p.p. decomposition $\hat{\Sigma}$ in $N \oplus \boldsymbol{Z}$ satisfying the above condition* (D). *Moreover,* $\varphi^{-1}(\mathrm{orb}(\boldsymbol{R}_{\geqq 0}1)) \simeq$

$\varphi^{-1}(\mathrm{orb}(\boldsymbol{R}_{\geqq 0}(-1)))$ is a toric variety intersecting itself along two disjoint divisors.

PROOF. First, we note that Σ_r^0 consists of representatives of r-dimensional cones of Σ modulo $g^{\boldsymbol{Z}}$, i.e., $\{g^l\sigma \mid l \in \boldsymbol{Z}, \sigma \in \Sigma_r^0\} = \{r\text{-dimensional cones in } \Sigma\}$ and $l = 0$ if $g^l\sigma = \tau$ for $\sigma, \tau \in \Sigma_r^0$. Take an element v in N so that $gv - v$ is contained in $\mathrm{Int}(gC)$ and let $\hat{C} = (\boldsymbol{R}_{\geqq 0}(v, 1) + C) \cup (\boldsymbol{R}_{\geqq 0}(v, -1) + C)$. Then $\hat{g}_+\hat{C}$ is contained in $\mathrm{Int}(\hat{C}) \cup \{0\}$. Let $\Sigma^0 = \{\text{faces of } \sigma \mid \sigma \in \Sigma_r^0\}$ and let $\hat{\Sigma}^0 = \{\hat{\sigma}_\pm, \tilde{\tau}_\pm \mid \sigma, \tau \in \Sigma^0, \tau \subset gC\}$, where $\hat{\sigma}_\pm = \boldsymbol{R}_{\geqq 0}(v, \pm 1) + \sigma$ and $\tilde{\tau}_\pm = \boldsymbol{R}_{\geqq 0}(v, \pm 1) + \boldsymbol{R}_{\geqq 0}(gv, \pm 1) + \tau$. Then we can verify that $g^l\eta \cap \lambda$ are faces of $g^l\eta$ and λ, for $\eta, \lambda \in \hat{\Sigma}^0$ and for $l \in \boldsymbol{Z}$ and that $|\hat{\Sigma}^0| = \hat{C} \setminus \mathrm{Int}(\hat{g}_+\hat{C})$. Hence $\hat{\Sigma} := \{\text{faces of } (\hat{g}_+)^l\lambda \mid \lambda \in \hat{\Sigma}^0, l \in \boldsymbol{Z}\}$ is an r.p.p. decomposition in $N \oplus \boldsymbol{Z}$ and satisfies the condition (D). The last assertion follows from the construction of $\hat{\Sigma}$. q.e.d.

COROLLARY 2.3. *Let g be in $K(N)$ and assume that there exist an r.p.p. decomposition Σ with $|\Sigma| = (H(g) \setminus L(g)) \cup \{0\}$, a positive integer l, a convex rational cone C and a set Σ_r^0 of r-dimensional cones in Σ such that $|\Sigma_r^0| = C \setminus \mathrm{Int}(g^lC)$ and that $g^lC \subset \mathrm{Int}(C) \cup \{0\}$. Then there exists an r.p.p. decomposition $\hat{\Sigma}$ satisfying the condition* (D).

PROOF. By Theorem 2.2, we have a $(\hat{g}_+)^{l\boldsymbol{Z}}$-invariant r.p.p. decomposition $\hat{\Xi}$ such that $|\hat{\Xi}| = ((H(g) \times \boldsymbol{R}) \setminus (L(g) \times \{0\})) \cup \{0\}$ and that $\{\sigma \in \hat{\Xi} \mid \sigma \subset N_{\boldsymbol{R}}\} = \Sigma$. Let $\hat{\Sigma} = \{h_1\sigma_1 \cap h_2\sigma_2 \cap \cdots \cap h_l\sigma_l \mid \sigma_i \in \hat{\Xi}\}$, where $h_i = (\hat{g}_+)^i$. Then $\hat{\Sigma}$ is $(\hat{g}_+)^{\boldsymbol{Z}}$-invariant, consists of rational cones, $|\hat{\Sigma}| = |\hat{\Xi}|$ and $\{\sigma \in \hat{\Sigma} \mid \sigma \subset N_{\boldsymbol{R}}\} = \Sigma$, because Σ is $g^{\boldsymbol{Z}}$-invariant. Hence it is sufficient to show that $\hat{\Sigma}$ is an r.p.p. decomposition. Let τ be a face of an element $\sigma = h_1\sigma_1 \cap h_2\sigma_2 \cap \cdots \cap h_l\sigma_l$ in $\hat{\Sigma}$. Then $\tau = \sigma \cap x^\perp$, for an element $x = x_1 + x_2 + \cdots + x_l$ in $\sigma^\vee = (h_1\sigma_1)^\vee + (h_2\sigma_2)^\vee + \cdots + (h_l\sigma_l)^\vee$ $(x_i \in (h_i\sigma_i)^\vee)$, where $\sigma^\vee$ is the dual cone of σ and $x^\perp = \{y \in N_{\boldsymbol{R}} \mid \langle x, y\rangle = 0\}$. Let $\tau_i = h_i\sigma_i \cap x_i^\perp$. Then τ_i is a face of $h_i\sigma_i$ and

$$\begin{aligned}\tau &= \{y \in \sigma \mid \langle x, y\rangle = 0\}\\ &= \{y \in \sigma \mid \langle x_1, y\rangle = \langle x_2, y\rangle = \cdots = \langle x_l, y\rangle = 0\} = \tau_1 \cap \tau_2 \cap \cdots \cap \tau_l \in \hat{\Sigma}\ ,\end{aligned}$$

