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GEVREY HYPOELLIPTICITY OF A CLASS OF PSEUDODIFFERENTIAL OPERATORS

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

TADATO MATSUZAWA

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Introduction. We consider a pseudodifferential equation:

$$(1) \qquad a(x, D)u = f$$

with data f having Gevrey index $s \geqq 1$. Here $a(x, D)$ is a pseudodifferential operator of type $S^m_{\rho,\delta}$ of Hörmander (cf. [4]). We are interested in the Gevrey regularity of solutions, more precisely, in which way the Gevrey index of solutions depends on ρ, δ and s.

In [3], we have given the definition of a class of hypoelliptic pseudodifferential operators of symbol class $S^m_{\rho,\delta,\sigma}(\Omega\times R^n)$, $\Omega\subset R^n$, $0\leqq\delta<\rho\leqq 1$, $\sigma\geqq 1$, which consists of symbols $a(x,\xi)\in S^m_{\rho,\delta}(\Omega\times R^n)$ satisfying

$$(2) \qquad |a(x, \xi)| \geqq c|\xi|^{m'}, \quad |\xi| \geqq B, \ -\infty < m' < \infty ,$$

$$(3) \qquad |a^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_0 C_1^{|\alpha+\beta|}\alpha!\ \beta!^{\sigma}|a(x, \xi)|(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|} ,$$
$$x\in\Omega,\ |\xi| \geqq B|\alpha|^{\theta},\ \theta = \sigma/(\rho-\delta) .$$

Under these conditions, we have constructed a parametrix b of $a(x, D)$ with symbol $b(x,\xi)\in S^{-m'}_{\rho,\delta,\sigma}(\Omega\times R^n)$. Here b is expressed by an infinite series of symbols, and the remainder $r = ba - I$ is an integral operator with a kernel of Gevrey function of index $\theta=\sigma/(\rho-\delta)$ (cf. Theorem 3.1 and Corollary 3.1 of [3]). Thus we have $\max(\sigma/(\rho-\delta), s)$ as the Gevrey index for solutions of the equation (1). This gives the best possible index when $\rho\equiv 1$, $0\leqq\delta<1$ as was shown by several examples in [3], but not necessarily the best possible when $0<\rho<1$.

It seems impossible to apply directly the method of [3] to obtain sharper results if $0<\rho<1$. We use a finite approximation of parametrix instead of infinite approximation used in [3]. The remainder term is not necessarily smooth, so we are forced to estimate all derivatives of solutions inductively. This method seems unusual in the study of hypoellipticity because it looks tedious. However, surprisingly this method provides a sharper result for Gevrey hypoellipticity. For the nonlinear problem such a method was used by Friedman [2] to get analyticity of solutions of elliptic and parabolic systems and by Volevič [16] for a class of pseudodifferential equations.

Now we would like to summarize the basic idea of this paper partly motivated by [14] without going into technical details. We consider the equation (1) with $a(x, \xi)$ satisfying

$$(2) \qquad |a(x, \xi)| \geqq c|\xi|^{m'}, \quad |\xi| \geqq B,$$

$$(3') \qquad |a_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_0 C_1^{|\alpha+\beta|}\alpha!\, B!^{\alpha}|a(x, \xi)|(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|}, \quad x \in \Omega, \quad |\xi| \geqq B.$$

The condition (3′) is slightly stronger than (3) but enough for applications. We first construct a left parametrix $b^N(x, D)$ of $a(x, D)$ consisting of a finite number of pseudodifferential operators. Then we reduce the equation (1) into an integral equation.

$$u = b^N f - r^N u, \quad (b^N a u = b^N f = u + r^N u).$$

By induction on $k = |\alpha|$, $k = 0, 1, \cdots$, we obtain successive estimates of type

$$(4) \qquad \sup_{x \in K \subset\subset \Omega} |D^{\alpha}u| \leqq C_0 C_1^{|\alpha|}\alpha!^{\max(s,\theta)}, \quad \theta = \max(1/\rho, \sigma/(1-\delta)),$$

(cf. Theorem 3.1).

Since we have $\sigma/(\rho-\delta) > \max(1/\rho, \sigma/(1-\delta))$ for $0 < \delta < \rho < 1$ and $\sigma \geqq 1$, this improves our previous result of [3].

The plan of the paper is as follows. In §1, we start with the precise definition of pseudodifferential operators considered in this paper. The regularity properties of their kernels and the pseudolocal property will be studied. In §2, we shall consider the symbolic calculus of a composed operator. In §3, we shall prove the main result (Theorem 3.1) on Gevrey hypoellipticity of pseudodifferential equations. In §4, we shall give some examples of differential operators.

Finally we remark that the same problem has been investigated in [7] and [14] recently. In [7], similar results have been obtained by constructing parametrices for a class of degenerate parabolic pseudodifferential operators, and in [14] by applying the theory of multiple products of pseudodifferential operators. Compared with these results our proof given here would be significantly elementary.

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1. A class of pseudodifferential operators. Let Ω be an open subset of R^n whose point is denoted by $x = (x_1, \cdots, x_n)$. We use general notation such as $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = 1/i\partial/\partial x_j$, $j = 1, \cdots, n$, etc.

DEFINITION 1.1. Let $u \in C^\infty(\Omega)$. Then we say that u is in G^s in Ω $(s \geqq 1)$ if for any compact set K of Ω there are positive constants C_0 and C_1 such that

$$\sup_{x \in K} |D^\alpha u(x)| \leqq C_0 C_1^{|\alpha|} |\alpha|^{s|\alpha|} , \quad \alpha \in Z_+^n . \tag{1.1}$$

DEFINITION 1.2. Let $-\infty < m < \infty$; $0 \leqq \delta < \rho \leqq 1$; $\sigma \geqq 1$. We denote by $S^m_{\rho,\delta,\sigma}(\Omega \times R^n)$ the set of all $a(x, \xi) \in C^\infty(\Omega \times R^n)$ such that for every compact set K of Ω there are positive constants C_0, C_1 and B such that

$$\sup_{x \in K} |a^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_0 C_1^{|\alpha+\beta|} \alpha!\beta!^\sigma |\xi|^{m-\rho|\alpha|+\delta|\beta|} , \quad |\xi| \geqq B , \tag{1.2}$$

where $a^{(\alpha)}_{(\beta)}(x, \xi) = \partial_\xi^\alpha D_x^\beta a(x, \xi)$, $\partial_\xi = (\partial/\partial\xi_1, \cdots, \partial/\partial\xi_n)$.

We associate with such a symbol $a(x, \xi)$ a pseudodifferential operator as usual:

$$a(x, D)u(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} a(x, \xi) u(y) dy d\xi , \quad u \in C_0^\infty(\Omega) .$$

Let $K(x, y) \in \mathscr{D}'(\Omega \times \Omega)$ be the distribution kernel of $a(x, D)$ expressed by the oscillatory integral:

$$K(x, y) = (2\pi)^{-n} \int e^{i\langle x-y,\xi\rangle} a(x, \xi) d\xi .$$

The following theorems strengthen Theorem 1.1 in [3].

THEOREM 1.1. *Let* $a(x, \xi) \in S^m_{\rho,\delta,\sigma}(\Omega \times R^n)$. *Then we have*

$$K(x, y) \in G^{\theta_0; 1/\rho}_{x,y}(\Omega \times \Omega \setminus \Delta) , \quad \Delta = \{(x, x); x \in \Omega\} , \quad \theta_0 = \max(1/\rho, \sigma + \delta/\rho) .$$

THEOREM 1.2. *If* $u \in \mathscr{E}'(\Omega)$ *which is in* G^s $(s \geqq 1)$ *in a neighborhood of* $x_0 \in \Omega$, *then we have* $a(x, D)u \in G^{\theta_1}$ *in the same neighborhood of* $x_0 \in \Omega$, *where* $\theta_1 = \max(s, \sigma + s\delta, 1/\rho, \sigma + \delta/\rho)$. *More precisely, we have*

(1) $\theta_1 = \max(1/\rho, \sigma + \delta/\rho)$ *if* $1 \leqq s \leqq \min(1/\rho, \sigma/(1-\delta))$;

(2) $\theta_1 = \sigma + s\delta$ *if* $1/\rho \leqq s \leqq \sigma/(1-\delta)$;

(3) $\theta_1 = 1/\rho$ *if* $\sigma/(1-\delta) \leqq s \leqq 1/\sigma$;

(4) $\theta_1 = s$ *if* $s \geqq \max(1/\rho, \sigma/(1-\delta))$.

