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# EIGENMAPS AND MINIMAL IMMERSIONS OF PROJECTIVE SPACES INTO SPHERES

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**Introduction.** In this paper we will consider the parameter spaces of eigenmaps and isometric minimal immersions of projective spaces into spheres.

A map $f: (M, g) \to S^m \subset \boldsymbol{R}^{m+1}$ is harmonic if $f$ satisfies $\Delta^{(M,g)}f = 2e(f)f$, where $\Delta^{(M,g)}$ is the Laplacian of $(M, g)$ and $e(f)$ is the energy density of $f$ (cf. [5]). In particular, if $2e(f) = \lambda$ is a constant, then $\lambda \in \mathrm{Spec}(M, g)$. Such a harmonic map is called an *eigenmap* [5]. By a theorem of Takahashi in [9], an eigenmap is an isometric minimal immersion if and only if it is an isometric immersion. An eigenmap $\phi: M \to S^m$ is said to be *full* if its image $\phi(M)$ is not contained in any great sphere in $S^m$. Let $\phi_1, \phi_2: M \to S^m$ be full eigenmaps. Then they are said to be equivalent if there exists an isometry $\rho$ of $S^m$ such that $\rho \circ \phi_1 = \phi_2$.

It is a fundamental problem on isometric minimal immersions to study to what extent they exist. In [3], do Carmo and Wallach showed that the set of equivalence classes of all full isometric minimal immersions of compact symmetric spaces into spheres are parametrized by a compact convex body in some vector space. It is also natural to consider a similar problem for eigenmaps. In fact in [12], Toth and d'Ambra showed that the set of equivalence classes of all full eigenmaps are also parametrized by a compact convex body in some vector space.

Before showing further results on specific spaces, we explain the standard construction of isometric minimal immersions of a compact irreducible symmetric space $(M, g)$ into spheres. Let $\Delta^{(M,g)}$ be the Laplacian of $(M, g)$ with such sign that all eigenvalues are non-negative. We denote by $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, the set of all distinct eigenvalues of $\Delta^{(M,g)}$, and by $V^k$ the eigenspace of $\Delta^{(M,g)}$ corresponding to $\lambda_k$. Put $\dim V^k = m(k) + 1$ and $\dim M = d$. For each $k \geqq 1$, define a canonical measure $d\mu$ on $M$ normalized by $\int_M d\mu = m(k) + 1$. Take an orthonormal base $\{f_0, f_1, \cdots, f_{m(k)}\}$ and define a mapping

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$$x_k: M \to \boldsymbol{R}^{m(k)+1};\ p \mapsto (f_0(p), f_1(p), \cdots, f_{m(k)}(p)) .$$

Then $x_k$ realizes an isometric minimal immersion of $(M, (\lambda_k/d)g)$ into the unit sphere in $\boldsymbol{R}^{m(k)+1}$, which we call the *standard isometric minimal immersion.*

The following theorem of do Carmo and Wallach [3] gives a description of the set of equivalence classes of all full isometric minimal immersions of compact irreducible symmetric spaces into spheres.

THEOREM 0.1. (i) *Assume that there exists a full isometric minimal immersion $\phi$ of $(M, c^2g)$ with a constant $c \neq 0$ into a unit sphere $S_1^q$. Then there exists $k \geqq 1$ such that $c^2 = \lambda_k/d$ and $q \leqq m(k)$.*

(ii) *The set of equivalence classes of full isometric minimal immersions of $(M, (\lambda_k/d)g)$ into $S_1^q$, $q \leqq m(k)$, is parametrized by a convex body $W_M$ in some vector space $L_M$ in such a way that the interior points of $W_M$ correspond to those $[\phi]$ with $q = m(k)$ and that the boundary points of $W_M$ correspond to those $[\phi]$ with $q < m(k)$.*

We will give the description of $W_M$ and explain how it parametrizes the set of equivalence classes of full isometric minimal immersions in § 2. A similar theorem holds for eigenmaps.

THEOREM 0.2 (Toth and d'Ambra [12]). *Let $\lambda \in \mathrm{Spec}(M, g)$. Then the set of equivalence classes of full eigenmaps $\phi$ of $(M, g)$ into $S_1^q$ with $2e(\phi) = \lambda$ can be parametrized by a convex body $W_E$ in some vector space $L_E$. The interior points of $W_E$ correspond to those $[\phi]$ with $q = m(k)$ while the boundary points correspond to those $[\phi]$ with $q < m(k)$.*

For specific spaces the dimensions of $L_M$ and $L_E$ are studied, since it is closely related to the following rigidity problem: Let $\phi$ be another full isometric minimal immersion (resp. eigenmap), then is it equivalent to $x_k$?

By Theorems 1 or 2, the rigidity problem is reduced to studying whether $\dim L_M$ or $\dim L_E$ is equal to zero or not. In fact, do Carmo and Wallach showed:

THEOREM 0.3 (do Carmo and Wallach [3]). *Le $(M, g)$ be the $d$-dimensional sphere with constant sectional curvature. Then*

(i) $\dim L_M \geqq 18$ *if* $d \geqq 3$ *and* $k \geqq 4$,

(ii) $\dim L_M = 0$ *if* $d = 2$ *or* $k \leqq 3$.

Thus the standard isometric minimal immersion $x_k$ of the $d$-dimensional sphere is rigid in the category of isometric minimal immersions if $d = 2$ or $k \leqq 3$. Toth and d'Ambra studied the parameter space $W_E$ when $M$

is also a $d$-dimensional sphere.

THEOREM 0.4 (Toth and d'Ambra [12]). *Let $(M, g)$ be the $d$-dimensional sphere with constant sectional curvature. Then*
(i) $\dim L_E \geqq 10$ *if* $d \geqq 3$ *and* $k \geqq 2$,
(ii) $\dim L_E = 0$ *if* $d = 2$ *or* $k = 1$.

Recently Urakawa obtained results on $\dim L_M$ for complex projective spaces and the quaternion projective plane. From his proof we can get information on $\dim L_E$ for complex projective spaces if $k \geqq 2$. We state it together with his original results on $\dim L_M$.

THEOREM 0.5 (Urakawa [14]). *Let $(M, g)$ be the complex projective space $P^n(C) = SU(n+1)/S(U(1)\times U(n))$ with an $SU(n+1)$-invariant Riemannian metric $g$. Then*
(i) $\dim L_M \geqq 91$ *if* $n \geqq 2$ *and* $k \geqq 4$,
(ii) $\dim L_E \geqq 28$ *if* $n \geqq 2$ *and* $k \geqq 2$.

THEOREM 0.6 (Urakawa [14]). *Let $(M, g)$ be the quaternion projective plane $P^2(\boldsymbol{H}) = Sp(3)/Sp(1)\times Sp(2)$ with an $Sp(3)$-invariant Riemannian metric $g$. Then* $\dim L_M \geqq 29007$ *if* $k \geqq 4$.

In this paper we prove the above theorem generally for quaternion projective spaces. Namely we prove the following:

THEOREM 0.7. *Let $(M, g)$ be the quaternion projective space $P^n(\boldsymbol{H}) = Sp(n+1)/Sp(1)\times Sp(n)$ with an $Sp(n+1)$-invariant Riemannian metric. Then*
(i) $\dim L_M \geqq 1386$ *if* $n \geqq 2$ *and* $k \geqq 3$,
$\dim L_M = 0$ *if* $n \geqq 2$ *and* $k = 1$.
(ii) $\dim L_E \geqq 1078$ *if* $n \geqq 2$ *and* $k \geqq 2$,
$\dim L_E \geqq 42$ *if* $n \geqq 3$ *and* $k = 1$,
$\dim L_E = 0$ *if* $n = 2$ *and* $k = 1$.

Furthermore we will consider a similar problem for the Cayley projective plane and prove the following:

THEOREM 0.8. *Let $(M, g)$ be the Cayley projective plane $P^2(Ca) = F_4/Spin(9)$ with an $F_4$-invariant Riemannian metric. Then*
(i) $\dim L_M \geqq 107406$ *if* $k \geqq 3$,
$\dim L_M = 0$ *if* $k = 1$.
(ii) $\dim L_E \geqq 19448$ *if* $k \geqq 2$,
$\dim L_E = 0$ *if* $k = 1$.

From the above theorems, the standard isometric minimal immersions $x_k$ of spheres $S^n$, $n \geqq 3$, complex projective spaces $P^n(C)$, $n \geqq 2$, quaternion projective spaces $P^n(\boldsymbol{H})$, $n \geqq 2$, or the Cayley projective plane are rigid if $k=1$ while they are not rigid if $k \geqq 4$.

After the author completed this work, Professor H. Urakawa informed him of the result of Z. Yiming [16], which states the following:

THEOREM. *Let* $(M, g)$ *be the quaternion projective space* $P^n(\boldsymbol{H}) = Sp(n+1)/Sp(1)\times Sp(n)$ *with an* $Sp(n+1)$*-invariant Riemanian metric. Then* $x_k$ *is rigid if* $k=1$. *If* $k>1$ *then* $\dim L_M \geqq 84$.

But no proof of the key Lemma 4.2 in [16] is given. Lemma 4.2 in [16] is proved as (4.6) in this paper. We cannot say anything about the case $k=2$ by using the theory of do Carmo and Wallach.

Thanks are due to Professor H. Urakawa for sending him a copy of Yiming's paper and to Professor G. Toth for pointing out some mistakes in the first draft.

**1. The standard isometric minimal immersions.** In this section we explain the construction of standard isometric minimal immersions.