because $\langle x_i, y\rangle \geqq 0$ for $y \in \sigma$. Next, let $\sigma = h_1\sigma_1 \cap h_2\sigma_2 \cap \cdots \cap h_l\sigma_l$ and $\tau = h_1\tau_1 \cap h_2\tau_2 \cap \cdots \cap h_l\tau_l$ be in $\hat{\Sigma}$. Then $\sigma \cap \tau = h_1(\sigma_1 \cap \tau_1) \cap h_2(\sigma_2 \cap \tau_2) \cap \cdots \cap h_l(\sigma_l \cap \tau_l)$. Since $h_i(\sigma_i \cap \tau_i)$ is a face of $h_i\sigma_i$, there exists an element x_i in $(h_i\sigma_i)^\vee$ with $h_i(\sigma_i \cap \tau_i) = h_i\sigma_i \cap x_i^\perp$. Hence

$$\begin{aligned}\sigma \cap \tau &= \{y \in \sigma \mid \langle x_1, y\rangle = \langle x_2, y\rangle = \cdots = \langle x_l, y\rangle = 0\}\\ &= \{y \in \sigma \mid \langle x_1 + x_2 + \cdots + x_l, y\rangle = 0\}\end{aligned}$$

is a face of σ, because $x_1 + x_2 + \cdots + x_l \in (h_1\sigma_1)^\vee + (h_2\sigma_2)^\vee + \cdots + (h_l\sigma_l)^\vee = \sigma^\vee$. q.e.d.

3. Global spherical shells. We keep the notation in Section 1.

DEFINITION 3.1. An open set S of a complex manifold U is a *global spherical shell,* if $U \setminus S$ is connected and if S is biholomorphic to $\{(z_1, z_2, \cdots, z_r) \in \boldsymbol{C}^r \mid \alpha < \sum_{k=1}^r |z_k|^2 < \beta\}$ for positive real numbers α and β with $0 < \alpha < \beta$.

See [5], for the properties of compact complex manifolds containing global spherical shells.

THEOREM 3.2. *If there exists an r-dimensional non-singular rational cone σ in $N_{\boldsymbol{R}}$ such that $g\sigma \setminus \{0\}$ is contained in the interior of σ and that $[\sigma] := \{faces\ of\ \sigma\} \setminus \{\sigma\}$ is contained in Σ, then U contains a global spherical shell.*

PROOF. Let $\{n_1, n_2, \cdots, n_r\}$ be a $\boldsymbol{Z}$-basis of N with $\sigma = \boldsymbol{R}_{\geqq 0} n_1 + \boldsymbol{R}_{\geqq 0} n_2 + \cdots + \boldsymbol{R}_{\geqq 0} n_r$ and let $\{m_1, m_2, \cdots, m_r\}$ be the $\boldsymbol{Z}$-basis of N^* dual to $\{n_1, n_2, \cdots, n_r\}$. Let z_i be the holomorphic function on $T\mathrm{emb}([\sigma])$ $(\simeq \boldsymbol{C}^r \setminus \{0\})$ which is the natural extension of the character $m_i \otimes 1_{\boldsymbol{C}^\times}$: $T \to \boldsymbol{C}^\times$ of m_i. Then $(z_1, z_2, \cdots, z_r)$ is a global coordinate on $T\mathrm{emb}([\sigma])$. Let $S = \{(z_1, z_2, \cdots, z_r) \in T\mathrm{emb}([\sigma]) \mid \gamma - \varepsilon < \sum_{k=1}^r |z_k|^2 < \gamma + \varepsilon\}$ for positive real numbers γ and ε with $\gamma - \varepsilon > 0$ and $\gamma + \varepsilon < 1$. Then we easily see that the image $\mathrm{ord}(S \setminus \tilde{X}) = \{u_1 n_1 + u_2 n_2 + \cdots + u_r n_r \mid \gamma - \varepsilon < \exp(-2u_1) + \exp(-2u_2) + \cdots + \exp(-2u_r) < \gamma + \varepsilon\}$ of $S \setminus \tilde{X}$ under the map ord: $T \to N_{\boldsymbol{R}}$ is contained in $\sigma \setminus \{0\} \subset H(g)$. Hence S is contained in $\tilde{U}$. Let $P_{\pm} = \{u_1 n_1 + u_2 n_2 + \cdots + u_r n_r \mid \exp(-2u_1) + \exp(-2u_2) + \cdots + \exp(-2u_r) < \gamma \pm \varepsilon\}$. Then the closure of gP_+ is contained in P_- for small enough ε, because $gn_j = a_{1j} n_1 + a_{2j} n_2 + \cdots + a_{rj} n_r$ with $a_{ij} \geqq 1$, for $j = 1$ through r. Since $S \setminus \tilde{X} = \mathrm{ord}^{-1}(P_+ \setminus \bar{P}_-)$ and since $S \cap \tilde{X} \subset T\mathrm{emb}([\sigma]) \setminus T$, the restriction to S of the quotient map $q: \tilde{U} \to U$ is injective. Moreover, the image $q(S)$ of S is global, i.e., $U \setminus q(S)$ is connected, because $U \setminus (q(S) \cup X)$ is the image under q of the connected set $\mathrm{ord}^{-1}(\bar{P}_- \setminus gP_+)$. q.e.d.

4. Invariants. We keep the notation in Section 1. Throughout this section, we assume that there exists an r.p.p. decomposition Σ satisfying the conditions of Corollary 2.3 and consisting of non-singular cones. Let $\Theta_{\tilde{U}}(-\log \tilde{X})$ and $\Theta_U(-\log X)$ be the logarithmic tangent sheaves of $(\tilde{U}, \tilde{X})$ and (U, X), respectively, and let $\Omega^1_{\tilde{U}}(\log \tilde{X})$ and $\Omega^1_U(\log X)$ be the dual sheaves of $\Theta_{\tilde{U}}(-\log \tilde{X})$ and $\Theta_U(-\log X)$, respectively. The first purpose of this section is to prove the following proposition.