PROOF OF THEOREM 1.1. Let U be any compact set of $\Omega \times \Omega \setminus \Delta$. For each $(x, y) \in U$ and $\alpha, \beta \in Z_+^n$, we have in the sense of oscillatory integral:

$$D_x^\alpha D_y^\beta K(x, y) = (2\pi)^{-n} \sum_{\gamma+\tau=\alpha} \frac{\alpha!}{\gamma!\tau!} \int e^{i\langle x-y,\xi\rangle} \xi^\gamma (-\xi)^\beta a_{(\tau)}(x, \xi) d\xi .$$

We have the estimate

$$\left| \int_{|\xi| \leqq B} e^{i\langle x-y,\xi\rangle} \xi^\gamma (-\xi)^\beta a_{(\tau)}(x, \xi) d\xi \right| \leqq C_0 C_1^{|\alpha+\beta|} \tau!^\sigma \tag{1.3}$$

with constants C_0 and C_1 independent of α, β and $\tau \leqq \alpha$. Next, by setting

$$N = [(|\gamma + \beta| + \delta|\tau| + m_+ + n + 2)/\rho] , \quad m_+ = \max(m, 0) ,$$

we have for a fixed i, $1 \leqq i \leqq n$,

$$\begin{aligned} (x_i - y_i)^N & \int_{|\xi| \geqq B} e^{i\langle x-y,\xi\rangle} \xi^\gamma (-\xi)^\beta a_{(\tau)}(x, \xi) d\xi \\ &= \int_{|\xi| \geqq B} e^{i\langle x-y,\xi\rangle} D_{\xi_i}^N \{\xi^\gamma (-\xi)^\beta a_{(\tau)}(x, \xi)\} d\xi \\ &\quad - \sum_{k=0}^{N-1} (x_i - y_i)^{N-1-k} \int_{|\xi|=B} e^{i\langle x-y,\xi\rangle} D_{\xi_i}^k \{\xi^\gamma (-\xi)^\beta a_{(\tau)}(x, \xi)\} dS_\xi \\ &\equiv G_1(x, y) - G_2(x, y) . \end{aligned}$$

By the hypothesis (1.2) the integrand of G_1 is estimated by

$$\begin{aligned} & \left| \sum_{k=0}^{\gamma_i+\beta_i} \frac{N!(\gamma_i + \beta_i)!}{k!(N-k)!\,(\gamma_i + \beta_i - k)!} \xi^{\gamma+\beta} \xi_i^{-k} D_{\xi_i}^{N-k} a_{(\tau)}(x, \xi) \right| \\ & \qquad \leqq C_0 C_1^{|\tau|+N} (\tau!)^\sigma N! \sum_{k=0}^{\gamma_i+\beta_i} \binom{\gamma_i + \beta_i}{k} |\xi|^{|\gamma+\beta|-k} |\xi|^{m_+ - \rho(N-k) + \delta|\tau|} . \end{aligned} \tag{1.4}$$

Taking another couple of constants C_0 and C_1 we can estimate this by

$$C_0 C_1^{|\alpha+\beta|} \tau!^\sigma N! (1 + |\xi|)^{-n-1} .$$

By the definition of the number N, we have

$$\tau!^\sigma N! \leqq C^{|\alpha+\beta|} |\beta|^{(|\beta|/\rho)} |\gamma|^{(|\gamma|/\rho)} |\tau|^{(\sigma+\delta/\rho)|\tau|}$$

with a constant C independent of α, β and $\tau \leqq \alpha$. Hence we have an estimate of type

$$|G_1(x, y)| \leqq C_0 C_1^{|\alpha+\beta|} |\alpha|^{\theta_0|\alpha|} |\beta|^{(|\beta|/\rho)} , \quad \theta_0 = \max(1/\rho, \sigma + \delta/\rho) . \tag{1.5}$$

Similar estimate holds for $G_2(x, y)$. Since U is a compact set of $\Omega \times \Omega \setminus \Delta$, one can find a direction i, $1 \leqq i \leqq n$, such that $|x_i - y_i| \geqq d > 0$ for any $(x, y) \in U$ and finally we have the estimate

$$\sup_{(x,y)\in U} |D_x^\alpha D_y^\beta K(x, y)| \leqq C_0 C_1^{|\alpha+\beta|} |\alpha|^{\theta_0|\alpha|} |\beta|^{(|\beta|/\rho)} , \quad \alpha, \beta \in Z_+^n , \tag{1.6}$$

where the constants C_0 and C_1 are independent of α and β.

PROOF OF THEOREM 1.2. We first remark that for $f\in C_0^\infty(\Omega')$, Ω' is a relatively compact open subset of Ω, and we have

$$(1.7)\qquad |D_x^\alpha\{a(x, D)f(x)\}| \leqq \sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}C^{|\tau|+1}(\tau!)^\sigma \mathrm{Vol}\{\mathrm{supp}\, f\} \times \sup_{x\in\Omega}|D_x^\gamma(1-\Delta)^N f(x)|\,,$$

where the constant C depends only on Ω' and $N=N(\tau)=[(\delta|\tau|+m_+ + n+2)/2]$. Indeed, we have

$$D_x^\alpha a(x, D)f(x) = (2\pi)^{-n}\sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}\iint e^{i\langle x-y,\xi\rangle}\xi^\gamma a_{(\tau)}(x,\xi)f(y)dyd\xi$$
$$= (2\pi)^{-n}\sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}\iint e^{i\langle x-y,\xi\rangle}(1+|\xi|^2)^{-N}a_{(\tau)}(x,\xi)(1-\Delta_y)^N D_y^\gamma f(y)dyd\xi$$

which gives the estimate (1.7).

Now we take $u\in\mathscr{E}'(\Omega)$ which is in G^s in a bounded neighborhood V of $x_0\in\Omega$. Let U be a neighborhood of x_0 such that $\bar U\subset V$. There is a positive number d such that $0<d<\mathrm{dis}(U, R^n\setminus V)$. Let $\{g_l\}_{l=0}^\infty$ be a series of functions in $C_0^\infty(V)$ such that $g_l(x)=1$ on $\{x; \mathrm{dis}(x, U)<d\}$ and $|D_x^\alpha g_l(x)|\leqq C^l\alpha!$ if $|\alpha|\leqq l$, where the constant C is independent of l (cf. [5]). Then we have for $x\in U$

$$(1.8)\qquad D_x^\alpha\{a(x, D)u(x)\} = D_x^\alpha\{a(x, D)g_l u(x)\} + D_x^\alpha\int K(x,y)\{1-g_l(y)\}u(y)dy\,.$$

By using (1.7) we have

$$|D_x^\alpha a(x, D)g_l u(x)| \leqq \sum_{\gamma+\tau=\alpha}\binom{\alpha}{\gamma}C^{|\tau|+1}(\tau!)^\sigma \sup_{x\in\Omega}|D_x^\gamma(1-\Delta)^N\{g_l(x)u(x)\}|\,,$$

where $N=[(\delta|\tau|+m_+ +n+2)/2]$. Taking $l=2|\alpha|$ we have

$$|D_x^\alpha a(x, D)g_l u(x)| \leqq C_1^{|\alpha|+1}\sum_{\gamma+\tau=\alpha}|\tau|^{(\sigma+s\delta)|\tau|}|\gamma|^{s|\gamma|} \leqq C_2^{|\alpha|+1}|\alpha|^{\max(s,\sigma+s\delta)|\alpha|}\,.$$

By Theorem 1.1, the last term of (1.8) is in G^{θ_0} in U, $\theta_0=\max(1/\rho, \sigma+\delta/\rho)$. Thus $a(x, D)u$ is in G^{θ_1} in U, where $\theta_1=\max(s, \sigma+s\delta, 1/\rho, \sigma+\delta/\rho)$. We only verify the case (4), the other cases being treated similarly. Namely, we assume $s\geqq\max(1/\rho, \sigma/(1-\delta))$. Then we have $s\geqq\sigma+s\delta$ and $s\geqq\sigma+\delta/\rho$, which proves the assertion (4).