Let $M = G/K$ be a $d$-dimensional irreducible Riemannian symmetric space of compact type and let $g$ be a $G$-invariant Riemannian metric on $M$. We denote by $\Delta^{(M,g)}$ the Laplacian on $(M, g)$ and by

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

the set of all eigenvalues of $\Delta^{(M,g)}$. We denote by $V^k$ the eigenspace of $\Delta^{(M,g)}$ corresponding to the eigenvalue $\lambda_k$ and denote its dimension by $\dim V^k = m(k)+1$. Let $d\mu$ be the canonical measure on $M$ normalized by $\int_M d\mu = m(k)+1$ and let $\{f_0, f_1, \cdots, f_{m(k)}\}$ be an orthonormal base of $V^k$ with respect to the $L^2$-inner product. Define a mapping $x_k$ by

$$x_k \colon M \to \boldsymbol{R}^{m(k)+1};\ p \mapsto (f_0(p), f_1(p), \cdots, f_{m(k)}(p))\ .$$

The action of $G$ on $M$ naturally induces an action of $G$ on $V^k$ by $(\sigma\cdot f)(p) = f(\sigma^{-1}\cdot p)$ for $\sigma \in G$, $p \in M$. Let $v_0 = \sum_{i=0}^{m(k)} f_i(p) f_i \in V^k$. Then

$$\sigma\cdot v_0 = \sum_{i=0}^{m(k)} f_i(p)(\sigma\cdot f_i) = \sum_{i=0}^{m(k)} f_i(\sigma\cdot p) f_i\ .$$

Thus we may regard $x_k$ as

$$x_k \colon M \to S_1 \subset V^k;\ \sigma K \to \sigma\cdot v_0\ .$$

Since $G$ preserves the $L^2$-inner product, the image $x_k(M)$ is contained in a sphere centered at the origin. Furthermore by integrating $\langle x_k(p), x_k(p)\rangle$ on $M$, we have

$$(m(k)+1)\langle x_k(eK), x_k(eK)\rangle = \int_M \langle x_k(p), x_k(p)\rangle d\mu$$

$$= \int_M \sum_{j=0}^{m(k)} (f_j(p))^2 d\mu$$
$$= m(k) + 1 .$$

Thus $x_k$ is a map of $M$ into the unit sphere in $\boldsymbol{R}^{m(k)+1}$ centered at the origin. An irreducible representation $V$ of $G$ is said to be of *class one* if it contains a non-zero $K$-fixed vector. We remark that $V^k$ is irreducible when $M$ is of rank one. The (0, 2)-tensor $x_k^* g_0$ on $M$ induced from the standard Euclidean metric $g_0$ on $\boldsymbol{R}^{m(k)+1}$ is $G$-invariant. Thus by the irreducibility of $M$, $x_k$ must be an isometric immersion with respect to $c^2 g$ for some constant $c \neq 0$. Since $\Delta^{(M,c^2g)} x_k = (\lambda_k / c^2) x_k$, a theorem of Takahashi [9] implies that $x_k$ realizes an isometric minimal immersion of $(M, c^2 g)$ into a sphere of radius $(dc^2/\lambda_k)^{1/2}$ Thus we have $c^2 = \lambda_k / d$.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to an Ad($G$)-invariant inner product in $\mathfrak{g}$. Then the tangent space $x_k^*(T_{\sigma K}(M))$ is

$$x_k^*(T_{\sigma K}(M)) = \{\sigma(X \cdot v);\ X \in \mathfrak{p}\} . \tag{1.1}$$

**2. Classification theorem.** In this section, we give a brief summary of the classification theorem of do Carmo and Wallach [3], and that of Toth and d'Ambra [12] stated in the introduction.

Let $\phi = (\phi_0, \phi_1, \cdots, \phi_q): (M, g) \to S_1^q \subset \boldsymbol{R}^{q+1}$ be a full eigenmap of an irreducible Riemannian symmetric space $(M, g)$ into the unit sphere $S_1^q$ with $\Delta^{(M,g)} \phi = \lambda_k \phi$, $\lambda_k \in \mathrm{Spec}(M, g)$. Since $\phi$ is a full eigenmap, $\phi_0, \phi_1, \cdots, \phi_q$ are linearly independent, i.e., $q \leqq m(k)$. Thus there exists a matrix $A$ of size $(m(k)+1) \times (m(k)+1)$ such that $(\phi_0, \phi_1, \cdots, \phi_q, 0, \cdots, 0) = (f_0, f_1, \cdots, f_{m(k)})A$. Taking the polar decomposition of $A$, we see that $i \circ \phi$ is equivalent to $S \circ x_k$, where $i$ is the canonical inclusion $S^q \subset S^{m(k)}$ and $S$ is a symmetric positive semi-definite matrix of size $(m(k)+1) \times (m(k)+1)$.

We identify the symmetric tensor product $S^2(V^k)$ with the space of all symmetric linear endomorphisms on $V^k$ by

$$u \cdot v(t) = (\langle u, t \rangle v + \langle v, t \rangle u)/2 , \quad u, v, t \in V^k .$$

The inner product ( , ) on $S^2(V^k)$, induced from the inner product $\langle\ ,\ \rangle$ on $V^k$ under the above identification, is $(A, B) = \mathrm{trace}\, AB$ for $A, B \in S^2(V^k)$. The induced action of $G$ on $S^2(V^k)$ is $\sigma \cdot A = \sigma A \sigma^{-1}$ for $\sigma \in G$, $A \in S^2(V^k)$. Furthermore, we have $\langle A(u), v \rangle = (A, u \cdot v)$ for $A \in S^2(V^k)$, $u, v \in V$.

Since $i \circ \phi$ is a map of $M$ into the unit sphere, we have $\langle S(x_k(p)), S(x_k(p)) \rangle = 1$ for $p \in M$, i.e.,

$$\langle S(x_k(\sigma K)), S(x_k(\sigma K)) \rangle = (S^2, \sigma \cdot v_0^2) = 1 , \quad \sigma \in G .$$

Since $(I, \sigma\cdot v_0^2) = 1$, we have

$$(S^2 - I, \sigma\cdot v_0^2) = 0 .$$

Let $W_0 = \{\{G\cdot v_0^2\}\}$ be the $\boldsymbol{R}$-linear span of $G\cdot v_0^2$ in $S^2(V^k)$ and let $L_E$ be its orthogonal complement $L_E = \{C \in S^2(V^k); C \perp \sigma\cdot v_0^2, \sigma \in G\}$. Then $C = S^2 - I$ is contained in $L_E$. Let $W_E = \{C \in L_E; C + I$ is positive semi-definite$\}$. Then the correspondence

$$W_E \ni C \mapsto (C + I)^{1/2}x_k$$

gives a parametrization of the set of equivalence classes of full eigenmaps. This is an outline of the proof of Theorem 0.2 stated in the introduction.

LEMMA 2.1 (do Carmo and Wallach [3]). *If each irreducible $K$-submodules of $V^k$ has multiplicity one, then $W_0$ is the sum of all class one submodules of $(G, K)$ in $S^2(V^k)$.*

For the proof of Lemma 2.1, we refer to do Carmo and Wallach [3] or Toth [11]. Although do Carmo and Wallach [3] proved Lemma 2.1 only for the case $M = S^n$, their proof works well under the assumption of Lemma 2.1.

REMARK 2.2. The assumption of Lemma 2.1 is satisfied if $M$ is a symmetric space of compact type and of rank one (cf. [8] and [11]).

Now we consider the case where an eigenmap $S\circ x_k$ is an isometric immersion. In this case, $S\circ x_k$ is an isometric minimal immersion. By (1.1), $S\circ x_k$ is an isometric immersion if and only if

$$\langle S(\sigma(X\cdot v_0)), S(\sigma(X\cdot v_0))\rangle = \langle \sigma(X\cdot v_0), \sigma(X\cdot v_0)\rangle \quad \text{for} \quad \sigma \in G , \quad X \in \mathfrak{p} .$$

By an argument similar to that on eigenmaps, the equivalence classes of full isometric minimal immersions of $(M, (\lambda_k/d)g)$ into spheres are parametrized by the convex set $W_M = \{C \in L_M; C + I$ is positive semi-definite$\}$ in $L_M = \{C \in S^2(V^k); C \perp \sigma(X\cdot v_0)^2, \sigma \in G, X \in \mathfrak{p}\}$.

Let $x_k$: $M \to S_1 \subset V^k$ be the $k$-th standard isometric minimal immersion and let $V_1 = \{X\cdot v_0; X \in \mathfrak{p}\}$. Then $S^2(V_1)$ is contained in $S^2(V^k)$ in a natural manner. Let $L'_M$ be the sum of all $G$-submodules of $S^2(V^k)$ which do not contain any $K$-irreducible factors of $S^2(V_1)$. Then we have:

LEMMA 2.3 (do Carmo and Wallach [3]). *$L'_M$ is contained in $L_M$.*

**3. Irreducible characters of compact Lie groups.** In this section we explain the way to express irreducible characters of a compact Lie group as polynomials of fundamental irreducible characters.

Let $G$ be a simple simply connected compact Lie group and $T$ be a maximal torus of $G$. We denote by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of $G$ and $T$, respectively, and we denote by $\langle\ ,\ \rangle$ a $G$-invariant inner product on $\mathfrak{g}$. Define and fix once for all a lexicographic order $<$ in $\mathfrak{t}$. Let $\Sigma^+(G)$ be the set of all positive roots of $\mathfrak{g}^C$ with respect to $\mathfrak{t}^C$ and $\{\alpha_1, \cdots, \alpha_n\}$ be the set of all simple roots, where $n$ is the rank of $\mathfrak{g}$. We put

$$\boldsymbol{D}(G) = \{H \in \mathfrak{t};\ \langle \alpha, H\rangle \in \boldsymbol{Z} \quad \text{for some} \quad \alpha \in \Sigma^+(G)\} .$$

Take a component $\mathfrak{h}$ of $\mathfrak{t}$-$\boldsymbol{D}(G)$ whose closure contains the origin $o \in \mathfrak{t}$. Then the restriction of the exponential map exp on $\mathfrak{h}$ is a diffeomorphism of $\mathfrak{h}$ onto $\exp(\mathfrak{h}) \subset G$. Let $\{\Lambda_1, \cdots, \Lambda_n\}$ be the system of fundamental weights, i.e., $2\langle \Lambda_i, \alpha_j\rangle/\langle \alpha_j, \alpha_j\rangle = \delta_{ij}$, $1 \leqq i, j \leqq n$. Then the equivalence classes of all complex irreducible representations of $G$ corresponds bijectively to

$$D(G) = \{\Sigma_{j=1}^n m_j \Lambda_j;\ m_j\text{'s are non-negative integers}\} .$$

We denote by $V(\Lambda)$ the corresponding irreducible $G$-module with highest weight $\Lambda \in D(G)$. For a complex $G$-module $V$, we denote by $\chi_V$ its character. For brevity, we denote also by $\chi_\Lambda$ the character $\chi_{V(\Lambda)}$ of $V(\Lambda)$. Put $z_j = \chi_{\Lambda_j}$. Then it is easily seen that each character $\chi_V$ is a polynomial in $z_1, z_2, \cdots, z_n$ with integral coefficients.