PROPOSITION 4.1.

$$H^i(U, \mathcal{O}_U) \simeq \begin{cases} \boldsymbol{C} & for \quad i = 0, 1 \\ 0 & for \quad i \geqq 2\ , \end{cases}$$

$$H^i(U, \Theta_U(-\log X)) \simeq \begin{cases} \ker(g-1) & \text{for } i=0 \\ \operatorname{coker}(g-1) & \text{for } i=1 \\ 0 & \text{for } i \geqq 2 \end{cases}$$

and

$$H^i(U, \Omega^1_U(\log X)) \simeq \begin{cases} \ker({}^t g-1) & \text{for } i=0 \\ \operatorname{coker}({}^t g-1) & \text{for } i=1 \\ 0 & \text{for } i \geqq 2 \, , \end{cases}$$

where $(g-1)\colon N_C \to N_C$ *and* $({}^t g - 1)\colon N_C^* \to N_C^*$ *are the* C*-linear maps sending* l *and* l^* *to* $gl - l$ *and* ${}^t gl^* - l^*$*, respectively.*

For the proof, we need some lemmas. Let $\mathscr{F} = q_*^{g^Z}\tilde{\mathscr{F}}$, for a locally free sheaf $\tilde{\mathscr{F}}$ on $\tilde{U}$ with an action of g^Z, where $g\colon \tilde{U} \to U = \tilde{U}/g^Z$ is the quotient map and $q_*^{g^Z}\tilde{\mathscr{F}}$ denotes the subsheaf of $q_*\tilde{\mathscr{F}}$ consisting of germs of g^Z-invariant sections. Then by [2, Corollary 3 to Theorem 5.3.1], we have the spectral sequence:

$$E_2^{p,q}(g^Z, \tilde{\mathscr{F}}) = H^p(g^Z, H^q(\tilde{U}, \tilde{\mathscr{F}})) \Rightarrow H^{p+q}(U, \mathscr{F}) \, .$$

Here we note that $\mathscr{F} = \mathscr{O}_U$, $\Omega^1_U(\log X)$ or $\Theta_U(-\log X)$, according as $\tilde{\mathscr{F}} = \mathscr{O}_{\tilde{U}}$, $\Omega^1_{\tilde{U}}(\log \tilde{X})$ or $\Theta_{\tilde{U}}(-\log \tilde{X})$. Since g^Z is a free group, we have $E_2^{p,q}(g^Z, \tilde{\mathscr{F}}) = 0$ for $p > 1$. Hence the above spectral sequence degenerates and $H^q(U, \mathscr{F}) \simeq E_2^{0,q}(g^Z, \tilde{\mathscr{F}}) \oplus E_2^{1,q-1}(g^Z, \tilde{\mathscr{F}})$. First, we calculate $E_2^{p,0}$, $(g^Z, \tilde{\mathscr{F}})$ for $p = 0, 1$ and for $\tilde{\mathscr{F}} = \mathscr{O}_{\tilde{U}}$, $\Theta_{\tilde{U}}(-\log \tilde{X})$, $\Omega^1_{\tilde{U}}(\log \tilde{X})$.

LEMMA 4.2. $H^0(\tilde{U}, \mathscr{O}_{\tilde{U}}) \simeq C$.

PROOF. Since $\tilde{U}$ is an open set of $T\mathrm{emb}(\Sigma)$, any holomorphic function f on $\tilde{U}$ is expressed as a series

$$f = \sum_{m \in N^*} c_m e(m) \, ,$$

where $e(m)$ is the natural extension to $T\mathrm{emb}(\Sigma)$ of the character $m \otimes 1_{C^\times}\colon T \to C^\times$ of m. Here c_m must vanish, if $\langle m, n\rangle < 0$ for a non-zero element $n \in N$ with $R_{\geqq 0} n \in \Sigma$, because $e(m)$ has poles along $\mathrm{orb}(R_{\geqq 0} n) \subset \tilde{U}$. However,

$$\begin{aligned} &\{m \in N^* \mid \langle m, n\rangle \geqq 0 \quad \text{for all} \quad n \in N \text{ with } R_{\geqq 0} n \in \Sigma\} \\ &\quad = \{m \in N^* \mid \langle m, y\rangle \geqq 0 \quad \text{for all} \quad y \text{ in } H(g)\} \\ &\quad = (L({}^t g) \cup \{0\}) \cap N^* = \{0\} \, , \end{aligned}$$

because $|\Sigma| = (H(g) \setminus L(g)) \cup \{0\}$. Hence $f = c_0$ is a constant function. Therefore, $H^0(\tilde{U}, \mathscr{O}_{\tilde{U}}) \simeq C$. q.e.d.

By [4, Proposition 1.12], there are g^Z-equivariant isomorphisms

$\Theta_{\tilde{U}}(-\log \tilde{X}) \simeq \mathcal{O}_{\tilde{U}} \otimes_{\mathbf{Z}} N$ and $\Omega^1_{\tilde{U}}(\log \tilde{X}) \simeq \mathcal{O}_{\tilde{U}} \otimes_{\mathbf{Z}} N^*$. Hence by the above lemma, we have:

LEMMA 4.3. *There exist $g^{\mathbf{Z}}$-equivariant isomorphisms* $H^0(\tilde{U}, \Theta_{\tilde{U}}(-\log \tilde{X})) \simeq N_{\mathbf{C}}$ *and* $H^0(\tilde{U}, \Omega^1_{\tilde{U}}(\log \tilde{X})) \simeq N^*_{\mathbf{C}}$.