2. **Symbolic calculus.** Let $a(x,\xi)\in S^{m'}_{\rho,\delta,\sigma}(\Omega\times R^n)$ and $b(x,\xi)\in S^{m''}_{\rho,\delta,\sigma}(\Omega\times R^n)$. Let Ω' be a relatively compact open subset of Ω and take $h\in C_0^\infty(\Omega)$ so that $h=1$ on a neighborhood $\tilde U$ of $\bar\Omega'$. Then the symbol of the operator $r(x, D)=a(x, D)hb(x, D)$ is given by

$$(2.1)\qquad r(x,\xi)=a(x,D+\xi)h(x)b(x,\xi)$$
$$=(2\pi)^{-n}\int_{R^n_\eta}\int_\Omega e^{i\langle x-y,\eta\rangle}a(x,\xi+\eta)h(y)b(y,\xi)dyd\eta\ .$$

We set

$$r^N(x,\xi)=\sum_{|\alpha|\leqq N}\frac{1}{\alpha!}a^{(\alpha)}(x,\xi)b_{(\alpha)}(x,\xi)\ ,\quad N=0,1,\cdots .$$

Then we easily see $r^N(x,\xi)\in S^m_{\rho,\delta,\sigma}(\Omega\times R^n)$, $m=m'+m''$, $N=0,1,\cdots$.

THEOREM 2.1. *We have*

$$r(x,D)=r^N(x,D)+F^N(x,D)\quad in\quad \Omega'\ ,$$

where $F^N(x,D)$ can be written as a sum of two operators, $F^N(x,D)=F_1^N+F_2^N$. F_1^N is an integral operator from $C^\infty(\bar\Omega')$ into $G^\theta(\Omega')$ with kernel $F_1^N(x,y)$, $(x,y)\in(\Omega'_x\times\Omega'_y)$, $\theta=\max(1/\rho,\sigma/(1-\delta))$. F_2^N is a pseudo-differential operator with symbol $F_2^N(x,\xi)$ satisfying the condition

$$(\mathrm{F})\qquad |D_x^\beta\partial_\xi^\gamma F_2^N(x,\xi)|\leqq C_0C_1^{N+|\beta+\gamma|}N!^\sigma\gamma!\beta!^\sigma|\xi|^{m_++n-(\rho-\delta)N-\rho|\gamma|}$$
$$\times\sum_{\tau\leqq\beta}\binom{\beta}{\tau}|\tau|^{\sigma\delta|\tau|}|\xi|^{\delta|\beta-\tau|+\delta^2|\tau|}$$
$$m_+=\max(m,0),\ x\in\Omega',\ |\xi|\leqq B\quad(\text{cf. }(1.3))\ .$$

More precisely, we can write

$$D_x^\beta F_2^N(x,\xi)=\sum_{\tau\leqq\beta}\binom{\beta}{\tau}u^\tau(x,\xi)\ ,$$

where each $u^\tau(x,\xi)$ satisfies an estimate of type

$$(\mathrm{F}')\qquad |\partial_\xi^\gamma u^\tau(x,\xi)|\leqq C_0C_1^{N+|\gamma+\beta|}N!^\sigma\gamma!\beta!^\sigma\tau!^{\sigma\delta}|\xi|^{m_++n-(\rho-\delta)N-\rho|\gamma|+\delta|\beta-\tau|+\delta^2|\tau|}\ ,$$
$$x\in\Omega'\ ,\quad |\xi|\geqq B\ .$$

PROOF. First we choose cut-off functions $h_l(x)\in C_0^\infty(\Omega)$, $=1$ on the neighborhood $\tilde U$ of $\bar\Omega'$, with support in a fixed compact set. Moreover, we assume

$$(2.2)\qquad |D^\alpha h_l(x)|\leqq C^l\alpha!\ ,\quad |\alpha|\leqq l+m'_++n+2\ ,\quad l=0,1,\cdots ,$$

where the constant C is independent of l. We express $F(x,\xi)=r(x,\xi)-r^N(x,\xi)$ by

$$(2.3)\qquad F^N(x,\xi)=r(x,\xi)-r_l(x,\xi)+r_l(x,\xi)-r_l^N(x,\xi)+r_l^N(x,\xi)-r^N(x,\xi)$$
$$=r_l^1(x,\xi)+r_l^2(x,\xi)+r_l^3(x,\xi)\ ,$$

where $r_l(x,\xi)=a(x,D+\xi)h_l(x)b(x,\xi)$ and

$$r_l^N(x, \xi) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!} a^{(\alpha)}(x, \xi) D_x^\alpha\{h_l(x) b(x, \xi)\} .$$

We note that $r_l^3(X, \xi) \equiv 0$ in $\Omega' \times R^n$. Let $K_a(x, y)$ and $K_b(x, y) \in \mathscr{D}'(\Omega \times \Omega)$ be the distribution kernels of $a(x, D)$ and $b(x, D)$, respectively. Then for $u \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} r_l^1(x, D)u(x) &= \int K_a(x, z)\{h(z) - h_l(z)\}\left\{\int K_b(z, y) u(y) dy\right\} dz \\ &= \int\left(\int K_a(x, z)\{h(z) - h_l(z)\} K_b(z, y) dz\right) u(y) dy . \end{aligned}$$

Hence the kernel of $r_l^1(x, D)$ is given by

$$K_l^1(x, y) = \int_\Omega K_a(x, z)\{h(z) - h_l(z)\} K_b(z, y) dz . \tag{2.4}$$

This is in $G_{x,y}^{\theta_0, 1/\rho}$ in $\Omega' \times \Omega'$ uniformly with respect to h_l, $l = 0, 1, \cdots$, (cf. Theorem 1.1), and so we shall use this as if it is not depending on l.

Next we observe $r_l^2(x, \xi)$. We have for $x, \xi \in \Omega' \times R^n$

$$r_l^2(x, \xi) = (2\pi)^{-n} \iint e^{i\langle x-y, \eta\rangle}\Big[a(x, \xi + \eta) - \sum_{|\alpha|\leq N} \frac{1}{\alpha!} a^{(\alpha)}(x, \xi)\eta^\alpha\Big] u(y, \xi) dy d\eta ,$$

where we have written $u(y, \xi) = h_l(y) b(y, \xi)$. We shall need the following cut-off functions $\chi_j(\xi) \in C_0^\infty(R^n)$, $j = 0, 1, \cdots$, such that $\chi_j(\xi) = 1$ for $|\xi| \leq 1/4$, $\chi_j(\xi) = 0$ for $|\xi| \geq 1/2$ and $|\chi_j^{(\alpha)}(\xi)| \leq C^j \alpha!$ for $|\alpha| \leq j + 1$, where the constant C is independent of j, $j = 0, 1, \cdots$. By using $\chi_j(\xi)$ we divide $r_l^2(x, \xi)$ into four parts:

$$\begin{aligned} r_l^2(x, \xi) = &(2\pi)^{-n} \iint e^{i\langle x-y, \eta\rangle}\Big[a(x, \xi+\eta) - \sum_{|\alpha|\leq N} \frac{1}{\alpha!} a^{(\alpha)}(x, \xi)\eta^\alpha\Big] \chi_j\Big(\frac{\eta}{|\xi|}\Big) u(y, \xi) dy d\eta \\ &+ (2\pi)^{-n} \sum_{|\alpha|\leq N} \frac{1}{\alpha!} a^{(\alpha)}(x, \xi) \iint e^{i\langle x-y, \eta\rangle}\eta^\alpha\Big[\chi_j\Big(\frac{\eta}{|\xi|}\Big) - 1\Big] u(y, \xi) dy d\eta \\ &+ (2\pi)^{-n} \iint e^{i\langle x-y, \eta\rangle} a(x, \xi + \eta)\Big[1 - \chi_j\Big(\frac{\eta}{|\xi|}\Big)\Big]\Big[1 - \chi_j\Big(\frac{\xi+\eta}{|\xi|}\Big)\Big] u(y, \xi) dy d\eta \\ &+ (2\pi)^{-n} \iint e^{i\langle x-y, \eta\rangle} a(x, \xi+\eta)\Big[1 - \chi_j\Big(\frac{\eta}{|\xi|}\Big)\Big] \chi_j\Big(\frac{\xi+\eta}{|\xi|}\Big) u(y, \xi) dy d\eta \\ = &I_1(x, \xi) + I_2(x, \xi) + I_3(x, \xi) + I_4(x, \xi) . \end{aligned}$$