Recall the following facts on characters:

(i) The characters are determined by their restriction on $\exp(\mathfrak{h})$.

(ii) An irreducible character is an eigenfunction of the Laplacian $\Delta$ of $G$ with respect to a bi-invariant Riemannian metric.

Let $g$ be the $G$-invariant metric on $G$ induced from the $\mathrm{Ad}(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Then the eigenvalue of $\Delta$ on $\chi_\Lambda$ is given by the following:

LEMMA 3.1. *The eigenvalue $C_\Lambda$ of $\Delta$ on $\chi_\Lambda$ is*

$$C_\Lambda = \langle \Lambda + 2\delta, \Lambda\rangle , \quad \Lambda \in D(G) ,$$

*where* $2\delta = \sum_{j=1}^n \Lambda_j$

For the proof we refer, for instance, to [6].

A function $h$ on $G$ is called a *class function* if it satisfies $h(\sigma x \sigma^{-1}) = h(x)$ for $x, \sigma \in G$. For example, characters are class functions. There exists a differential operator $\partial(\Delta)$ on $\exp(\mathfrak{h})$, called the *radial part* of $\Delta$, such that

$$(\Delta h)|_{\exp(\mathfrak{h})} = \partial(\Delta)(h|_{\exp(\mathfrak{h})}) ,$$

if $h$ is a class function. An explicit expression for $\partial(\Delta)$ is known (cf. [1]). But we will employ another expression.

Consider a polynomial in $n$ variables $z_1, z_2, \cdots, z_n$. For any $\Lambda = \sum_{j=1}^{n} m_j\Lambda_j \in D(G)$, we denote by $z^\Lambda$ the monomial $z_1^{m_1} \cdots z_n^{m_n}$. A polynomial $P(z_1, z_2, \cdots, z_n)$ is said to be of degree $\Lambda$ if

$$P(z_1, z_2, \cdots, z_n) = \sum_{\lambda \leqq \Lambda} a_\lambda z^\lambda \quad \text{with} \quad a_\Lambda \neq 0 .$$

Since $V(\Lambda)$ is contained in $V(\Lambda_1)^{\otimes m_1} \otimes \cdots \otimes V(\Lambda_n)^{\otimes m_n}$ exactly once and the character of $V(\Lambda)^{\otimes m_1} \otimes \cdots \otimes V(\Lambda_n)^{\otimes m_n}$ is $z^\Lambda$, the character $\chi_\Lambda$ of $V(\Lambda)$ is the following monic polynomial of degree $\Lambda$

$$\chi_\Lambda = \sum_{\lambda \leqq \Lambda} a_\lambda z^\lambda , \quad a_\Lambda = 1 , \tag{3.1}$$

Let $\{t_1, \cdots, t_n\}$ be a linear coordinate system on $\mathfrak{h}$. Then it defines a coordinate system on $\exp(\mathfrak{h})$. We take another coordinate system on $\exp(\mathfrak{h})$. In general, characters are complex-valued functions. But if $z_i$ is not real-valued, then there exists $z_j$ such that $z_i = \bar{z}_j$, $i \neq j$ (cf. [4]). So we define $x_1, x_2, \cdots, x_n$ by

$$x_i = \begin{cases} z_i & \text{if} \quad z_i \text{ is real-valued} , \\ \operatorname{Re} z_i & \text{if} \quad z_i = \bar{z}_j , \quad i < j , \\ \operatorname{Im} z_i & \text{if} \quad z_i = \bar{z}_j , \quad j < i . \end{cases}$$

LEMMA 3.2 (Vretare [15]).

$$\partial(x_1, x_2, \cdots, x_n)/\partial(t_1, t_2, \cdots, t_n) \neq 0 \quad on \quad \exp(\mathfrak{h}) .$$

Thus $x = (x_1, x_2, \cdots, x_n)$ defines a local coordinate system on $\exp(\mathfrak{h})$ and $\partial(\Delta)$ is expressed as

$$\partial(\Delta) = \sum_{1 \leqq i \leqq j \leqq n} a_{ij}\partial^2/\partial x_i \partial x_j + \sum_{i \leqq j \leqq n} b_j \partial/\partial x_j , \tag{3.2}$$

where $a_{ij}$ and $b_j$ are $C^\infty$ functions.

LEMMA 3.3. *Assume that $z_1, z_2, \cdots, z_n$ are real-valued. Then we have the following*:

(i) $b_j = C_{\Lambda_j} z_j$ *for* $1 \leqq j \leqq n$.

(ii) *For any $\Lambda \in D(G)$, $\partial(\Delta)z^\Lambda$ is a polynomial of degree $\Lambda$ with the highest term $C_\Lambda z^\Lambda$.*

(iii) *Put $\chi_{\Lambda_i+\Lambda_j} = z_i z_j + \sum_{\lambda < \Lambda_i+\Lambda_j} a_\lambda z^\lambda$. Then we have*

$$(1 + \delta_{ij})a_{ij} = (C_{\Lambda_i+\Lambda_j} - C_{\Lambda_i} - C_{\Lambda_j}) z_i z_j + (C_{\Lambda_i+\Lambda_j} - \partial(\Delta))\Big(\sum_{\lambda < \Lambda_i+\Lambda_j} a_\lambda z^\lambda\Big) . \tag{3.3}$$

PROOF. (i) is clear, since $z_j$ is an eigenfunction of $\partial(\Delta)$ corresponding to the eigenvalue $C_{\Lambda_j}$.

(ii) is proved by induction. Assume that (ii) holds for $\lambda \in D(G)$, $\lambda < \Lambda$. Then, since $\chi_\Lambda$ is a monic polynomial of degree $\Lambda$ by (3.1) and is an eigenfunction of $\partial(\Delta)$ corresponding to the eigenvalue $C_\Lambda$, we have

$$\partial(\Delta)\chi_\Lambda = \partial(\Delta)z^\Lambda + \partial(\Delta)(\sum_{\lambda<\Lambda} a_\lambda z^\Lambda) = C_\Lambda(z^\Lambda + \sum_{\lambda<\Lambda} a_\lambda z^\lambda) .$$

Comparing both sides and then by the induction hypothesis, we have

$$\partial(\Delta)z^\Lambda = C_\Lambda z^\Lambda + (\text{polynomial of degree} < \Lambda) .$$

Namely, (ii) holds for $\Lambda \in D(G)$. Obviously (ii) holds for $\Lambda = 0$. Thus (ii) is proved.

(iii) Since the character $\chi_{\Lambda_i+\Lambda_j}$ is an eigenfunction of $\partial(\Delta)$ corresponding to the eigenvalue $C_{\Lambda_i+\Lambda_j}$, we have (3.3). q.e.d.

REMARK. (i) $a_{ij}$ a polynomial of degree $\Lambda_i + \Lambda_j$, $1 \leqq i \leqq j \leqq n$, since the second term on the right hand side of (3.3) is a polynomial of degree less than $\Lambda_i + \Lambda_j$ by (ii) and $(C_{\Lambda_i+\Lambda_j} - C_{\Lambda_i} - C_{\Lambda_j}) = 2\langle \Lambda_i, \Lambda_j\rangle \neq 0$, $1 \leqq i \leqq j \leqq n$.

(ii) By Lemma 3.3, we can inductively determine the coefficients $a_{ij}$ and $b_j$ in (3.2).

(iii) The assumption of Lemma 3.3 is not essential. But for our purpose it is sufficient.

Now we explain the way of calculating the coefficients $a_\lambda$'s in the expression (3.1) of $\chi_\Lambda$. Let us number $\lambda$'s $\in D(G)$, which appear in (3.1), as

$$\Lambda = \lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_N .$$

Note that $\lambda_0, \lambda_1, \cdots, \lambda_N$ must be the weights of $V(\Lambda)$. We know that $a_{\lambda_0} = 1$. We go on inductively. Assume that we have first $r$ coefficients $1 = a_{\lambda_0}, a_{\lambda_1}, \cdots, a_{\lambda_{r-1}}$, $1 \leqq r \leqq N$. Put $P_r = \sum_{j=0}^{r-1} a_{\lambda_j} z^{\lambda_j}$ and $Q_r = \sum_{j=r}^{N} a_{\lambda_j} z^{\lambda_j}$. Since $\chi_\Lambda = P_r + Q_r$ is an eigenfunction of $\partial(\Delta)$ corresponding to the eigenvalue $C_\Lambda$, we have

$$\partial(\Delta)P_r - C_\Lambda P_r = -\partial(\Delta)Q_r + C_\Lambda Q_r . \tag{3.4}$$

Let $\alpha z^\mu$ be the highest term on the left hand side. Since $\partial(\Delta)Q_r$ is a polynomial in $z_1, z_2, \cdots, z_n$ of degree $\lambda_r$ and $C_\Lambda - C_{\Lambda_r} \neq 0$ [6, p. 191], the highest term on the right hand side is $(C_\Lambda - C_{\lambda_r})a_{\lambda_r} z^{\lambda_r}$. Comparing the highest terms of both sides of (3.4), we have $\mu = \lambda_r$ and $a_{\lambda_r} = \alpha/(C_\Lambda - C_{\lambda_r})$. Thus we have the following:

LEMMA 3.4. *Let $V(\Lambda)$ be the irreducible G-module with highest weight $\Lambda \in D(G)$. Assume that*

$$\chi_\Lambda = \sum_{j=0}^{r-1} a_{\lambda_j} z^{\lambda_j} + (\textit{terms of degree} < \lambda_{r-1}) ,$$
$$a_{\lambda_0} = 1 , \quad \Lambda = \lambda_0 > \lambda_1 > \cdots > \lambda_r .$$

*Put* $P_r = \sum_{j=0}^{r-1} a_{\lambda_j} z^{\lambda_j}$ *and let* $\alpha z^{\mu}$ *be the highest term of* $\partial(\Delta)P_r - C_\Lambda P_r$. *Then we have*

(i) $\mu = \lambda_r$.