LEMMA 4.4. $H^p(g^{\mathbf{Z}}, \mathbf{C}) \simeq \mathbf{C}$ *for* $p = 0, 1$,

$$H^p(g^{\mathbf{Z}}, N_{\mathbf{C}}) \simeq \begin{cases} \ker(g-1) & \text{for } p = 0 \\ \operatorname{coker}(g-1) & \text{for } p = 1 \end{cases}$$

and

$$H^p(g^{\mathbf{Z}}, N^*_{\mathbf{C}}) \simeq \begin{cases} \ker({}^tg-1) & \text{for } p = 0 \\ \operatorname{coker}({}^tg-1) & \text{for } p = 1 . \end{cases}$$

PROOF. Clearly, $H^0(g^{\mathbf{Z}}, N_{\mathbf{C}}) = (N_{\mathbf{C}})^{g^{\mathbf{Z}}} = \ker(g-1)$ and $H^0({}^tg^{\mathbf{Z}}, N^*_{\mathbf{C}}) = (N^*_{\mathbf{C}})^{{}^tg^{\mathbf{Z}}} = \ker({}^tg-1)$. Since $g^{\mathbf{Z}}$ (resp. ${}^tg^{\mathbf{Z}}$) is generated by g (resp. ${}^tg$), we have $Z^1(g^{\mathbf{Z}}, N_{\mathbf{C}}) \simeq N_{\mathbf{C}}$ (resp. $Z^1({}^tg^{\mathbf{Z}}, N^*_{\mathbf{C}}) \simeq N^*_{\mathbf{C}}$) and $B^1(g^{\mathbf{Z}}, N_{\mathbf{C}}) \simeq \operatorname{Im}(g-1)$ (resp. $B^1({}^tg^{\mathbf{Z}}, N^*_{\mathbf{C}}) \simeq \operatorname{Im}({}^tg-1)$). Hence $H^1(g^{\mathbf{Z}}, N_{\mathbf{C}}) \simeq \operatorname{coker}(g-1)$ (resp. $H^1({}^tg^{\mathbf{Z}}, N^*_{\mathbf{C}}) \simeq \operatorname{coker}({}^tg-1)$). Since $g^{\mathbf{Z}}$ acts on $H^0(\tilde{U}, \mathcal{O}_{\tilde{U}}) \simeq \mathbf{C}$ trivially, we get $H^0(g^{\mathbf{Z}}, \mathbf{C}) \simeq \mathbf{C}$ and $H^1(g^{\mathbf{Z}}, \mathbf{C}) = \operatorname{Hom}(g^{\mathbf{Z}}, \mathbf{C}) \simeq \mathbf{C}$. q.e.d.

Next, we show that $E_2^{p,q}(g^{\mathbf{Z}}, \tilde{\mathscr{F}}) = 0$ for $q \geqq 1$. Let l be an integer such that g^l and Σ satisfy the condition of Theorem 2.2 and let $U' = \tilde{U}/g^{l\mathbf{Z}}$ (resp. $X' = \tilde{X}/g^{l\mathbf{Z}}$). Then U' (resp. X') is an l-sheeted unramified covering of U (resp. X). By Theorem 2.2, we have a degeneration $\varphi: \mathscr{U} \to \mathbf{P}^1$ of U' and a divisor $\mathscr{X}$ on $\mathscr{U}$ such that $\varphi^{-1}(t) \simeq U'$ ($\varphi^{-1}(t) \cap \mathscr{X} \simeq X'$) for $t \neq 0, \infty$ and that $U_0 := \varphi^{-1}(0)$ is an irreducible variety we obtain by identifying two disjoint divisors of a toric variety. Let $\Theta_{\mathscr{U}}(-\log \mathscr{X})$ be the subsheaf of the tangent sheaf $\Theta_{\mathscr{U}}$ of $\mathscr{U}$ consisting of germs of holomorphic derivatives δ with $\delta I \subset I$ and let $\Omega^1_{\mathscr{U}}(\log \mathscr{X})$ be the dual sheaf of $\Theta_{\mathscr{U}}(-\log \mathscr{X})$, where $I \subset \mathcal{O}_{\mathscr{U}}$ is the ideal of definition for $\mathscr{X}$.

LEMMA 4.5.

$$\dim H^p(U_0, \mathcal{O}_{U_0}) = \begin{cases} 1 & \text{for } p = 0, 1 \\ 0 & \text{for } p \geqq 2 , \end{cases}$$

$$\dim H^p(U_0, \Theta_{\mathscr{U}}(-\log \mathscr{X})_0) = \begin{cases} \dim N_{\mathbf{C}}^{g^{l\mathbf{Z}}} + 1 & \text{for } p = 0, 1 \\ 0 & \text{for } p \geqq 2 \end{cases}$$

and

$$\dim H^p(U_0, \Omega^1_{\mathscr{U}}(\log \mathscr{X})_0) = \begin{cases} \dim (N^*_{\mathbf{C}})^{{}^tg^{l\mathbf{Z}}} + 1 & \text{for } p = 0, 1 \\ 0 & \text{for } p \geqq 2 . \end{cases}$$

PROOF. Let D be the double locus of U_0 and let $\hat{U}_0$ be the normalization of U_0. Then we have an exact sequence

$$0 \to \mathcal{O}_{U_0} \to \mathcal{O}_{\hat{U}_0} \to \mathcal{O}_D \to 0 .$$

Since $\hat{U}_0$ and D are compact toric varieties and since $\mathscr{G} \otimes \mathcal{O}_{\hat{U}_0}$ and $\mathscr{G} \otimes \mathcal{O}_D$ are free sheaves on $\hat{U}_0$ and D, respectively, we have $H^p(\hat{U}_0, \mathscr{G} \otimes \mathcal{O}_{\hat{U}_0}) = H^p(D, \mathscr{G} \otimes \mathcal{O}_D) = 0$ for $p > 0$, where $\mathscr{G} = \mathcal{O}_{\mathscr{U}}, \Theta_{\mathscr{U}}(-\log \mathscr{X})$ or $\Omega^1_{\mathscr{U}}(\log \mathscr{X})$. Hence $H^0(U_0, \mathscr{G} \otimes \mathcal{O}_{U_0}) = \ker(d)$, $H^1(U_0, \mathscr{G} \otimes \mathcal{O}_{U_0}) = \operatorname{coker}(d)$ and $H^p(U_0, \mathscr{G} \otimes \mathcal{O}_{U_0}) = 0$ for $p \geqq 2$, where $d: H^0(\hat{U}_0, \mathscr{G} \otimes \mathcal{O}_{\hat{U}_0}) \to H^0(D, \mathscr{G} \otimes \mathcal{O}_D)$. Here $H^0(\hat{U}_0, \mathscr{G} \otimes \mathcal{O}_{\hat{U}_0}) = H^0(D, \mathscr{G} \otimes \mathcal{O}_D) = \boldsymbol{C}$, $N_{\boldsymbol{C}} \oplus \boldsymbol{C}$ or $N^*_{\boldsymbol{C}} \oplus \boldsymbol{C}^*$ and $d = 0$, $(\hat{g}_+)^l - 1$ $(=(g^l - 1, 0))$ or $({}^t\hat{g}_+)^l - 1$ $(=({}^tg^l - 1, 0))$, according as $\mathscr{G} = \mathcal{O}_{\mathscr{U}}, \Theta_{\mathscr{U}}(-\log \mathscr{X})$ or $\Omega^1_{\mathscr{U}}(\log \mathscr{X})$. q.e.d.