Concerning $I_1(x, \xi)$, we have

$$\begin{aligned} D_x^\beta \partial_\xi^\gamma I_1(x, \xi) = &(2\pi)^{-n} \sum_{\substack{\tau\leq\beta \\ \mu\leq\gamma}} \binom{\beta}{\tau}\binom{\gamma}{\mu} \iint e^{i\langle x-y, \eta\rangle} \sum_{|\alpha|=N+1} \frac{N+1}{\alpha!} \int_0^1 (1-t)^N \\ &\times a_{(\tau)}^{(\alpha+\mu)}(x, \xi + t\eta) dt D_y^{\alpha+\beta-\tau} \partial_\xi^{\gamma-\mu}\Big[\chi_j\Big(\frac{\eta}{|\xi|}\Big) u(y, \xi)\Big] dy d\xi . \end{aligned}$$

We have $|\xi|/2 \geqq |\xi + t\eta| \leqq 3|\xi|/2$ and $|\eta| \leqq |\xi|/2$ when $\chi_j(\eta/|\xi|) \neq 0$ and $0 \leqq t \leqq 1$. By using this and taking $j = N + |\gamma| + 1$ and $l = N + |\beta|$, we have an estimate of type

(2.5) $$|D_x^\beta \partial_\xi^\gamma I_1(x, \xi)| \leqq C_0 C_1^{N+|\beta+\gamma|} N!\gamma!\beta!^\sigma |\xi|^{m+n-(\rho-\delta)N-\rho|\gamma|+\delta|\beta|}, \quad x \in \Omega', \quad |\xi| \geqq B.$$

Next we consider $I_2(x, \xi)$. We have

$$D_x^\beta \partial_\xi^\gamma I_2 = \sum_{|\alpha|\leqq N} \sum_{\substack{\tau\leqq\beta \\ \mu\leqq\gamma}} \binom{\beta}{\tau}\binom{\gamma}{\mu} \frac{1}{\alpha!} a_{(\tau)}^{(\alpha+\mu)}(x, \xi)\Big[\int u_{(\alpha+\beta-\tau)}^{(\gamma-\mu)}\Big(x - \frac{y}{|\xi|}, \xi\Big)\boldsymbol{\chi}_j(y)dy - u_{(\alpha+\beta-\tau)}^{(\gamma-\mu)}(x, \xi)\Big]$$
$$+ \sum_{|\alpha|\leqq N} \sum_{\substack{\tau\leqq\beta \\ \mu\leqq\gamma}} \binom{\beta}{\tau}\binom{\gamma}{\mu} \frac{1}{\alpha!} a_{(\tau)}^{(\alpha+\mu)}(x, \xi) \sum_{\substack{\lambda\leqq\gamma-\mu \\ \lambda\neq 0}} \binom{\gamma-\mu}{\lambda} \iint e^{i\langle x-y, \eta\rangle} u_{(\alpha+\beta-\lambda)}^{(\gamma-\mu-\lambda)}(y, \xi) \times \partial_\xi^\lambda \chi_j\Big(\frac{\eta}{|\xi|}\Big) dy d\eta,$$

where

$$\boldsymbol{\chi}_j(x) = (2\pi)^{-n}\int e^{i\langle x, \eta\rangle}\chi_j(\eta)d\eta.$$

Noting that $\int \boldsymbol{\chi}_j(x)dx = \chi_j(0) = 1$ and that the other moment of χ_j is equal to zero, we see that the first sum on the right hand side is equal to

$$\sum_{|\alpha|\leqq N} \sum_{\substack{\tau\leqq\beta \\ \mu\leqq\gamma}} \binom{\beta}{\tau}\binom{\gamma}{\mu} \frac{1}{\alpha!} a_{(\tau)}^{(\alpha+\mu)}(x, \xi) \sum_{|\kappa|=N+1-|\alpha|} \frac{N+1-\alpha}{\kappa!} \int (y/|\xi|)^\kappa \times u_{(\alpha+\beta-\tau+\kappa)}^{(\gamma-\mu)}(x - ty/|\xi|, \xi)(1-t)^{N-|\alpha|+1} dt \boldsymbol{\chi}_j(y)dy.$$

We have by the definition of χ_j, $j = N + |\gamma| + 1$,

$$\int |y^\alpha \boldsymbol{\chi}_j(y)| dy \leqq C^{N+|\gamma|+1}\alpha!, \quad |\alpha| \leqq N + 1.$$

We recall that we have taken $l = N + |\beta|$. Then we have $|\alpha+\beta-\tau+\kappa| \leqq N + |\beta| + 1$ and noting that $\operatorname{supp} \partial_\xi^\lambda \chi_j(\eta/|\xi|) \subset \{|\xi|/4 \leqq |\eta| \leqq |\xi|/2\}$, $\lambda \neq 0$, we have finally

(2.6) $$|D_x^\beta \partial_\xi^\gamma I_2| \leqq C_0 C_1^{N+|\alpha+\beta|} N!^\sigma \gamma!\beta!^\sigma |\xi|^{m-(\rho-\delta)N-\rho|\gamma|+\delta|\beta|}, \quad x \in \Omega', \quad |\xi| \geqq B,$$

where the constants C_0 and C_1 are independent of N, β and γ.

Now we consider $I_3(x, \xi)$. We have

$$D_x^\beta \partial_\xi^\gamma I_3(x, \xi) = (2\pi)^{-n} \sum_{\substack{\tau\leqq\beta \\ \mu\leqq\gamma}} \binom{\beta}{\tau}\binom{\gamma}{\mu} \iint e^{i\langle x-y, \eta\rangle} a_{(\tau)}^{(\mu)}(x, \xi+\eta) \times \partial_\xi^{\gamma-\mu}\Big\{\Big[1 - \chi_j\Big(\frac{\eta}{|\xi|}\Big)\Big]\cdot\Big[1 - \chi_j\Big(\frac{\xi+\eta}{\xi}\Big)\Big] u_{(\beta-\tau)}(y, \xi)\Big\} dy d\eta$$

$$= (2\pi)^{-n} \sum_{\substack{\tau \leqq \beta \\ \mu \leqq \gamma \\ \lambda \leqq \gamma - \rho}} \binom{\beta}{\tau}\binom{\gamma}{\mu}\binom{\gamma-\mu}{\lambda} \sum_{\substack{|\alpha|=N+m_+ +n+2 \\ =: N'}} \frac{N'!}{\alpha!} \int H_{\alpha,\mu,\tau,\lambda}(x, x-y, \xi)$$

$$\times (-\Delta_y)^{[\delta|\tau|/2]} u^{(\lambda)}_{(\alpha+\beta-\tau)}(y, \xi) dy \ ,$$

where

$$H_{\alpha,\mu,\tau,\lambda}(x, x-y, \xi) = \int e^{i\langle x-y,\eta\rangle} \partial_\xi^{\gamma-\mu-\lambda} \left\{ \left[1 - \chi_j\left(\frac{\eta}{|\xi|}\right)\right] \right.$$

$$\left. \times \left[1 - \chi_j\left(\frac{\xi+\eta}{|\xi|}\right)\right] a^{(\mu)}_{(\tau)}(x, \xi+\eta) \right\} \eta^\alpha |\eta|^{-2N'-2[\delta|\tau|/2]} d\eta \ .$$