(ii) $\chi_\Lambda = \sum_{j=0}^{r-1} a_{\lambda_j} z^{\lambda_j} + (\alpha/(C_\Lambda - C_{\lambda_r}))z^{\lambda_r} + (\text{terms of degree} < \lambda_r)$.

In order to decompose the symmetric tensor product $S^2(V^k)$, we need the following:

LEMMA 3.5. *Let* $\chi_\Lambda^{(2)}$ *be the character of* $S^2(V(\Lambda))$. *Then*

$$\chi_\Lambda^{(2)}(\sigma) = (\chi_\Lambda(\sigma)^2 + \chi_\Lambda(\sigma^2))/2 \quad \text{for} \quad \sigma \in G . \tag{3.5}$$

For the proof of Lemma 3.5, we refer to [14].

**4. Quaternion projective spaces.** In this section, we use the following notation:

$$G = Sp(n) = \{\sigma \in U(2n);\, {}^t\sigma J_n \sigma = J_n\}\,, \quad n \geqq 3\,,$$

where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and $I_n$ is the $n \times n$ identity matrix.

$$K = Sp(1)\times Sp(n-1) = \left\{\begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix};\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(1),\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n-1)\right\}$$

$$\begin{aligned} \mathfrak{g} = \mathfrak{sp}(n) &= \{X \in \mathfrak{u}(2n);\, {}^tXJ_n + J_nX = 0\} \\ &= \left\{\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix};\, A,\, B \in M_n(C),\, {}^t\bar{A} + A = 0,\, B = {}^tB\right\}, \end{aligned}$$

$$\begin{aligned} \mathfrak{k} &= \mathfrak{sp}(1)\times\mathfrak{sp}(n-1) \\ &= \left\{\begin{pmatrix} x & 0 & y & 0 \\ 0 & X & 0 & Y \\ -\bar{y} & 0 & \bar{x} & 0 \\ 0 & -\bar{Y} & 0 & \bar{X} \end{pmatrix};\ \begin{matrix} x \in (-1)^{1/2}\boldsymbol{R},\ y \in \boldsymbol{C},\ X,\ Y \in M_{n-1}(\boldsymbol{C}), \\ {}^tX + \bar{X} = 0,\ Y = {}^tY \end{matrix}\right\} \end{aligned}$$

$$B(X, Y) = -\mathrm{Trace}(XY)\,, \quad X,\, Y \in \mathfrak{g}$$

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & Z & 0 & W \\ -{}^t\bar{Z} & 0 & {}^tW & 0 \\ 0 & -\bar{W} & 0 & \bar{Z} \\ -{}^t\bar{W} & 0 & -{}^tZ & 0 \end{pmatrix};\ Z, W \in M(1, n-1, \boldsymbol{C})\right\}:$$

the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $B$,

$$T = \left\{ \begin{pmatrix} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_n & & & \\ & & & \alpha_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \alpha_n^{-1} \end{pmatrix} ;\ \alpha_i \in \boldsymbol{C},\ \alpha_i \bar{\alpha}_i = 1,\ 1 \leqq i \leqq n \right\} ,$$

$$\mathfrak{t} = \{H(x_1, \cdots, x_n);\ x_i \in \boldsymbol{R},\ 1 \leqq i \leqq n\}:$$

the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$, where

$$H(x_1, \cdots, x_n) = (-1)^{1/2} \begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_n & & & \\ & & & -x_1 & & \\ & & & & \ddots & \\ & & & & & -x_n \end{pmatrix} .$$

We can identify $P^{n-1}(\boldsymbol{H})$ with $G/K$ and introduce a $G$-invariant Riemannian metric induced from the inner product $B(X, Y)$, $X, Y \in \mathfrak{p}$.

Define an element $\varepsilon_i$ of $\mathfrak{t}$ by

$$\varepsilon_i = H(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$$

and introduce a lexicographic order $>$ in $\mathfrak{t}$ by

$$\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > 0 .$$

Let $\Sigma^+(G)$ (resp. $\Sigma^+(K)$) be the set of positive roots of the complexification $\mathfrak{g}^C$ (resp. $\mathfrak{k}^C$) with respect to $\mathfrak{t}^C$. Then we have

$$\Sigma^+(G) = \{\varepsilon_i \pm \varepsilon_j;\ 1 \leqq i < j \leqq n\} \cup \{2\varepsilon_i;\ 1 \leqq i \leqq n\} ,$$
$$\Sigma^+(K) = \{\varepsilon_i \pm \varepsilon_j;\ 2 \leqq i < j \leqq n\} \cup \{2\varepsilon_i;\ 1 \leqq i \leqq n\} .$$

Then the dominant integral forms for $G$ (resp. $K$) with respect to $>$ are

$$D(G) = \left\{\sum_{i=1}^n a_i\varepsilon_i;\ a_i \in \boldsymbol{Z},\ a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0\right\} ,$$
$$D(K) = \left\{\sum_{i=1}^n b_i\varepsilon_i;\ b_1 \in \boldsymbol{Z},\ b_1 \geqq 0,\ b_2 \geqq b_3 \geqq \cdots \geqq b_n \geqq 0\right\} .$$

We put

$$\mathfrak{h} = \left\{\sum_{i=1}^n a_i\varepsilon_i;\ 1 > a_1 > a_2 > \cdots > a_n > 0\right\} ,$$

$$\delta_G = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n .$$

The complexification $\mathfrak{p}^C$ of $\mathfrak{p}$ is the irreducible $K$-module with highest weight $\varepsilon_1 + \varepsilon_2$. Then the symmetric tensor product $S^2(\mathfrak{p}^C)$ is decomposed as a $K$-module as (cf. [14])

$$S^2(\mathfrak{p}^C) = V(2\varepsilon_1 + 2\varepsilon_2) + V(\varepsilon_2 + \varepsilon_3) + V(0) .$$

LEMMA 4.1 (Urakawa [14]). (1) *Let* $n = 3$. *Then every $G$-module over $\boldsymbol{C}$ which contains one of the $K$-irreducible factors of* $S^2(\mathfrak{p})^C$ *has the highest weight* $\sum_{i=1}^3 a_i\varepsilon_i$, *where the triple* $(a_1, a_2, a_3)$ *is one of the following*:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a_1$ | $k+2$ | $k+3$ | $k+1$ | $k+4$ | $k+2$ | $k$ |
| $a_2$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ |
| $a_3$ | 2 | 1 | 1 | 0 | 0 | 0 |
| | $k \geqq 2$ | $k \geqq 1$ | $k \geqq 1$ | $k \geqq 0$ | $k \geqq 1$ | $k \geqq 0$ |

(2) *Let* $n \geqq 4$. *If* $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$ *satisfy one of the conditions*

(i) $a_3 \geqq 3$ (ii) $a_4 \geqq 2$ *or* (iii) $a_i \geqq 1$ *for some* $5 \leqq i \leqq n$, *then the $G$-module* $V(\sum_{i=1}^n a_i\varepsilon_i)$ *contains none of the $K$-irreducible components of* $S^2(\mathfrak{p}^C)$.

Now we describe the radial part of the Laplacian $\Delta$ of $Sp(n)$ with respect to the fundamental irreducible characters. We put $\Lambda_j = \sum_{i=1}^j \varepsilon_i \in D(G)$. Then $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_n\}$ is the fundamental weight system of $\mathfrak{sp}(n)$. It is known that each character $z_i$ of $V(\Lambda_i)$ is real-valued. Thus we denote by $x_i$ the character of $V(\Lambda_i)$. We also denote by $x_i$ the restriction of $x_i$ to $\exp(\mathfrak{h})$ and its pull back on $\mathfrak{h}$ by $\exp\colon \mathfrak{h} \to T$.

Let $g$ be the $G$-invariant Riemannian metric on $G$ induced by $B$. We denote by $\Delta^{(G,g)}$ the Laplacian of $(G, g)$. Then we have the following:

LEMMA 4.2. *The character* $\chi_\Lambda$ *of* $V(\Lambda)$ *for* $\Lambda \in D(G)$ *is an eigenfunction of* $\Delta^{(G,g)}$ *with eigenvalue*

$$C_\Lambda = \sum_{i=1}^n (a_i^2 + 2(n+1-i)a_i) , \quad \Lambda = \sum_{i=1}^n a_i\varepsilon_i .$$

The radial part $\partial(\Delta^{(G,g)})$ is a differential operator of second order with polynomial coefficients. We have the first order term of $\partial(\Delta)$ easily by Lemma 3.3(i). But to get an explicit form of the second order terms we need the following:

LEMMA 4.3 (Tsukamoto [13]). *An* $Sp(n)$-*module* $V(\Lambda_r) \otimes V(\Lambda_s)$, $1 \leqq r \leqq s \leqq n$, *decomposes into irreducible modules as*

$$V(\Lambda_r)\otimes V(\Lambda_s) = \sum_{(i,j)\in S} V(\Lambda_i + \Lambda_j)\ ,$$

*where the set S consists of pairs of non-negative integers* $(i, j)$ *satisfying* $s - r \leqq j - i \leqq 2n - s - r$, $i + j \leqq r + s$ *and* $i + j \equiv r + s \pmod 2$.