Since $\varphi^{-1}(\boldsymbol{C}^\times) \simeq U' \times \boldsymbol{C}^\times$ $(\varphi^{-1}(\boldsymbol{C}^\times) \cap \mathscr{X} \simeq X' \times \boldsymbol{C}^\times)$, we see that $\Theta_{\mathscr{U}}(-\log \mathscr{X})_t \simeq \Theta_{U'}(-\log X') \oplus \mathcal{O}_{U'}$ and that $\Omega^1_{\mathscr{U}}(\log \mathscr{X})_t \simeq \Omega^1_{\mathscr{U}}(\log X') \oplus \mathcal{O}_{U'}$ for each $t \in \boldsymbol{C}^\times$. Hence by the upper semi-continuity [1, Theorem 4.12], we have

$$\dim H^p(U', \mathcal{O}_{U'}) \leqq \dim H^p(U_0, \mathcal{O}_{U_0}) ,$$

$$\dim H^p(U', \Omega^1_{U'}(\log X')) + \dim H^p(U', \mathcal{O}_{U'}) \leqq \dim H^p(U_0, \Omega^1_{\mathscr{U}}(\log \mathscr{X})_0)$$

and

$$\dim H^p(U', \Theta_{U'}(-\log X')) + \dim H^p(U', \mathcal{O}_{U'}) \leqq \dim H^p(U_0, \Theta_{\mathscr{U}}(-\log \mathscr{X})_0) .$$

On the other hand, by Lemma 4.3, $\dim H^p(g^{l\boldsymbol{Z}}, F) = \dim E_2^{p,0}(g^{l\boldsymbol{Z}}, \tilde{\mathscr{F}}) \leqq \dim H^p(U', \mathscr{F}')$, where $F = \boldsymbol{C}$, $N_{\boldsymbol{C}}$ or $N^*_{\boldsymbol{C}}$, and $\mathscr{F}' = \mathcal{O}_{U'}, \Theta_{U'}(-\log X')$ or $\Omega^1_{U'}(\log X')$, according as $\mathscr{F} = \mathcal{O}_{\tilde{U}}, \Theta_{\tilde{U}}(-\log \tilde{X})$ or $\Omega^1_{\tilde{U}}(\log \tilde{X})$. Hence by Lemmas 4.3, 4.4 and 4.5, we obtain the equalities $\dim E_2^{p,0}(g^{l\boldsymbol{Z}}, \tilde{\mathscr{F}}) = \dim H^p(U', \mathscr{F}')$, because $\dim(N_{\boldsymbol{C}})^{g^{l\boldsymbol{Z}}} = \dim \ker(g^l - 1) = \dim \operatorname{coker}(g^l - 1)$ and $\dim(N^*_{\boldsymbol{C}})^{{}^tg^{l\boldsymbol{Z}}} = \dim \ker({}^tg^l - 1) = \dim \operatorname{coker}({}^tg^l - 1)$. Therefore, we have $E_2^{p,q}(g^{l\boldsymbol{Z}}, \tilde{\mathscr{F}}) = 0$ for $q \geqq 1$. Then by the Hochschild-Serre exact sequence, we have $E_2^{p,q}(g^{\boldsymbol{Z}}, \tilde{\mathscr{F}}) = H^p(g^{\boldsymbol{Z}}, H^q(\tilde{U}, \tilde{\mathscr{F}})) = H^p(g^{\boldsymbol{Z}}/g^{l\boldsymbol{Z}}, H^q(\tilde{U}, \tilde{\mathscr{F}})^{g^{l\boldsymbol{Z}}}) = 0$ for $q \geqq 1$. Hence $H^p(U, \mathscr{F}) = E_2^{p,0}(g^{\boldsymbol{Z}} \tilde{\mathscr{F}})$. Thus we complete the proof of Proposition 4.1, by Lemmas 4.3 and 4.4.

PROPOSITION 4.6.

$$\dim H^i(U, \Omega^1_U) = \begin{cases} 0 & \textit{for} \quad i \neq 1 \\ s & \textit{for} \quad i = 1 , \end{cases}$$

where s is the number of the irreducible components of X.