Observing that the support of the integrand is in the domain $|\eta| \geqq |\xi|/4$ and $|\xi + \eta| \geqq |\xi|/4$, we have the estimate

$$|H_{\alpha,\mu,\tau,\lambda}(x, x-y, \xi)| \leqq C^{N+|\gamma-\lambda|+|\tau|+1} |\gamma - \lambda|! \cdot \tau!^\sigma |\xi|^{-N-\rho|\gamma-\lambda|} \ ,$$

where the constant C is independent of N, μ and τ. On the other hand, since we have taken $l = N + |\beta|$ and $\tau \leqq \beta$, $|\alpha| = N'$, we have an estimate of type

$$|(-\Delta_y)^{[\delta|\tau|/2]} u^{(\lambda)}_{(\alpha+\beta-\tau)}(y, \xi)| \leqq C_0 C_1^{N+|\lambda|+|\beta-\tau|} N!^\sigma \lambda! (\beta-\tau)!^\sigma$$

$$\times [\delta|\tau|]!^\sigma |\xi|^{m''-\rho|\lambda|+\delta(N+|\beta-\tau|)+\delta|\tau|)} \ .$$

From these estimates we have

$$|D_x^\beta \partial_\xi^\gamma I_3| \leqq C_0 C_1^{N+|\gamma+\beta|} N!^\sigma \gamma! \beta!^\sigma |\xi|^{m''-(1-\delta)N-\rho|\gamma|} \tag{2.7}$$

$$\times \sum_{\tau \leqq \beta} \binom{\beta}{\tau} [\delta|\tau|]!^\sigma |\xi|^{\delta|\beta-\tau|+\delta^2|\tau|} \ .$$

Finally we shall consider $I_4(x, \xi)$. We rewrite

$$I_4(x, \xi) = (2\pi)^{-n} \iint e^{i\langle x-y,\eta\rangle} a(x, \xi+\eta)[h_l(y) - h_k(y)] b(y, \xi) dy d\eta$$

$$- (2\pi)^{-n} \iint e^{i\langle x-y,\eta\rangle} a(x, \xi+\eta)[h_l(y) - h_k(y)] b(y, \xi) \chi_j\left(\frac{\eta}{|\xi|}\right) dy d\eta$$

$$- (2\pi)^{-n} \iint e^{i\langle x-y,\eta\rangle} a(x, \xi+\eta)[h_l(y) - h_k(y)] b(y, \xi)$$

$$\times \left[1 - \chi_j\left(\frac{\eta}{|\xi|}\right)\right]\left[1 - \chi_j\left(\frac{\xi+\eta}{|\xi|}\right)\right] dy d\eta$$

$$+ (2\pi)^{-n} \iint e^{i\langle x-y,\eta\rangle} a(x, \xi+\eta) h_k(y) b(y, \xi) \left[1 - \chi_j\left(\frac{\eta}{|\xi|}\right)\right]$$

$$\times \left[1 - \chi_j\left(\frac{\xi+\eta}{|\xi|}\right)\right] dy d\eta$$

$$\equiv I_{4,1} - I_{4,2} - I_{4,3} + I_{4,4} \ .$$

We can easily see that $I_{4,1}(x, D)$ is an integral operator with kernel in $G^{\theta;1/\rho}_{x,z}$ in $\Omega'\times\Omega'$ uniformly with respect to $l, k\in Z_+$ (cf. (2.4)).

Next we have

$$D^\beta_x\partial^\gamma_\xi I_{4,2}(x, \xi) = (2\pi)^{-n}\sum_{\substack{\tau\leq\beta\\ \mu\leq\gamma}}\binom{\beta}{\tau}\binom{\gamma}{\mu}\iint e^{i\langle x-y,\eta\rangle}(1+|\eta|^2)^{-q}\eta^{\beta-\tau}$$
$$\times a^{(\mu)}_{(\tau)}(x, \xi+\eta)(1-\Delta_y)^q\partial^{\gamma-\mu}_\xi\left\{\chi_j\left(\frac{\eta}{|\xi|}\right)[h_l(y)-h_k(y)]b(y, \xi)\right\}dyd\eta\ .$$

By taking $q = q(\tau) = [(|\beta-\tau|+N)/2]$, we have an estimate of type

$$|D^\beta_x\partial^\gamma_\xi I_{4,2}(x, \xi)| \leqq C_0C_1^{N+|\beta+\gamma|}N!^\sigma\gamma!\beta!^\sigma|\xi|^{m_++n-(\rho-\delta)N-\rho|\gamma|+\delta|\beta|}\ , \quad x\in\Omega'\ ,\quad |\xi|\geqq B\ . \tag{2.8}$$

We can handle $I_{4,3}(x, \xi)$ similarly as in $I_3(x, \xi)$. It remains to consider $I_{4,4}$. We have

$$D^\beta_x I_{4,4}(x, \xi) = (2\pi)^{-n}\sum_{\tau\leq\beta}\binom{\beta}{\tau}\iint e^{i\langle x-y,\eta\rangle}(1+|\eta|^2)^{-r}\eta^{\beta-\tau}a_{(\tau)}(x, \xi+\eta)$$
$$\times\chi_j\left(\frac{\xi+\eta}{|\xi|}\right)\left[1-\chi_j\left(\frac{\eta}{|\xi|}\right)\right](1-\Delta_y)^r\{h_k(y)b(y, \xi)\}dyd\eta\ .$$

Taking $r = [(|\beta-\tau|+\delta|\tau|+s+n+2)/2(1-\delta)]$ depending on τ and $s = 0, 1, \cdots$, and taking $k = 2r$, we have the estimate

$$|I_{4,4(\beta)}(x, \xi)| \leqq C_0C_1^{s+|\beta|}(s+|\beta|)!^{\sigma/(1-\delta)}|\xi|^{-s-(n+1)}\ , \quad x\in\Omega'\ ,\quad |\xi|\geqq B\ ,\quad s\in Z_+\ . \tag{2.9}$$

By virtue of (2.9), the kernel of $I_{4,4}(x, D)$ given by

$$I_{4,4}(x, z) = (2\pi)^{-n}\int e^{i\langle x-z,\xi\rangle}I_{4,4}(x, \xi)d\xi$$

is in $G^\theta(\Omega'_x\times\Omega'_z)$, $\theta = \max(1/\rho, \sigma/(1-\delta))$. Indeed, we have

$$D^\alpha_x D^\gamma_x I_{4,4}(x, z) = (2\pi)^{-n}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\int e^{i\langle x-y,\xi\rangle}\xi^{\alpha-\beta+\gamma}I_{4,4(\beta)}(x, \xi)d\xi$$

and we have the estimate (taking $s = |\alpha-\beta+\gamma|$)

$$|D^\alpha_x D^\gamma_z I_{4,4}(x, z)| \leqq C_0C_1^{|\alpha+\gamma|}|\alpha+\gamma|!^{\sigma/(1-\delta)}\ ,\quad (x, z)\in\Omega'\times\Omega'\ ,$$

where the constants C_0 and C_1 are independent of $\alpha, \gamma\in Z^n_+$. Summing up, we can split the operator as $F^N(x, D) = F^N_1+F^N_2$, where F^N_1 is an integral operator from $G^\theta(\Omega')$ into $G^\theta(\Omega')$ and F^N_2 is a pseudodifferential operator with symbol satisfying the condition (F). By the above argument the property (F′) is obviously verified.

3. Gevrey hypoellipticity.

THEOREM 3.1. (cf. [3, Theorem 3.1].) *Let* $a(x, \xi) \in S^{m}_{\rho,\delta,\sigma}(\Omega \times R^n)$ *(see Definition* 1.2), *and assume that there are positive constants c and B and* $-\infty < m' < \infty$ *such that*

$$(H_1) \qquad |a(x, \xi)| \geqq c|\xi|^{m'}, \quad x \in \Omega, \quad |\xi| \geqq B.$$

Assume also that for any compact set $K \subset \Omega$, *there are positive constants* C_0 *and* C_1 *such that*

$$(H_2) \qquad |a^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_0 C_1^{|\alpha+\beta|} \alpha! \beta!^{\sigma} |a(x, \xi)| |\xi|^{-\rho|\alpha|+\delta|\beta|}, \quad x \in K, \quad |\xi| \geqq B.$$

Then the operator $a(x, D)$ *is Gevrey hypoelliptic of order* θ, $\theta = \max(1/\rho, \sigma/(1-\delta))$, *that is, if* $u \in \mathscr{E}'(\Omega)$ *and* $a(x, D)u$ *is in* G^s *in* Ω', $\Omega' \subset \Omega$, *then u is also in* G^s *in* Ω' *for* $s \geqq \theta$.