Using the above lemma we can express the character $\chi_{\Lambda_i+\Lambda_j}$ as a polynomial in $x_1, x_2, \cdots, x_n$ and by (3.3) we can obtain the coefficients of the second order terms of $\partial(\Delta^{(Sp(n),g)})$.

LEMMA 4.4. *The radial part* $\partial(\Delta^{(Sp(n),g)})$ *is*

$$\begin{aligned}
&\partial(\Delta^{(Sp(n),g)})\\
&\quad = (2n+1)x_1\partial/\partial x_1 + 4nx_2\partial/\partial x_2 + (6n-3)x_3\partial/\partial x_3 + (8n-8)x_4\partial/\partial x_4\\
&\qquad + (10n-15)x_5\partial/\partial x_5 + (\textit{terms in } \partial/\partial x_6, \cdots, \partial/\partial x_n)\\
&\qquad + x_1^2\partial^2/\partial x_1^2 + 2x_1x_2\partial^2/\partial x_1\partial x_2 + 2x_1x_3\partial^2/\partial x_1\partial x_3\\
&\qquad + 2x_1x_4\partial^2/\partial x_1\partial x_4 + 2x_1x_5\partial^2/\partial x_1\partial x_5\\
&\qquad + (2x_2^2 - 2x_1x_3 - 2nx_1^2)\partial^2/\partial x_2^2 + (4x_2x_3 - 6x_1x_4 - (4n-2)x_1x_2)\partial^2/\partial x_2\partial x_3\\
&\qquad + (4x_2x_4 - 8x_1x_5 - 4(n-1)x_1x_3)\partial^2/\partial x_2\partial x_4\\
&\qquad + (6x_2x_5 - 10x_1x_6 - (4n-6)x_1x_4)\partial^2/\partial x_2\partial x_5\\
&\qquad + \left.\begin{cases} 3x_3^2 - 2x_2x_4 - 2(n-1)x_2^2 - 4x_1x_5 - 2nx_1^2\ , & n \geqq 4\\ 3x_3^2 - 4x_2^2 + 4x_1x_3 - 6x_1^2\ , & n = 3\end{cases}\right\}\partial^2/\partial x_3^2\\
&\qquad + (6x_3x_4 - 16x_2x_5 - (10n+1)x_2x_3 + 10x_1x_6 - (2n+3)x_1x_4\\
&\qquad + (4n-2)x_1x_2)\partial^2/\partial x_3\partial x_4\\
&\qquad + (\textit{terms in } \partial^2/\partial x_1\partial x_6, \cdots, \partial^2/\partial x_2\partial x_6, \cdots, \partial^2/\partial x_3\partial x_5, \cdots)\ ,
\end{aligned}$$

*where the terms of degree* $<\Lambda_n$ *are omitted in the coefficients of the second order terms.*

Let $V^k$ be the $k$-th eigenspace of $\Delta^{(M,g)}$ and $(V^k)^C$ be its complexification. Then $(V^k)^C$ is an irreducible $Sp(n)$-module with highest weight $k(\varepsilon_1 + \varepsilon_2) = k\Lambda_2$. Thus the restriction to $\mathfrak{h}$ of its character is a polynomial of degree $k\Lambda_2$.

We look for all irreducible $Sp(n)$-submodules of $S^2(V(k\Lambda_2))$ whose highest weights are greater than or equal to $4\Lambda_1 + (k-8)\Lambda_2 + 4\Lambda_3$. For this purpose, we express $\chi^{(2)}_{k\Lambda_2}$ as a polynomial in $x_1, \cdots, x_n$ in the following manner:

( i ) We calculate the character $\chi_{k\Lambda_2}$ as a polynomial in $x_1, \cdots, x_n$ by using Lemmas 3.4 and 4.4.

(ii) We denote by $y_j$ the function on $\mathfrak{h}$ defined by $y_j(H) = x_j(2H)$ for $H \in \mathfrak{h}$ and find the expression for $y_j$ as a polynomial in $x_1, \cdots, x_n$ for $1 \leqq j \leqq n$.

(iii) By Lemma 3.6, the character $\chi^{(2)}_{k\Lambda_2}$ is the polynomial in $x_1, \cdots, x_n$ given by

$$\chi^{(2)}_{k\Lambda_2} = (1/2)((\chi_{k\Lambda_2}(x_1, \cdots, x_n))^2 + \chi_{k\Lambda_2}(y_1, \cdots, y_n)) . \tag{4.1}$$

By Lemma 3.4 and 4.4, we calculate inductively the coefficients of the character $\chi_{k\Lambda_2}$ as a polynomial in $x_1, \cdots, x_n$ up to $x_1^4 x_2^{k-8} x_3^4$.

$$\begin{aligned}\chi_{k\Lambda_2} = {} & x_2^k - (k-1)x_1x_2^{k-3}x_3 + ((k-2)(k-3)/2)x_1^2x_2^{k-4}x_3^2 \\ & + (k-2)x_1^2x_2^{k-3}x_4 - x_1^2x_2^{k-2} \\ & - ((k-3)(k-4)(k-5)/6)x_1^3x_2^{k-6}x_3^3 \\ & - (k-3)(k-4)x_1^3x_2^{k-5}x_3x_4 \\ & - (k-3)x_1^3x_2^{k-4}x_5 + \begin{Bmatrix} (k-3)x_1^3x_2^{k-4}x_3 , & n > 4 \\ 2(k-3)x_1^3x_2^{k-4}x_3 , & n = 3 \end{Bmatrix} \\ & + ((k-4)(k-5)(k-6)(k-7)/24)x_1^4x_2^{k-8}x_3^4 \\ & + (\text{terms of degree} < 4\Lambda_1 + (k-8)\Lambda_2 + 4\Lambda_3) .\end{aligned} \tag{4.2}$$

Since degree of the terms which appear in the expression for $\chi_{k\Lambda_2}$ are weights of $V(k\Lambda_2)$, we know that $x_2^k, x_1x_2^{k-2}x_3, \cdots, x_1^4x_2^{k-8}x_3^4$ and the terms of degree $< 4\Lambda_1 + (k-8)\Lambda_2 + 4\Lambda_3$ appear in the expression for $\chi_{k\Lambda_2}$. When we apply the terms of $\partial(\Delta^{(Sp(n),g)})$ which is not given explicitly in Lemma 4.4 to the monomials $x_2^k, \cdots, x_1^4x_2^{k-8}x_3^4$, the degree will be lower than $4\Lambda_1 + (k-8)\Lambda_2 + 4\Lambda_3$. Thus to obtain (4.2), the expression for $\partial(\Delta^{(Sp(n),g)})$ in Lemma 4.4 is sufficient.

Next we find the expression for $y_j$ as a polynomial in $x_1, \cdots, x_n$. For any $\lambda \in \mathfrak{t}$, we denote by by $e^{\lambda}$ the function on $\mathfrak{t}$ defined by $e^{\lambda}(H) = e^{2\pi i\langle H, \lambda\rangle}$ for $H \in \mathfrak{t}$. Put $\omega_{\lambda} = \sum_{\lambda \in W} e^{\sigma\lambda}$, where $W$ is the Weyl group of $G$. Counting the multiplicity of the weights of $V(\Lambda_1)$ (cf. [6]), we have $x_1 = \omega_1$. Thus we have $y_1(H) = x_1(2H) = \omega_{2\Lambda_1}(H)$. On the other hand, we have

$$(x_1(H))^2 = (\omega_{2\Lambda_1} + 2\omega_2 + 2n)(H) ,$$
$$x_2(H) = (\omega_{2\Lambda_1} + 2(n-1))(H) .$$

Thus we have,

$$y_1 = x_1^2 - 2x_2 + 2 .$$

Similarly we have,

$$y_2 = x_2^2 - 2x_1x_3 - 2x_1^2 + (\text{terms of degree} < \Lambda_n) ,$$

$$y_3 = \begin{cases} x_3^2 - 2x_2x_4 - 2x_2^2 + x_1x_5 + 4x_1x_3 + 2x_1^2 + (\text{terms of degree} < \Lambda_n) & \text{if } n \geqq 5 , \\ x_3^2 - 2x_2x_4 - 2x_2^2 + 4x_1x_3 + 2x_1^2 + (\text{terms of degree} < \Lambda_n) & \text{if } n = 4 , \\ x_3^2 - 2x_2^2 + 2x_1x_3 + 2x_1^2 + (\text{terms of degree} < \Lambda_n) & \text{if } n = 3 . \end{cases}$$

Note that $y_j$ is a polynomial in $x_1, \cdots, x_n$ of degree $2\Lambda_j$. When we substitute $y_j$'s into (4.2) instead of $x_j$'s, the degree of $y_1^2 y_2^{k-3} y_4$ is $4\Lambda_1 + 2(k-3)\Lambda_2 + 2\Lambda_4 = 2k\varepsilon_1 + (2k-4)\varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4$ which is less than $4\Lambda_1 + (2k-4)\Lambda_2 + 4\Lambda_3 = 2k\varepsilon_1 + (2k-4)\varepsilon_2 + 4\varepsilon_3$. Thus, for our purpose, there are no need to have expressions for $y_4, y_5, \cdots$ as polynomials in $x_1, x_2, \cdots, x_n$ by (4.2). Substitute $y_i$'s into (4.2). Then, by (4.1), we have

$$
\begin{aligned}
(4.3) \qquad \chi^{(2)}_{k\Lambda_2} = {} & x_2^{2k} - (2k-1)x_1x_2^{2k-2}x_3 + (2k^2-5k+4)x_1^2x_2^{2k-4}x_3^2 \\
& + (2k-3)x_1^2x_2^{2k-3}x_4 - 2x_1^2x_2^{2k-2} \\
& - ((4k^3-24k^2+53k-45)/3)x_1^3x_2^{2k-6}x_3^3 \\
& - (4k^2-16k+18)x_1^3x_2^{2k-5}x_3x_4 - (2k-4)x_1^3x_2^{2k-4}x_5 \\
& + (4k-6)x_1^3x_2^{2k-4}x_3 \\
& + ((4k^4-44k^3+191k^2-397k+342)/6)x_1^4x_2^{2k-8}x_3^4 \\
& + (\text{terms of degree} < 4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3)\,.
\end{aligned}
$$