PROOF. Let $\boldsymbol{R}_{\geqq 0}n_1 + \boldsymbol{R}_{\geqq 0}n_2 + \cdots + \boldsymbol{R}_{\geqq 0}n_r$ be an r-dimensional nonsingular cone in Σ and let $\{m_1, m_2, \cdots, m_r\}$ be the $\boldsymbol{Z}$-basis of N^* dual to

$\{n_1, n_2, \cdots, n_r\}$. Let $e(m)$ be the same as in the proof of Lemma 4.2 and let $\omega_j = de(m_j)/e(m_j)$ for $j = 1$ through r. Then $\{\omega_1, \omega_2, \cdots, \omega_r\}$ is a $\boldsymbol{C}$-basis of $H^0(\tilde{U}, \Omega^1_{\tilde{U}}(\log \tilde{X})) \simeq N^*_{\boldsymbol{C}}$. Here we note that ω_j has poles along $\mathrm{orb}(\boldsymbol{R}_{\geq 0}n_j)$ and does not have poles along $\mathrm{orb}(\boldsymbol{R}_{\geq 0}n_k)$ with $k \neq j$, because $\langle m_j, n_k\rangle = \delta_{jk}$. Hence any non-zero element of $H^0(U, \Omega^1_U(\log X)) \simeq H^0(\tilde{U}, \Omega^1_{\tilde{U}}(\log \tilde{X}))^{g^{\boldsymbol{Z}}}$ has poles along X. Thus we conclude that $H^0(U, \Omega^1_U) = 0$. Next, consider the long exact sequence of the cohomology groups arising from the short exact sequence

$$0 \to \Omega^1_U \to \Omega^1_U(\log X) \to \bigoplus_{k=1}^s \mathcal{O}_{\hat{X}_k} \to 0 ,$$

where $\hat{X}_k$ are the normalizations of the irreducible components X_k of $X = X_1 + X_2 + \cdots + X_s$. Since each $\hat{X}_k$ is a compact toric variety, we have $H^i(\hat{X}_k, \mathcal{O}_{\hat{X}_k}) = 0$ for $i > 0$ and $H^0(\hat{X}_k, \mathcal{O}_{\hat{X}_k}) \simeq \boldsymbol{C}$. Hence by Proposition 4.1, we get $H^i(U, \Omega^1_U) = 0$ for $i > 1$ and $\dim H^1(U, \Omega^1_U) = \dim H^0(U, \Omega^1_U) - \dim \ker({}^tg - 1) + s + \dim \mathrm{coker}({}^tg - 1) = s$. q.e.d.

When $r = 3$, we can determine the dimensions of $H^q(U, \Omega^p_U)$ for all p and q by the Serre duality. In particular, $\dim H^0(U, \Omega^1_U) = \dim H^3(U, \Omega^2_U) = 0$ and $\dim H^1(U, \mathcal{O}) = \dim H^2(U, \Omega^3_U) = 1$. Since $b_1(U) = b_5(U) = 1$, the maps $E_1^{0,1} \to E_1^{1,1}$ and $E_1^{2,2} \to E_1^{3,2}$ must be zero-maps and hence the spectral sequence $E_1^{p,q} = H^q(U, \Omega^p_U) \Rightarrow H^{p+q}(U, \boldsymbol{C})$ degenerates. Thus we have:

THEOREM 4.7. *When* $r = 3$, *U has the following Betti-numbers*: $b_0(U) = b_1(U) = b_5(U) = b_6(U) = 1$, $b_2(U) = b_4(U) = s$ *and* $b_3(U) = 0$. Hence *the Euler-Poincaré characteristic of U is* $\chi(U) = 2s$.

5. Deformations. We keep the notation and the assumption in the previous section. Let Θ_U be the tangent sheaf of U.

PROPOSITION 5.1. *Assume that the dual graph of* $X = X_1 + X_2 + \cdots + X_s$ *is a triangulation. Then* $H^i(U, \Theta_U) \simeq \bigoplus_{k=1}^s H^i(X_k, \mathcal{O}_{X_k}(X_k))$, *for* $i \geqq 2$, $H^0(U, \Theta_U) \simeq \ker(g - 1)$ *and there exists an exact sequence*

$$0 \to H^1(U, \Theta_U(-\log X)) \to H^1(U, \Theta_U) \to \bigoplus_{k=1}^s H^1(X_k, \mathcal{O}_{X_k}(X_k)) \to 0 .$$

PROOF. Consider the long exact sequence of cohomology groups arising from the short exact sequence of sheaves

$$0 \to \Theta_U(-\log X) \to \Theta_U \to \bigoplus_{k=1}^s \mathcal{O}_{X_k}(X_k) \to 0 .$$

Then by Proposition 4.1, it is sufficient to show that $H^0(X_k, \mathcal{O}_{X_k}(X_k)) = 0$, for each irreducible component X_k of X. Let Y be an irreducible component of $\tilde{X}$ such that the image $q(Y)$ of Y under the quotient map $q: \tilde{X} \to X$ is X_k. Then Y is the closure of the orbit $\mathrm{orb}(\boldsymbol{R}_{\geq 0}n)$ corresponding to a one-dimensional cone $\boldsymbol{R}_{\geq 0}n$ in Σ. Let $n_1, n_2, \cdots$ and n_t be the link of n in Σ, i.e., $\boldsymbol{R}_{\geq 0}n + \boldsymbol{R}_{\geq 0}n_i$ ($i = 1$ through t) are two-dimensional cones in Σ. Then the closures Y_i of the orbits $\operatorname{orb}(\boldsymbol{R}_{\geqq 0}n_i)$ are the irreducible components of $\tilde{X}$ with $Y_i \cap Y \neq \emptyset$. We easily see that $\dim H^0(Y, \mathcal{O}_Y(Y)) = \sharp\{m \in N^* \mid \langle m, n\rangle = -1,\ \langle m, n_i\rangle \geqq 0 \text{ for } 1 \leqq i \leqq t\}$. Suppose that there exists an element m in N^* such that $\langle m, n\rangle = -1$ and that $\langle m, n_i\rangle \geqq 0$. Then the convex hull of $\{n, n_1, \cdots, n_t\}$ contains the origin, a contradiction to the fact that $\{n, n_1, \cdots, n_t\} \subset |\Sigma| \setminus \{0\} \subset H(g)$. Therefore, $H^0(X_k, \mathcal{O}_{X_k}(X_k)) \simeq H^0(Y, \mathcal{O}_Y(Y)) = 0$. q.e.d.

Since the dimension of each irreducible component X_k of X is equal to $r-1$, we get $H^r(X_k, \mathcal{O}_{X_k}(X_k)) = 0$. Hence we have:

COROLLARY 5.2. $H^r(U, \Theta_U) = 0$.