For the proof we need two lemmas. We first define the symbol of a left parametrix of $a(x, D)$ as usual by:

$$(3.1) \qquad b_0(x, \xi) = 1/a(x, \xi), \quad x \in \Omega, \quad |\xi| \geqq B,$$

$$(3.2) \qquad b_j(x, \xi) = -b_0(x, \xi) \sum_{1 \leqq |\alpha| \leqq j} b^{(\alpha)}_{j-|\alpha|} a_{(\alpha)}(x, \xi), \quad x \in \Omega, \quad |\xi| \geqq B, \quad j = 1, 2, \cdots.$$

Take a function $\chi(\xi) \in C^\infty(R^n)$ such that $\chi(\xi) = 0$ for $|\xi| \leqq B$ and $\chi(\xi) = 1$ for $|\xi| \geqq B + 1$, and set

$$b^N(x, \xi) \equiv \chi(\xi) \sum_{j=0}^{N} b_j(x, \xi).$$

Then we have $b^N(x, \xi) \in S^{m'}_{\rho,\delta,\sigma}(\Omega \times R^n)$.

LEMMA 3.1. *Let* Ω' *be a relatively compact open subset of* Ω, *and take a function* $h \in C_0^\infty(\Omega)$ *such that* $h = 1$ *in a neighborhood of* $\bar{\Omega}'$. *Then we have*

$$b^N(x, D) h a(x, D) = I + R^N(x, D) \quad in \quad \Omega',$$
$$R^N(x, D) = R_1^N + R_2^N,$$

where R_1^N *is an integral operator from* $C^\infty(\bar{\Omega}')$ *into* $G^\theta(\bar{\Omega}')$ *with kernel* $R_1^N(x, y)$ *and* R_2^N *is a pseudodifferential operator with symbol* $R_2^N(x, \xi)$ *satisfying the conditions* (F) *and* (F′) *in Theorem* 2.1.

PROOF. By Theorem 2.1 we have

$$b^N(x, D) h a(x, D) = r^N(x, D) + F^N \quad \text{in} \quad \Omega',$$

where the symbol of $r^N(x, D)$ is given by

$$r^N(x, \xi) = \sum_{k=0}^{N} \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial_\xi^\alpha \Big(\sum_{j=0}^{N} b_j(x, \xi) \Big) a_{(\alpha)}(x, \xi), \quad x \in \Omega', \quad |\xi| \geqq B.$$

By the definition of $b_j(x, \xi)$, $j = 0, 1, \cdots$, we have

$$r^N(x, \xi) = 1 + r^N(x, \xi)' , \quad x \in \Omega' , \quad |\xi| \geqq B ,$$
$$\chi(\xi) r^N(x, \xi)' \in S^{m_+ - (\rho-\delta)N}_{\rho,\delta,\sigma}(\Omega' \times R^n) .$$

All the symbols of the class $S^{m_+ - (\rho-\delta)N}_{\rho,\delta,\sigma}(\Omega' \times R^n)$ satisfy the conditions (F) and (F'). Hence we have the assertion of Lemma 3.1.

LEMMA 3.2. *Let $R_2^N(x, y)$ be the kernel of the operator $R_2^N(x, D)$ given in Lemma 3.1. Then*

$$R_2^N(x, y) \in G^{\theta, 1/\rho}_{x,y}(\Omega' \times \Omega' \setminus \Delta) , \quad \Delta = \{(x, x);\ x \in \Omega'\} ,$$

where $\theta = \max(1/\rho, \sigma/(1-\delta))$.

PROOF. For simplicity we assume N is so large that $N \geqq (m_+ + n + 2)/(\rho - \delta)$. Let U be a relatively compact open subset of $\Omega' \times \Omega' \setminus \Delta$. We shall estimate

$$\sup_U |D_x^\alpha D_y^\gamma R_2^N(x, y)| .$$

By virtue of the construction of R^N and by the fact $\theta \geqq \theta_0$, as in the proof of Theorem 1.1 the problem is reduced to estimating each term of the form

$$I = \int e^{i\langle x-y,\xi\rangle} \xi^{\alpha-\beta}(-\xi)^\gamma u^\tau(x, \xi) d\xi$$
$$= (x_i - y_i)^{-[(|\alpha-\beta|+|\gamma|+\delta|\beta-\tau|+\delta^2|\tau|)/\rho]} \int e^{i\langle x-y,\xi\rangle}$$
$$\times D_{\xi_i}^{[(|\alpha-\beta|+|\gamma|+\delta|\beta-\tau|+\delta^2|\tau|)\rho]} \{\xi^{\alpha-\beta}(-\xi)^\gamma u^\tau(x, \xi)\} d\xi ,$$

for a fixed i, $1 \leqq i \leqq n$ and $\tau \leqq \beta \leqq \alpha$.
By using the property (F'), we have an estimate of the form

$$|I| \leqq C_0 C_1^{N+|\alpha+\beta|} N!^\sigma \gamma!^{1/\rho} (\alpha-\beta)!^{1/\rho} \beta!^\sigma \tau!^{\sigma\delta} (\beta-\tau)!^{\delta/\rho} \tau!^{\delta^2/\rho} . \tag{3.3}$$

For the right hand side of (3.3) we have an estimate of type

$$(\alpha-\beta)!^{1/\rho} \beta!^\sigma \tau!^{\sigma\delta} (\beta-\tau)!^{\delta/\rho} \tau!^{\delta^2/\rho} \leqq C^{|\alpha|} (\alpha-\tau)!^{\theta_1} \tau!^{(\sigma+\sigma\delta+\delta^2/\rho)} .$$

Observing that we have

$$\theta = \max(1/\rho, \sigma/(1-\delta)) \geqq 1/\rho,\ \sigma + \delta/\rho,\ \sigma + \sigma\delta + \delta^2/\rho .$$

The last term is estimated by $C^{|\alpha|}\alpha!^\theta$. Thus we obtain an estimate of the form

$$\sup_U |D_x^\alpha D_y^\gamma R_2^N(x, y)| \leqq C_0 C_1^{|\alpha+\gamma|} \alpha!^\theta \gamma!^{1/\rho} . \tag{3.4}$$

We remark that we must take $l = |\alpha| + N$ in the construction of R_2^N.

However $R_1^N(x, y)$ is in G^θ in $\Omega'\times\Omega'$ uniformly with respect to l, $l=0, 1, \cdots$, and so we may use the operator R_2^N as if it is not depending on l.

PROOF OF THEOREM 3.1. We are considering the equation

$$a(x, D)u = f\ ,\quad u\in\mathscr{E}'(\Omega)\ ,\quad f\in\mathscr{D}'(\Omega)\ ,$$

where f is assumed to be G^s in $\Omega'\subset\subset\Omega$, $s\geqq\theta$. For simplicity we shall prove the case where $s=\theta=\max(1/\rho, \sigma/1-\rho)$. It is well known (cf. [4]) that u is in $C^\infty(\Omega')$ under the hypotheses of Theorem 3.1. Now take an arbitrary point $x_0\in\Omega'$ and a small neighborhood $U_d=\{x: |x-x_0|<d\}\subset\Omega'$, $d>0$. Let $\varphi\in C_0^\infty(U_d)$ be such that $\varphi(x)=1$ on $U_{d/2}$. Then we have by Theorem 1.1

$$a(x, D)\varphi u = f - a(x, D)(1-\varphi)u \equiv f_1\in C^\infty(\Omega')\cap G^\theta(U_{d/2})\ .$$