By (4.2) and (4.3), we have

$$
\begin{aligned}
(4.4) \qquad \chi^{(2)}_{k\Lambda_2} - \chi_{2k\Lambda_2} = {} & x_1^2x_2^{2k-4}x_3^2 - x_1^2x_2^{2k-3}x_4 - x_1^2x_2^{2k-2} \\
& - (2k-5)x_1^2x_2^{2k-6}x_3^3 + (2k-6)x_1^3x_2^{2k-5}x_3x_4 \\
& + x_1^3x_2^{2k-4}x_5 + \begin{Bmatrix} (2k-3)x_1^3x_2^{2k-4}x_3\,, & n \geqq 4 \\ (2k-4)x_1^3x_2^{2k-4}x_3\,, & n = 3 \end{Bmatrix} \\
& + (2k^2-13k+22)x_1^4x_2^{2k-8}x_3^4 \\
& + (\text{terms of degree} < 4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3)\,.
\end{aligned}
$$

Thus we have the following decomposition for $k \geqq 2$;

$$S^2(V(k\Lambda_2)) = V(2k\Lambda_2) + V(2\Lambda_1 + 2(k-2)\Lambda_2 + 2\Lambda_3) + \cdots\,.$$

Since $V(2\Lambda_1 + 2(k-2)\Lambda_2 + 2\Lambda_3)$ is not a class one representation of $(Sp(n), Sp(1)\times Sp(n-1))$, $L_E^C$ contains it by Lemma 2.1. By Weyl's dimension formula we have

$$\dim_C V(2\Lambda_1 + 2(k-2)\Lambda_2 + 2\Lambda_3) \geqq \dim_C V(2\Lambda_1 + 2\Lambda_3) \geqq 1078\,,$$

if $k \geqq 2$ and $n \geqq 3$. On the other hand, we have

$$S^2(V(\Lambda_2)) = \begin{cases} V(2\Lambda_2) + V(\Lambda_2) + V(0) & \text{if} \quad n = 3\,, \\ V(2\Lambda_2) + V(\Lambda_4) + V(\Lambda_2) + V(0) & \text{if} \quad n \geqq 4\,. \end{cases}$$

Thus, when $k = 1$, we have $L_E^C = 0$ for $n = 3$ and $\dim_C L_E^C = \dim_C V(\Lambda_4) \geqq 42$ for $n \geqq 4$. Summing up, we have:

THEOREM A. *Let $M = Sp(n)/Sp(1)\times Sp(n-1)$ be the quaternion projective space $P^{n-1}(\boldsymbol{H})$ with an $Sp(n)$-invariant Riemannian metric. Then*

(i) $\dim L_E = 0$ *if* $k = 1$ *and* $n = 3$,
$\dim L_E \geqq 42$ *if* $k = 1$ *and* $n \geqq 4$,
(ii) $\dim L_E \geqq 1078$ *if* $k \geqq 2$ *and* $n \geqq 3$.

Furthermore, we calculate the character of $V(2\Lambda_1 + (2k-4)\Lambda_2 + 2\Lambda_3)$ as

$$\begin{aligned}\chi_{2\Lambda_1+(2k-4)\Lambda_2+2\Lambda_3} = {} & x_1^2x_2^{2k-4}x_3^2 - x_1^2x_2^{2k-3}x_4 - x_1^2x_2^{2k-2} \\ & - (2k-5)x_1^3x_2^{2k-6}x_3^3 + (2k-6)x_1^3x_2^{2k-5}x_3x_4 \\ & + x_1^3x_2^{2k-4}x_5 + \begin{Bmatrix}(2k-4)x_1^3x_2^{2k-4}x_3\,, & n \geqq 4 \\ (2k-5)x_1^3x_2^{2k-4}x_3\,, & n = 3\end{Bmatrix} \\ & + (k-3)(2k-7)x_1^4x_2^{2k-8}x_3^4 \\ & + (\text{terms of degree} < 4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3)\,.\end{aligned}$$

Thus we have from (4.4)

$$\begin{aligned}\chi_{k\Lambda_2}^{(2)} - \chi_{2k\Lambda_2} - \chi_{2\Lambda_1+(2k-4)\Lambda_2+2\Lambda_3} = {} & x_1^3x_2^{2k-4}x_3 + x_1^4x_2^{2k-8}x_3^4 \\ & + (\text{terms of degree} < 4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3)\,.\end{aligned} \tag{4.5}$$

By a simple calculation, we have

$$\begin{aligned}\partial(\Delta)x_1^3x_2^{2k-4}x_3 = {} & C_{3\Lambda_1+(2k-4)\Lambda_2+\Lambda_3}x_1^3x_2^{2k-4}x_3 \\ & + (\text{terms of degree} < 4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3)\,.\end{aligned}$$

Thus by (4.5) we have

$$\begin{aligned}& \chi_{k\Lambda_2}^{(2)} - \chi_{2k\Lambda_2} - \chi_{2\Lambda_1+(2k-4)\Lambda_2+2\Lambda_3} - \chi_{3\Lambda_1+(2k-4)\Lambda_2+\Lambda_3} \\ & \quad = x_1^4x_2^{2k-8}x_3^4 + (\text{terms of degree} < 4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3)\,.\end{aligned}$$

Finally we have the following decomposition if $k \geqq 4$:

$$\begin{aligned}S^2(V(k\Lambda_2)) = {} & V(2k\Lambda_2) + V(2\Lambda_1 + (2k-4)\Lambda_2 + 2\Lambda_3) \\ & + V(3\Lambda_1 + (2k-4)\Lambda_2 + \Lambda_3) + V(4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3) + \cdots\,.\end{aligned} \tag{4.6}$$

By Lemma 4.1, $V(4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3) = V(2k\varepsilon_1 + (2k-4)\varepsilon_2 + 4\varepsilon_3)$ contains none of the $K$-irreducible components of $S^2(\mathfrak{p}^C)$. By Weyl's dimension formula, we have

$$\dim_C V(4\Lambda_1 + (2k-8)\Lambda_2 + 4\Lambda_3) \geqq \dim_C V(4\Lambda_1 + 4\Lambda_3) \geqq 41140\,,$$

if $n \geqq 3$ and $k \geqq 4$. When $k = 3$, we have the following decomposition if $n \geqq 4$:

$$S^2(V(3\Lambda_2)) = \begin{cases} V(6\Lambda_2) + V(2\Lambda_1 + 2\Lambda_2 + 2\Lambda_3) + V(3\Lambda_1 + 2\Lambda_2 + \Lambda_3) \\ \quad + V(4\Lambda_2 + 2\Lambda_2) + V(5\Lambda_2) + V(\Lambda_1 + 3\Lambda_2 + \Lambda_3) \\ \quad + V(2\Lambda_1 + \Lambda_2 + 2\Lambda_3) + V(3\Lambda_1 + \Lambda_2 + \Lambda_3) + V(4\Lambda_1 + \Lambda_2) \\ \quad + V(4\Lambda_2) + V(\Lambda_1 + 3\Lambda_3) + \cdots, \quad \text{if} \quad n = 3 , \\ V(6\Lambda_2) + V(2\Lambda_1 + 2\Lambda_2 + 2\Lambda_3) + V(3\Lambda_1 + 2\Lambda_2 + \Lambda_3) \\ \quad + V(4\Lambda_1 + 2\Lambda_2) + V(4\Lambda_2 + \Lambda_4) + (5\Lambda_2) \\ \quad + V(2\Lambda_1 + 2\Lambda_3 + \Lambda_4) + \cdots . \end{cases}$$

By Lemma 4.1, $V(\Lambda_1 + 3\Lambda_3) = V(4\varepsilon_1 + 3\varepsilon_2 + 3\varepsilon_3)$ for $n = 3$ and $V(2\Lambda_1 + 2\Lambda_3 + \Lambda_4) = V(5\varepsilon_1 + 3\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4)$ for $n \geqq 4$ contain none of the $K$-irreducible components of $S^2(\mathfrak{p}^C)$. By Weyl's dimension formula, we have

$$\dim_C V(\Lambda_1 + \Lambda_3) = 1386 \qquad \text{if} \quad n = 3 ,$$
$$\dim_C V(2\Lambda_1 + 2\Lambda_3 + \Lambda_4) \geqq 21344 \quad \text{if} \quad n \geqq 4 .$$

Thus by Lemma 2.3, we have the following:

THEOREM B. *Let $M = P^{n-1}(\boldsymbol{H})$ be the quaternion projective space with an $Sp(n)$-invariant Riemannian metric. Then*

(i) $\dim L_M = 0$ *if* $k = 1$ *and* $n \geqq 3$,

(ii) $\dim L_M \geqq 1386$ *if* $k \geqq 3$ *and* $n \geqq 3$.

REMARK. When $k = 2$ and $n = 3$, we have the decomposition

$$S^2(V(2\Lambda_2)) = V(4\Lambda_2) + V(2\Lambda_1 + 2\Lambda_3) + V(3\Lambda_1 + \Lambda_3) + V(4\Lambda_1) + V(3\Lambda_2) + V(\Lambda_1 + \Lambda_2 + \Lambda_3) + V(2\Lambda_2) + V(\Lambda_2) + V(0) .$$

Thus by Lemma 4.1, we have $\dim L'_M = 0$. But we cannot say anything about $\dim L_M$.