COROLLARY 5.3. *When $r = 2$, i.e., U is a hyperbolic Inoue surface or a half Inoue surface, we have $\dim H^1(U, \Theta_U) = 2s$, where s is the number of the irreducible components of X.*

PROOF. Note that when $r = 2$, any g in $K(N)$ and any $g^{\boldsymbol{Z}}$-invariant r.p.p. decomposition Σ with $|\Sigma| = (H(g) \setminus L(g)) \cup \{0\}$ satisfy the conditions of Corollary 2.3. Since g has two real eigenvalues both of which are not equal to one, we have $H^1(U, \Theta_U(-\log X)) \simeq \operatorname{coker}(g-1) = 0$. On the other hand, by the Riemann-Roch Theorem, we have

$$\dim H^1(Y, \mathcal{O}_Y(Y)) = \dim H^0(Y, \mathcal{O}_Y(Y)) - 1 - \deg \mathcal{O}_Y(Y) = -1 - Y^2 ,$$

for each irreducible component Y of X, because Y is a rational curve with $Y^2 < 0$. Hence

$$\begin{aligned} \dim H^1(U, \Theta_U) &= \sum_{k=1}^{s} \dim H^1(X_k, \mathcal{O}_{X_k}(X_k)) \\ &= \sum_{k=1}^{s} (-1 - X_k^2) = -s - X^2 + 2s = 2s , \end{aligned}$$

because $-X^2 = s$, by Nakamura's duality [7]. q.e.d.

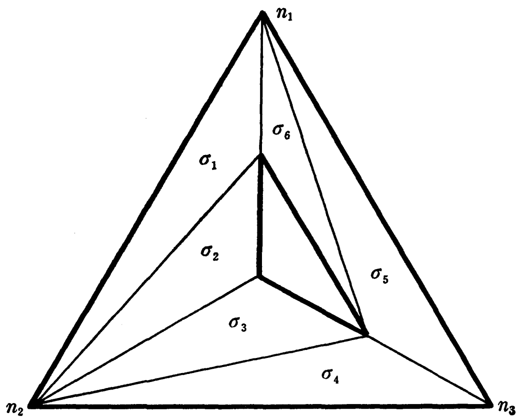
Since $H^2(U, \Theta_U(-\log X)) = 0$, there exists a universal family $\pi\colon (\mathscr{U}, \mathscr{X}) \to D$ of deformations for the pair $(U, X) \simeq (\pi^{-1}(0), \pi^{-1}(0) \cap \mathscr{X})$ over a polydisk D, i.e., the Kodaira-Spencer map $\rho\colon T_0(D) \to H^1(U, \Theta_U(-\log X))$ is bijective. In fact, we can construct such a family as follows. By Proposition 4.1 and Lemma 4.4, we have the canonical isomorphisms $H^1(U, \Theta_U(-\log X)) \simeq H^1(g^{\boldsymbol{Z}}, N_C) \simeq \operatorname{coker}(g-1)$. Here we note that $N_C = \ker(g-1) \oplus \operatorname{Im}(g-1)$. Let $\hat{g}^{\boldsymbol{Z}}$ be the automorphism group of $T\operatorname{emb}(\Sigma) \times \ker(g-1)$ generated by $\hat{g}\colon (x, t) \mapsto (e(t) \cdot gx, t)$, where $e\colon N_C \to T$ is the map induced by $\exp(2\pi\sqrt{-1}?)\colon \boldsymbol{C} \to \boldsymbol{C}^\times$. Then $\hat{g}^{\boldsymbol{Z}}$ preserve the open set $\tilde{U} \times \ker(g-1)$ and has no fixed point on it. Hence $\mathscr{U} := (\tilde{U} \times D)/\hat{g}^{\boldsymbol{Z}}$ is a complex manifold and the natural projection $\mathscr{U} \to D$ onto D is a proper smooth map, for a small enough polydisk D in $\ker(g-1)$.

6. Examples. We give five 3-dimensional examples and show a list of analytic invariants for them. Let $\{n_1, n_2, n_3\}$ be a $\boldsymbol{Z}$-basis of $\boldsymbol{Z}^3$.

EXAMPLE 1. $gn_1=2n_1+n_2+n_3$, $gn_2=n_1+n_2+n_3$ and $gn_3=n_1+n_2+2n_3$. $\Sigma=\{\text{faces of } g^l\sigma_i \mid l \in \boldsymbol{Z},\ i=1 \text{ through } 6\}$, where

$$\sigma_1 = \boldsymbol{R}_{\geqq 0}n_1 + \boldsymbol{R}_{\geqq 0}n_2 + \boldsymbol{R}_{\geqq 0}(2n_1 + n_2 + n_3)\ ,$$
$$\sigma_2 = \boldsymbol{R}_{\geqq 0}n_2 + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + n_3) + \boldsymbol{R}_{\geqq 0}(2n_1 + n_2 + n_3)\ ,$$
$$\sigma_3 = \boldsymbol{R}_{\geqq 0}n_2 + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + n_3) + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + 2n_3)\ ,$$
$$\sigma_4 = \boldsymbol{R}_{\geqq 0}n_2 + \boldsymbol{R}_{\geqq 0}n_3 + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + 2n_3)\ ,$$
$$\sigma_5 = \boldsymbol{R}_{\geqq 0}n_1 + \boldsymbol{R}_{\geqq 0}n_3 + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + 2n_3) \quad \text{and}$$
$$\sigma_6 = \boldsymbol{R}_{\geqq 0}n_1 + \boldsymbol{R}_{\geqq 0}(2n_1 + n_2 + n_3) + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + 2n_3)\ .$$

(See Figure 1.)



FJGURE 1

EXAMPLE 2. $gn_1=2n_1+n_2+n_3$, $gn_2=n_1+n_2+2n_3$ and $gn_3=n_1+n_2+n_3$. $\Sigma=\{\text{faces of } g^l\sigma_i \mid l \in \boldsymbol{Z},\ i=1 \text{ through } 6\}$, where σ_i are the same as in Example 1.