Next take N sufficiently large so that $N\geqq(m_+ + n + 2)/(\rho-\delta)$, and take $h\in C_0^\infty(\Omega)$ so that $h=1$ on Ω'. Then by Lemma 3.1, we have

$$b^N(x, D)ha(x, D)\varphi u = \varphi u + R^N(x, D)\varphi u = b^N(x, D)hf_1\ ;$$

namely, we have an integral equation with respect to φu:

$$\begin{aligned}(3.5)\qquad \varphi u &= b^N(x, D)hf_1 - \int R_1^N(x, y)\varphi(y)u(y)dy - \int R_2^N(x, y)\varphi(y)u(y)dy\\ &\equiv g(x) - \int R_2^N(x, y)\varphi(y)u(y)dy\ ,\end{aligned}$$

where $g(x)$ is a function in $G^\theta(U_{d/2})$. We set $R(x, y)=R_2^N(x, y)$ and denote its symbol by $R(x, \xi)$. Let $\omega=U_{d/4}$ and assume $0<d\leqq 1$. We denote by ω_ε the open set of points in ω at distance $>\varepsilon$ from the complement of ω denoted by ω^c. Then $\omega_\varepsilon=\emptyset$ if $\varepsilon>1/4$. We want to prove that there exists a constant B such that for every $\varepsilon>0$ and every integer $j>0$ we have

$$(3.6)\qquad \varepsilon^{\theta|\alpha|}\sup_{\omega_{|\alpha|\varepsilon}}|D^\alpha u(x)|\leqq B^{|\alpha|+1}\quad\text{if}\quad |\alpha|\leqq j\ ,$$

and

$$(3.6)'\qquad \varepsilon^{\theta k}\sup_{\omega_{k\varepsilon}}|(1-\Delta)^{[k/2]}u(x)|\leqq B^{k+1}\quad\text{if}\quad k\leqq j\ .$$

It follows from (3.6) or (3.6)′ that u is in G^θ in ω. Indeed, let K be a compact subset of ω and choose $c>0$ so that $K\subset\omega_c$. Setting $j=|\alpha|$ and $\varepsilon=c/|\alpha|$ in (3.6), we obtain

$$\sup_K|D^\alpha u|\leqq\sup_{\omega_c}|D^\alpha u|\leqq(B/c^\theta)^{|\alpha|+1}|\alpha|^{\theta|\alpha|}\ ,$$

which proves that $u\in G^\theta(\omega)$.

We shall prove (3.6) by induction on j. This is obviously true when $j=1$ if B is sufficiently large. Assuming that (3.6) is proved for $j-1$, $j \geqq 2$, we shall show that (3.6) follows for j if B is sufficiently large and independent of j. To do so we only have to estimate the derivatives $D^\alpha u$ with $|\alpha|=j$ $(j \geqq 2)$. Differentiation of (3.5) gives

$$\varepsilon^{\theta j} \sup_{\omega_{j\varepsilon}} |D^\alpha u(x)| \leqq A^{j+1} + \varepsilon^{\theta j} \sup_{x \in \omega_{j\varepsilon}} \left| D_x^\alpha \int R(x, y)\varphi(y)u(y)dy \right| ,$$

where A is a positive constant independent of j. By observing the construction of $R = R_2^N$ we have

$$D_x^\alpha \int R(x, y)\varphi(y)u(y)dy = \sum_{0 \neq \beta \leqq \alpha} \binom{\alpha}{\beta} \int K_\beta(x, y) D_y^{\alpha-\beta}(\varphi u)dy + \int R(x, y) D_y^\alpha(\varphi u)dy . \tag{3.7}$$

By virtue of the property (F′) we have

$$K_\beta(x, y) = \sum_{\tau \leqq \beta} \binom{\beta}{\tau} \int e^{i\langle x-y, \xi\rangle} u^\tau(x, \xi)d\xi .$$

First we shall treat the last term in (3.7). From the proof of Lemma 3.2, we may assume that there is a constant $C>0$ such that

$$|D_y^\gamma(x, y)| \leqq C^{|\gamma|+1}\gamma!^\theta |x-y|^{(-|\gamma|+1)/\rho} , \quad (x, y) \in U_d \times U_d \setminus \Delta . \tag{3.8}$$

For $|\alpha|=j$, rewrite $\alpha = \alpha' + \alpha''$ with $|\alpha'| = j-1$, $|\alpha''|=1$. We have

$$\begin{aligned}
\varepsilon^{\theta j} \sup_{\omega_{j\varepsilon}} \left| \int R(x, y) D_y^\alpha(\varphi(y)u(y))dy \right| &= \varepsilon^{\theta j} \sup_{\omega_{j\varepsilon}} \left| \int D_y^{\alpha''} R(x, y) D_y^{\alpha'}(\varphi(y)u(y))dy \right| \\
&\leqq \varepsilon^{\theta j} c^2 \sup_{\omega_{(j-1)\varepsilon}} |D^{\alpha'} u| + \varepsilon^{\theta j} \sup_{x \in \varepsilon_j} \left| \int_{\omega^c_{(j-1)\varepsilon}} D_y^{\alpha''} R(x, y) D_y^{\alpha''}(\varphi(y)u(y))dy \right| \\
&\equiv I_1 + I_2 .
\end{aligned}$$

By assumption we have

$$I_1 \leqq \varepsilon^\theta C' C^2 B^j ,$$

where C' is a small constant depending on ω. Denoting symbolically by D_y^k the differentiation of order $k \geqq 0$, we have

$$\begin{aligned}
I_2 &\leqq \varepsilon^{\theta j} \sum_{k=1}^{j-1} \sup_{x \in \omega_{j\varepsilon}} \left| \int_{\partial\omega_{(j-k)\varepsilon}} D_y^k R(x, y) D_y^{j-k-1} u dS_y \right| \\
&\quad + \varepsilon^{\theta j} \sum_{k=1}^{j-1} \sup_{x \in \omega_{j\varepsilon}} \left| \int_{\omega_{(j-k-1)\varepsilon} \setminus \omega_{(j-k)\varepsilon}} D_y^{k+1} R(x, y) D_y^{j-k-1} u(y)dy \right| \\
&\quad + \varepsilon^{\theta j} \sup_{x \in \omega_{j\varepsilon}} \left| \int_{\omega^c} D_y^\alpha R(x, y)\varphi(y)u(y)dy \right| \\
&\equiv I_{2,1} + I_{2,2} + I_{2,3} .
\end{aligned}$$

By the induction assumption and by using (3.8), we have

$$I_{2,1} \leqq C_\varepsilon \varepsilon^\theta \sum_{k=1}^{j-1} C^{k+1} k!^\theta \varepsilon^{k\theta}(k\varepsilon)^{(-k+1)/\rho} B^{j-k} \leqq \varepsilon^\theta C_\varepsilon C B^j \sum_{k=1}^{j-1} (C/B)^k .$$

Similarly we have

$$I_{2,2} \leqq \varepsilon^\theta C' C^2 B^j \sum_{k=1}^{j-1} (C/B)^{k-1}$$

and

$$I_{2,3} \leqq \varepsilon^\theta C' C^{j+1} .$$

Hence if we take B greater than $2C$ we have an estimate of the form

$$I_1 + I_2 \leqq \varepsilon^\theta C_\varepsilon (C^2 B^j + C^{j+1}) , \quad |\alpha| = j . \tag{3.9}$$

We want to apply the same method to estimate the first sum on the right hand side of (3.7). By virtue of the property (F′) we have

$$K_\beta(x, y) = \sum_{\tau \leqq \beta} \binom{\beta}{\tau} \int e^{i\langle x-y, \xi\rangle} u^\tau(x, \xi) d\xi .$$

Setting

$$K_\beta^\tau(x, y) = \int e^{i\langle x-y, \xi\rangle} u^\tau(x, \xi)(1 + |\xi|^2)^{-(\delta|\beta-\tau|+\delta^2|\tau|)/2} d\xi ,$$

we have the following estimate as in (3.8):

$$|D_y^\gamma K_\beta^\tau(x, y)| \leqq C^{|\beta+\gamma|+1} \beta!^\sigma \gamma!^\theta \tau!^{\sigma\delta} |x-y|^{-(|\gamma|+1)/\rho} , \quad (x, y) \in U_d \times U_d \setminus \Delta . \tag{3.10}$$