**5. The Cayley projective plane.** Let $G = F_4$, $K = Spin(9)$ and let $T$ be a maximal torus of $Spin(9)$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$ the Lie algebras of $G$, $K$ and $T$, respectively. Let $B$ be a $G$-invariant inner product in $\mathfrak{g}$ and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $B$. Then we can identify the Cayley projective plane $P^2(Ca)$ with $G/K$ and introduce a $G$-invariant Riemannian metric induced from the inner product $B(X, Y)$ for $X, Y \in \mathfrak{p}$.

Under suitable choise of an orthogonal base $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ of $\mathfrak{t}$, the set $\Sigma^+(G)$ (resp. $\Sigma^+(K)$) of positive roots of $G$ (resp. $K$) with respect to the lexicographic order defined by $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \varepsilon_4 > 0$ are

$$\Sigma^+(G) = \{\varepsilon_i ; 1 \leqq i \leqq 4\} \cup \{\varepsilon_i \pm \varepsilon_j ; 1 \leqq i < j \leqq 4\} \cup \left\{(1/2)\sum_{i=1}^{4} a_i\varepsilon_i ; a_i = \pm 1, 1 \leqq i \leqq 4\right\} ,$$

$$\Sigma^+(K) = \{\varepsilon_i \pm \varepsilon_j; 1 \leqq i < j \leqq 4\}$$
$$\cup \left\{(1/2)\sum_{i=1}^{4} a_i\varepsilon_i; a_i = \pm 1, 1 \leqq i \leqq 4, \prod_{i=1}^{4} a_i = -1\right\} .$$

The set of dominant integral forms for $G$ (resp. $K$) are

$$D(G) = \left\{\sum_{i=1}^{4} a_i\varepsilon_i; a_1 \geqq a_2 \geqq a_3 \geqq a_4 \geqq 0, a_1 \geqq a_2 + a_3 + a_4, \right.$$
$$\left. 2a_1, a_1 - a_2, a_2 - a_3, a_3 - a_4 \in \boldsymbol{Z}\right\} ,$$
$$D(K) = \left\{\sum_{i=1}^{4} b_i\varepsilon_i; b_1 \geqq b_2 \geqq b_3 \geqq |b_4|, b_1 \geqq b_2 + b_3 + b_4, \right.$$
$$\left. 2b_1, b_1 - b_2, b_2 - b_3, b_2 - b_4 \in \boldsymbol{Z}\right\} .$$

We put

$$\mathfrak{h} = \left\{\sum_{i=1}^{4} a_i\varepsilon_i; 1 \geqq a_1 + a_2, a_2 \geqq a_3 \geqq a_4 \geqq 0, a_1 \geqq a_2 + a_3 + a_4\right\} ,$$
$$\delta_G = (11\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_1)/2 .$$

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then $\mathfrak{p}^C$ is the irreducible $K$-module with highest weight $\varepsilon_1$ and the symmetric tensor product $S^2(\mathfrak{p}^C)$ is decomposed as

$$S^2(\mathfrak{p}^C) = V(2\varepsilon_1) + V((\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2) + V(0) .$$

LEMMA 5.1 (Mashimo [7]). *Every $G$-module over $\boldsymbol{C}$ which contains one of the $K$-irreducible component of $S^2(\mathfrak{p}^C)$ has the highest weight $\sum_{i=1}^{4} a_i\varepsilon_i$, where the quadruple $(a_1, a_2, a_3, a_4)$ is one of the following*:

| | | | | | | |
|---|---|---|---|---|---|---|
| $a_1$ | $k/2$ | $k/2$ | $k$ | $k$ | $k$ | $k$ |
| $a_2$ | 3/2 | 1/2 | 1 | 2 | 1 | 0 |
| $a_3$ | 1/2 | 1/2 | 1 | 0 | 0 | 0 |
| $a_4$ | 1/2 | 1/2 | 1 | 0 | 0 | 0 |
| | $k \geqq 5$ | $k \geqq 3$ | $k \geqq 3$ | $k \geqq 2$ | $k \geqq 2$ | $k \geqq 0$ |

Now we describe the radial part of the Laplacian $\Delta$ of $F_4$ with respect to the fundamental irreducible characters. We put

$$\Lambda_1 = \varepsilon_1 + \varepsilon_2 ,$$
$$\Lambda_2 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 ,$$
$$\Lambda_3 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2 ,$$
$$\Lambda_4 = \varepsilon_1 .$$

Then $\{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$ is the fundamental weight system of $\mathfrak{g}$. It is known that each character $z_i$ of $V(\Lambda_i)$ is real-valued. So we denote it by $x_i$. We denote also by $x_i$ the restriction of $x_i$ to $\exp(\mathfrak{h})$ and its pull back on $\mathfrak{h}$ by $\exp\colon \mathfrak{h} \to T$.

Let $g$ be the $G$-invariant Riemannian metric on $G$ induced by $B$. We denote by $\Delta^{(G,g)}$ the Laplacian of $(G, g)$. Then we have the following:

LEMMA 5.2. *The character* $\chi_\Lambda$ of $V(\Lambda)$ *for* $\Lambda = \sum_{i=1}^{4} a_i\varepsilon_i \in D(G)$ *is an eigenfunction of* $\Delta^{(G,g)}$ *with eigenvalue*

$$C_\Lambda = a_1^2 + a_2^2 + a_3^2 + a_4^2 + 11a_1 + 5a_2 + 3a_3 + a_4 .$$

LEMMA 5.3. *The radial part of* $\partial(\Delta^{(F_4,g)})$ *is*

$$\begin{aligned}
\partial(\Delta^{(F_4,g)}) = {} & 18x_1\partial/\partial x_1 + 36x_2\partial/\partial x_2 + 24x_3\partial/\partial x_3 + 12x_4\partial/\partial x_4 \\
& + (2x_1^2 - 7x_4^2 - 4x_1 - 2x_2 + 7x_3 + 7x_4 - 13)\partial^2/\partial x_1^2 \\
& + (6x_1x_2 - 6x_3^2 + 6x_2x_4 - 6x_1x_4 + 6x_4^3 + 6x_1x_3 - 21x_3x_4 - 8x_1^2 + 17x_2 \\
& - x_4^2 - x_3 - 2x_1 - 13x_4 + 13)\partial^2/\partial x_1\partial x_2 \\
& + (4x_1x_3 - 7x_3x_4 - 7x_1x_4 - 13x_4^2 + 20x_1 + 7x_2 + 5x_3 - 7x_4 + 13)\partial^2/\partial x_1\partial x_3 \\
& + (2x_1x_4 - 8x_3 - 20x_4)\partial^2/\partial x_1\partial x_4 \\
& + (12x_2^2 - 8x_1^3 + 6x_1^2 - 30x_3^2 - 6x_4^4 - 20x_4^3 + 18x_4^2 - 4x_1x_3^2 + 12x_1x_4^3 \\
& - 6x_1^2x_4^2 + 8x_2x_4^2 + 6x_1x_4^2 + 40x_3x_4^3 + 30x_1x_2 + 6x_1^2x_3 + 8x_2x_3 - 26x_1x_3 \\
& + 16x_1^2x_4 - 8x_3^2x_4 - 32x_2x_4 + 4x_1x_2x_4 - 20x_1x_4 - 34x_1x_3x_4 + 42x_3x_4 - 22x_2 \\
& + 2x_1 - 32x_3 + 20x_4 - 26)\partial^2/\partial x_2^2 \\
& + (8x_2x_3 - 5x_1x_3x_4 - 7x_3x_4^2 + 5x_1x_2 - 7x_1^2x_4 - 2x_3^2 + 17x_2x_4 + 6x_1x_4^2 + 7x_4^3 \\
& - 15x_1x_3 - x_3x_4 + 8x_1^2 - 20x_2 - 7x_1x_4 - 7x_4^2 + 22x_3 + x_1 - 7x_4)\partial^2/\partial x_2\partial x_3 \\
& + (4x_2x_4 - 6x_1x_3 - 7x_3x_4 - 7x_1x_4 + 13x_4^2 \\
& - 6x_1 + 7x_2 - 21x_3 + 7x_4 - 13)\partial^2/\partial x_2\partial x_4 \\
& + (3x_3^2 - x_2x_4 - 3x_1x_4^2 - 6x_4^3 + 4x_3x_4 - 4x_1^2 + 3x_2 \\
& - x_1x_4 - x_4^2 + 2x_3 + 4x_1 + 14x_4 - 13)\partial^2/\partial x_3^2 \\
& + (3x_3x_4 - 7x_1x_4 - 13x_4^2 - 8x_1 - 3x_2 + 5x_3 - 7x_4 + 13)\partial^2/\partial x_3\partial x_4 \\
& + (x_4^2 - 4x_1 - x_3 - 7x_4 - 13)\partial^2/\partial x_4^2
\end{aligned}$$

PROOF. The first order terms of $\partial(\Delta)$ are easily obtained by Lemmas 3.3 and 5.2. The second order terms are also obtained by Lemma 3.3. We omit the lengthy and tedious calculation. q.e.d.