EXAMPLE 3. $gn_1=n_2$, $gn_2=n_1+n_3$ and $gn_3=n_1$. $\Sigma=\{\text{faces of } g^l\tau_1 \text{ and } g^l\tau_2 \mid l \in \boldsymbol{Z}\}$, where $\tau_1 = \boldsymbol{R}_{\geqq 0}n_3 + \boldsymbol{R}_{\geqq 0}(n_1 + n_3) + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + n_3)$ and $\tau_2 = \boldsymbol{R}_{\geqq 0}n_2 + \boldsymbol{R}_{\geqq 0}n_3 + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + n_3)$. (See Figure 2.)

EXAMPLE 4. $gn_1=n_1+n_3$, $gn_2=n_1$ and $gn_3=n_2$. $\Sigma=\{\text{faces of } g^l\tau_1 \text{ and } g^l\tau_2 \mid l \in \boldsymbol{Z}\}$, where τ_1 and τ_2 are the same as in Example 3.

EXAMPLE 5. $gn_1=n_1+n_2+n_3$, $gn_2=n_3$ and $gn_3=n_1$. $\Sigma=\{\text{faces of } g^l\mu_1 \text{ and } g^l\mu_2 \mid l \in \boldsymbol{Z}\}$, where $\mu_1 = \boldsymbol{R}_{\geqq 0}n_1 + \boldsymbol{R}_{\geqq 0}n_2 + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + n_3)$ and $\mu_2 =$

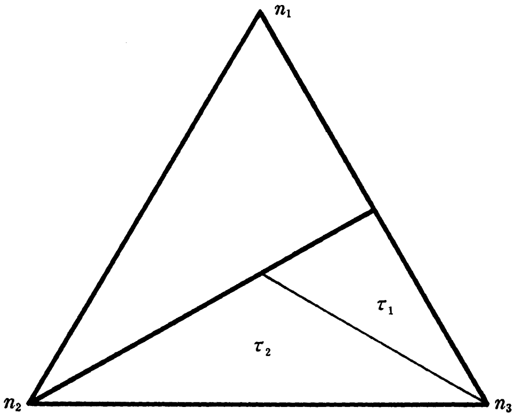


FIGURE 2

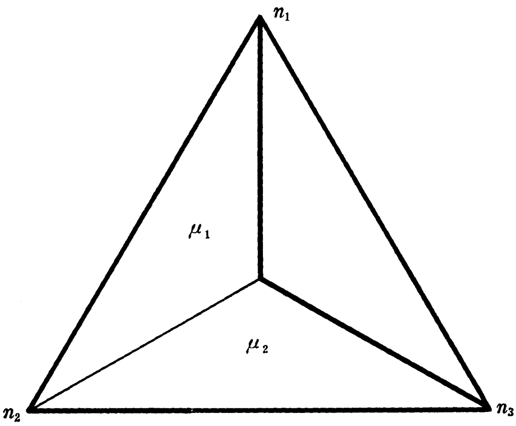


FIGURE 3

$\boldsymbol{R}_{\geqq 0}n_2 + \boldsymbol{R}_{\geqq 0}n_3 + \boldsymbol{R}_{\geqq 0}(n_1 + n_2 + n_3)$. (See Figure 3.)

We easily see that all the above examples satisfy the condition of Definition 1.1. The complex manifolds we obtain from g and Σ in Example 1 and Example 2 contain global spherical shells by Theorem 3.2. The complex manifold we obtain from g and Σ in Example 4 is bimeromorphic to that in [6]. Although our examples do not satisfy the assumptions of Theorem 5.1, we can calculate the dimensions $h^i(U, \Theta_U)$ of $H^i(U, \Theta_U)$ as follows. There are positive integers l such that the dual graphs of $X' := \tilde{X}/g^{l\boldsymbol{Z}}$ are triangulations. Then U and X are quotients of $U' := \tilde{U}/g^{l\boldsymbol{Z}}$ and X', respectively, by the finite cyclic groups $G = g^{\boldsymbol{Z}}/g^{l\boldsymbol{Z}}$, which have no fixed points on U'. Hence $\Theta_U/\Theta_U(-\log X)$ are the subsheaves of $q_*(\Theta_{U'}/\Theta_{U'}(-\log X')) \simeq q_*(\bigoplus_{k=1}^{sl} \mathcal{O}_{X'_k}(X'_k))$ consisting of germs of G-invariant sections, where $X' = X'_1 + X'_2 + \cdots + X'_{sl}$ and $q: X' \to X$ is the quotient map. Therefore, $\dim H^i(U, \Theta_U/\Theta_U(-\log X)) = \dim(\bigoplus_{k=1}^{sl} H^i(X'_k, \mathcal{O}_{X'_k}(X'_k)))^G = (1/l)\sum_{k=1}^{sl} \dim H^i(X'_k, \mathcal{O}_{X'_k}(X'_k))$. Then by Proposition 4.1, Theorem 5.1 and its proof, we have $h^0(U, \Theta_U) = \dim\ker(g-1)$, $h^1(U, \Theta_U) = \dim\operatorname{coker}(g-1) + (1/l)\sum_{k=1}^{sl} \dim H^1(X'_k, \mathcal{O}_{X'_k}(X'_k))$ and $h_i(U, \Theta_U) = (1/l)\sum_{k=1}^{sl} \dim H^i(X'_k, \mathcal{O}_{X'_k}(X'_k))$, for $i \geqq 2$.

	$h^0(U, \Theta)$	$h^1(U, \Theta)$	$h^2(U, \Theta)$	$C_3 = \chi(U)$	$C_1^3 = X^3$
Example 1	1	8	1	6	−18
Example 2	0	7	1	6	−18
Example 3	0	0	0	2	−2
Example 4	0	1	0	2	−4
Example 5	0	3	0	2	−8

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