Our purpose is to estimate $\varepsilon^{\theta j} \times (3.7)$ in the form

$$\varepsilon^{\theta j} \sum_{k=1}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \sup_{\substack{x \in \omega_{j\varepsilon} \\ k=|\beta| \\ l=|\tau|}} \left| \int K_\beta^\tau(x, y) D_y^{j-k} (1 - \Delta_y)^{(\delta(k-l)+\delta^2 l)/2} (\varphi u) dy \right. , \tag{3.11}$$

where $j = |\alpha|$. As before, we have for $1 \leqq k \leqq j$ and $0 \leqq l \leqq k$,

$$\begin{aligned} &\binom{j}{k}\binom{k}{l} \sup_{x \in \omega_{j\varepsilon}} \left| \int K_\beta^\tau(x, y) D_y^{j-k} (1 - \Delta_y)^{(\delta(k-l)+\delta^2 l)/2} (\varphi u) dy \right| \\ &\leqq \binom{j}{k}\binom{k}{l} C^{k+1} k!^\sigma l!^{\sigma\delta} \sup_{\omega_{(j-1)\varepsilon}} |D^{j-k-1}(1 - \Delta_x)^{(\delta(k-l)+\delta^2 l)/2} u(x)| \\ &\quad + \binom{j}{k}\binom{k}{l} \sup_{x \in \omega_{j\varepsilon}} \left| \int_{\omega_{(j-1)\varepsilon}^c} D_y K_\beta^\tau(x, y) D_y^{j-k-1} (1 - \Delta_y)^{(\delta(k-l)+\delta^2 l)/2} (\varphi u) dy \right| \\ &\equiv I_{1,k,l} + I_{2,k,l} . \end{aligned}$$

By assumption we have

$$\varepsilon^{\theta j} I_{1,k,l} \leqq C' C^{k+1} B^j \binom{j}{k} \binom{k}{l} k!^\sigma l!^{\sigma\delta} \varepsilon^{\theta((1-\delta)k+\delta(1-\delta)l)} B^{-(1-\delta)k-1+\delta(1-\delta)l} .$$

Since $\varepsilon \leqq 1/4j$, we have

$$\begin{aligned} \varepsilon^{\theta j} I_{1,k,l} &\leqq C_\varepsilon C^{k+1} B^j B^{-(1-\delta)k-1} \frac{j!k!^\sigma l!^{\sigma\delta}}{(j-k)!(k-l)!l!} \left(\frac{1}{4j}\right)^{\theta((1-\delta)k+\delta(1-\delta)l)} B^{\delta(1-\delta)l} \\ &\leqq C_\varepsilon C^{k+1} B^j B^{-(1-\delta)k-1} B^{\delta(1-\delta)l} . \end{aligned}$$

Thus we have

$$\varepsilon^{\theta j} \sum_{l=0}^{k} I_{1,k,l} \leqq C_\varepsilon C B^j [C/B^{(1-\delta)^2}]^k .$$

Hence if $B^{(1-\delta)^2} \leqq 2C$, we have

$$\varepsilon^{\theta j} \sum_{k=1}^{j} \sum_{l=0}^{k} I_{1,k,l} \leqq C_\varepsilon C B^j . \tag{3.12}$$

In the same way as we treated I_2, we have

$$\begin{aligned} &I_{2,k,l} \\ &\leqq \binom{j}{k}\binom{k}{l} \sum_{s=1}^{j-k-1} \sup_{\omega_{j\varepsilon}} \left| \int_{\partial\omega_{(j-1)\varepsilon}} D_y^s K_\beta^\tau(x, y) D^{j-k-s-1} (1-\Delta_y)^{[(\delta(k-l)+\delta^2 l)/2]} u(y) dS \right| \\ &\quad + \binom{j}{k}\binom{j}{l} \sum_{s=1}^{j-k-1} \sup_{x\in\omega_{j\varepsilon}} \left| \int_{\omega_{(j-s-1)\varepsilon} \setminus \omega_{(j-s)\varepsilon}} D_y^s K_\beta^\tau(x, y) D_y^{j-k-s} (1-\Delta_y)^{[(\delta(k-l)+\delta^2 l)/2]} u(y) dy \right| \\ &\quad + \binom{j}{k}\binom{k}{l} \sup_{x\in\omega_{j\varepsilon}} \left| \int_{\omega_{k\varepsilon}^c} D_y^{j-k} K_\beta^\tau(x, y) (1-\Delta_y)^{[(\delta(k-l)+\delta^2 l)/2]} (\varphi u) dy \right| \\ &\equiv I_{2,k,l}^1 + I_{2,k,l}^2 + I_{2,k,l}^3 . \end{aligned}$$

We have as before

$$\begin{aligned} \varepsilon^{\theta j} I_{2,k,l}^1 &\leqq C_\varepsilon C^{k+1} B^j \binom{j}{k}\binom{k}{l} B^{-(1-\delta)k} \cdot \varepsilon^{(k+1-\delta(k-l)+\delta^2 l)\theta} \\ &\qquad \times B^{-\delta(1-\delta)l} k!^\sigma l!^{\sigma\delta} \sum_{s=0}^{j-k-1} (C/B)^s s!^\theta \varepsilon^{s\theta} (\varepsilon s)^{(-s+1)/\rho} \\ &\leqq C_\varepsilon C^{k+1} B^j B^{-(1-\delta)^2 k} = C_\varepsilon C B^j [C/B(1-\delta)^2]^k . \end{aligned}$$

Hence we have

$$\varepsilon^{\theta j} \sum_{k=1}^{j} \sum_{l=0}^{k} I_{2,k,l}^1 \leqq C_\varepsilon C B^j , \tag{3.13}$$

if $B^{(1-\delta)^2} \geqq 2C$. In the same way, we have an estimate of the form

$$\varepsilon^{\theta j} \sum_{k=1}^{j} \sum_{l=0}^{k} (I_{2,k,l}^2 + I_{2,k,l}^3) \leqq 2 C_\varepsilon C B^j . \tag{3.14}$$

Combining the estimates (3.9), (3.12), (3.13) and (3.14) we have

$$\varepsilon^{\theta j} \sup_{\omega_{j\varepsilon}} |D^\alpha u(x)| \leqq A^{j+1} + C_\varepsilon C((C+5)B^j + C^j) .$$

Hence if we can take B so large that

$$A^{j+1} + C_\varepsilon C((C+5)B^j + C^j) \leqq B^{j+1}$$

the proof of (3.6) is completed. This condition is fulfilled for every j if $B \geqq \max(1, (2C)^{1/(1-\delta)^2}, 2A, A + C_\varepsilon C(C+6))$. q.e.d.

4. **Examples.** We consider the following differential operators:

(4.1) $$P_1 = -x^4 d^2/dx^2 + 1 \quad \text{in} \quad -\infty < x < \infty ,$$

(4.2) $$P_2 = x^4(\partial/\partial y - \partial^2/\partial x^2) + 1 \quad \text{in} \quad R^2_{x,y} .$$

We can easily verify that the characteristic polynomial $P_1(x, \xi) = x^4\xi^2 + 1$ satisfies the conditions (H_1) and (H_2) in Theorem 3.1 with $\sigma = \rho = 1$ and $\delta = 1/2$. Hence we have $\theta = 2$. We have a solution

$$v(x) = \begin{cases} xe^{-1/x} & x > 0 , \\ 0 & x \leqq 0 \end{cases}$$

of the equation $P_1 v(x) = 0$ and $v(x)$ is in G^2 in any neighborhood of the origin of R^1.

We can also easily verify that $P_2(x, y; \xi, \eta) = v^4(i\eta + \xi^2) + 1$ satisfies (H_1) and (H_2) with $\sigma = 1$, $\rho = 1/2$ and $\delta = 1/4$. Hence we have $\theta = 2$ also for P_2. We have a solution of the equation $P_2 u = 0$ as a function expressed by $u(x, y) = v(x)$, where $v(x)$ is the function given above.

We remark that we have only $\theta = 1/(\rho - \delta) = 4$ by the result of [3].

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Department of Mathematics
Faculty of Science
Nagoya University
Nagoya 464
Japan