Let $V^k$ be the $k$-th eigen-space of $\Delta^{(M,g)}$ and $(V^k)^C$ be its complexification. Then $(V^k)^C$ is an irreducible $F_4$-module with highest weight $k\Lambda_4 = k\varepsilon_1$. Thus the restriction to $\mathfrak{h}$ of its character is a polynomial of degree

$k\Lambda_4$. By Lemmas 3.4 and 5.3, we can calculate inductively its coefficients up to $x_3^4 x_4^{k-8}$.

$$
\begin{aligned}
(5.1) \qquad \chi_{k\Lambda_4} = {} & x_4^k - (k-1)x_3x_4^{k-2} + ((k-2)(k-3)/2)x_3^2x_4^{k-4} \\
& + (k-2)x_2x_4^{k-3} - x_1x_4^{k-2} - x_4^{k-1} \\
& - ((k-3)(k-4)(k-5)/6)x_3^3x_4^{k-6} - (k-3)(k-4)x_2x_3x_4^{k-5} \\
& + (k-3)x_1x_3x_4^{k-4} + (k-3)x_3x_4^{k-3} \\
& + ((k-4)(k-5)(k-6)(k-7)/24)x_3^4x_4^{k-8} \\
& + (\text{terms of degree} < 4\Lambda_3 + (k-8)\Lambda_4) .
\end{aligned}
$$

We calculate the character of $S^2(V(\Lambda_4))$ as a polynomial in $x_1, \cdots, x_4$ up to $x_3^4x_2^{2k-8}$ by a similar manner to that used in § 4. We put $y_j(H) = x_j(2H)$ for $H \in \mathfrak{t}$. Then by Lemma 3.5, the character $\chi^{(2)}_{k\Lambda_4}$ of $S^2(V(k\Lambda_4))$ is

$$\chi^{(2)}_{k\Lambda_4} = (1/2)((\chi_{k\Lambda_4}(x_1, x_2, x_3, x_4))^2 + \chi_{k\Lambda_4}(y_1, y_2, y_3, y_4)) .$$

When we substitute $y_i$'s into (5.1) instead of $x_i$'s, the degree of $y_2y_4^{k-3}$ is less than that of $x_3^4x_4^{2k-8}$. Thus we need only explicit expression for $y_3$ and $y_4$ as polynomials in $x_1$, $x_2$, $x_3$ and $x_4$, which can be obtained similarly as in § 4 as follows:

$$
\begin{aligned}
y_3 &= x_3^2 - 2x_2x_4 - 2x_1x_4^2 + 4x_1x_3 + 2x_1^2 + 2x_1 , \\
y_4 &= x_4^2 - 2x_1 - 2x_3 .
\end{aligned}
$$

Multiplicities of weights, which we need in the calculation, are found in [2]. Substituting $y_i$'s, we have

$$
\begin{aligned}
(5.2) \qquad \chi^{(2)}_{k\Lambda_4} = {} & x_4^{2k} - (2k-1)x_3x_4^{2k-2} + (2k^2-5k+4)x_3^2x_4^{2k-8} \\
& + (2k-3)x_2x_4^{2k-3} - 2x_1x_4^{2k-2} - x_4^{2k-1} \\
& - ((4k^3 - 24k^2 + 53k - 45)/3)x_3^3x_4^{2k-6} \\
& - (4k^2 - 16k + 18)x_2x_3x_4^{2k-5} + (4k-6)x_1x_3x_4^{2k-4} \\
& + (2k-4)x_3x_4^{2k-3} \\
& + ((4k^4 - 44k^3 + 191k^2 - 397k + 342)/6)x_3^4x_4^{2k-8} \\
& + (\text{terms of degree} < 4\Lambda_3 + (2k-8)\Lambda_4) .
\end{aligned}
$$

By (5.1) and (5.2), we have

$$
\begin{aligned}
(5.3) \qquad \chi^{(2)}_{k\Lambda_4} - \chi_{2k\Lambda_4} = {} & x_3^2x_4^{2k-4} - x_2x_4^{2k-3} - x_1x_4^{2k-2} \\
& - (2k-5)x_3^3x_4^{2k-6} + (2k-6)x_2x_3x_4^{2k-5} \\
& + (2k-3)x_1x_3x_4^{2k-4} - x_3x_4^{2k-3} \\
& + (2k^2 - 13k + 22)x_3^4x_4^{2k-8} \\
& + (\text{terms of degree} < 4\Lambda_3 + (2k-8)\Lambda_4) .
\end{aligned}
$$

Thus we have the following decomposition for $k \geqq 2$:

$$S^2(V(k\Lambda_4)) = V(2k\Lambda_4) + V(2\Lambda_3 + 2(k-2)\Lambda_4) + \cdots .$$

Since $V(2\Lambda_3 + 2(k-2)\Lambda_4)$ is not a class one representation of $(F_4, Spin(9))$, $L_E^C$ contains it by Lemma 2.1. On the other hand, we have $L_E^C = 0$ for $k = 1$, Since $S^2(V(\Lambda_4)) = V(2\Lambda_4) + V(\Lambda_4) + V(0)$. By Weyl's dimension formula we have

$$\dim_C V(2\Lambda_3 + 2(k-2)\Lambda_4) \geqq \dim_C V(2\Lambda_3) = 19448$$

if $k \geqq 2$. Thus we have the following:

THEOREM C. *Let $M = F_4/Spin(9)$ be the Cayley projective plane $P^2(Ca)$ with an $F_4$-invariant Riemannian metric. Then*

(i) $\dim L_E = 0$ *if* $k = 1$,

(ii) $\dim L_E \geqq 19448$ *if* $k \geqq 2$.

Furthermore, we calculate the character of $V(2\Lambda_3 + (2k-4)\Lambda_4)$ as

$$\begin{aligned}\chi_{2\Lambda_3+(2k-4)\Lambda_4} = {} & x_3^2x_4^{2k-4} - x_2x_4^{2k-3} - x_1x_4^{2k-2} - x_4^{2k-1} \\ & - (2k-5)x_3^3x_4^{2k-6} + (2k-6)x_2x_3x_4^{2k-5} \\ & + (2k-4)x_1x_3x_4^{2k-4} + (2k-3)x_3x_4^{2k-3} \\ & + (k-3)(2k-7)x_3^4x_4^{2k-8} \\ & + (\text{terms of degree} < 4\Lambda_3 + (2k-8)\Lambda_4) .\end{aligned}$$

Thus we have from (5.3)

$$\begin{aligned}(5.4)\qquad & \chi^{(2)}_{k\Lambda_4} - \chi_{2k\Lambda_4} - \chi_{2\Lambda_3+(2k-4)\Lambda_4} \\ & = x_4^{2k-1} + x_1x_3x_4^{2k-4} - (2k+4)x_3x_4^{2k-3} + x_3^4x_4^{2k-8} \\ & \quad + (\text{terms of degree} < 4\Lambda_3 + (2k-8)\Lambda_4) .\end{aligned}$$

The character of $V(\Lambda_1 + \Lambda_3 + (2k-4)\Lambda_4)$ is

$$\begin{aligned}\chi_{\Lambda_1+\Lambda_3+(2k-4)\Lambda_4} = {} & x_1x_3x_4^{2k-4} - x_3x_4^{2k-3} \\ & + (\text{terms of degree} < 4\Lambda_3 + (2k-8)\Lambda_4) .\end{aligned}$$

Thus by (5.1) and (5.4), we have

$$\begin{aligned}& \chi^{(2)}_{k\Lambda_4} - \chi_{2k\Lambda_4} - \chi_{2\Lambda_3+(2k-4)\Lambda_4} - \chi_{(2k-1)\Lambda_4} - \chi_{\Lambda_1+\Lambda_3+(2k-4)\Lambda_4} \\ & = x_3^4x_4^{2k-8} + (\text{terms of degree} < 4\Lambda_3 + (2k-8)\Lambda_4) .\end{aligned}$$

Finally, we have the following decomposition if $k \geqq 4$:

$$\begin{aligned}(5.5)\qquad S^2(V(k\Lambda_4)) = {} & V(2k\Lambda_4) + V(2\Lambda_3 + (2k-4)\Lambda_4) + V((2k-1)\Lambda_4) \\ & + V(\Lambda_1 + \Lambda_3 + (2k-4)\Lambda_4) + V(4\Lambda_3 + (2k-8)\Lambda_4) + \cdots .\end{aligned}$$

By Lemma 5.1, $V(4\Lambda_3 + (2k-8)\Lambda_4) = V((2k-2)\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4)$

contains none of the $K$-irreducible components of $S^2(\mathfrak{p}^C)$. By Weyl's dimension formula have

$$\dim_C V(4\Lambda_3 + (2k-8)\Lambda_4) \geqq \dim_C V(4\Lambda_3) = 11955216$$

if $k \geqq 4$. When $k = 3$, the symmetric tensor product $S^2(V(3\Lambda_4))$ is decomposed as

$$\begin{aligned} S^2(V(3\Lambda_4)) = V(6\Lambda_4) &+ V(2\Lambda_3 + 2\Lambda_4) + V(5\Lambda_4) + V(\Lambda_1 + \Lambda_3 + 2\Lambda_4) \\ &+ V(\Lambda_3 + 3\Lambda_4) + V(2\Lambda_1 + 2\Lambda_4) + V(2\Lambda_3 + \Lambda_4) + V(4\Lambda_4) \\ &+ V(\Lambda_2 + \Lambda_3) + \cdots . \end{aligned}$$

By Lemma 5.1, $V(\Lambda_2 + \Lambda_3) = V((7\varepsilon_1 + 3\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4)/2)$ contains none of the $K$-irreducible components of $S^2(\mathfrak{p}^C)$. By Weyl's dimension formula, we have

$$\dim_C V(\Lambda_2 + \Lambda_3) = 107406 .$$

Thus by Lemma 2.3, we have the following:

THEOREM D. *Let $M = P^2(Ca)$ be the Cayley projective plane with an $F_4$-invariant Riemannian metric. Then*

(i) $\dim L_M = 0$ *if* $k = 1$,

(ii) $\dim L_M \geqq 107406$ *if* $k \geqq 3$.

REMARK. When $k = 2$, $S^2(V(2\Lambda_4))$ is decomposed as

$$\begin{aligned} S^2(V(2\Lambda_4)) = V(4\Lambda_4) &+ V(2\Lambda_3) + V(3\Lambda_4) + V(\Lambda_1 + \Lambda_3) \\ &+ V(\Lambda_3 + \Lambda_4) + V(2\Lambda_1) + V(2\Lambda_4) + V(\Lambda_4) + V(0) . \end{aligned}$$

By Lemma 5.1, we have $L'_M = 0$. But we cannot say anything about $\dim L_M$.

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