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THE HELICOIDAL SURFACES AS BONNET SURFACES

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1. Introduction. In this paper we deal with the following question: which surfaces in the Euclidean space E^3 admit a mean-curvature-preserving isometry which is not an isometry of the whole space? This question has been studied by a series of mathematicians beginning with Bonnet [1] and including Cartan [2] and Chern [3]. These surfaces are special and have been classified into three types:

1. The surfaces of constant mean curvature other than the plane and the sphere;
2. Certain surfaces of nonconstant mean curvature that admit a one-parameter family of geometrically distinct nontrivial isometries;
3. Certain surfaces of nonconstant mean curvature that admit a single nontrivial isometry unique up to an isometry of the whole space.

A surface that belongs to one of the above types is called a *Bonnet surface*.

By a nontrivial isometry of a surface we mean an isometry of the surface to another surface or to itself which does not extend to an isometry of the whole space. Two isometries are said to be geometrically distinct if one is not the composition of the other followed by a space-isometry.

A *helicoidal surface* in E^3 is the locus of an appropriately chosen curve under a *helicoidal motion* with pitch in the interval $(0, \infty)$. Such a motion is described by a one-parameter group of isometries in E^3. The orbits of this motion (helices) through the initial curve foliate the helicoidal surface. More details can be found in various places in the literature—for instance in [4].

The main result of this work states:

THEOREM. *The helicoidal surfaces are necessarily Bonnet surfaces and they represent all three types.*

REMARKS. 1. The helicoidal surfaces of the third type provide a negative answer to the following conjecture, posed by Lawson and Tribuzy [5]:

We consider a Riemannian surface Σ and a smooth function $H\colon \Sigma \to \boldsymbol{R}$.

If a nontrivial family of isometric immersions with mean curvature H does not exist, then there are at most two noncongruent ones. Then the conjecture states: "In the absence of a nontrivial family the immersion must be unique." (see Final comment 2 in [5].)

2. By the main result in [5] the Bonnet surfaces of the second type cannot be compact.

3. It seems that not all Bonnet surfaces of the third type are helicoidal surfaces, since a helicoidal surface is determined by one real-valued function of one variable, while the Bonnet surfaces of the third type depend on four functions of one variable ([2]) and therefore have a greater degree of generality.

4. The helicoidal surfaces of the first type have been thoroughly examined in [4] along with their isometric deformation under preservation of the constant mean curvature. Our arguments here lead to the following new interesting geometric characterization of them: A helicoidal surface has constant mean curvature if and only if its principal directions make an angle constant with the orbits.

At this point I wish to express my gratitude to my academic advisor Professor William Pohl for his valuable guidance.

2. The Equations of Codazzi. We consider a surface M^2 in E^3, orientable and sufficiently smooth. We consider a well-defined field of orthonormal frames (x, e_1', e_2', e_3') over M^2, such that $x \in M^2$ and e_1', e_2' comprise an orthonormal basis of the tangent space of M^2 at x. Then we have

$$\eta_i = dx \cdot e_i' \,,$$
$$\eta_{ij} = de_i' \cdot e_j' \,,$$
$$\eta_{ij} = -\eta_{ji} \,, \quad (\text{so } \eta_{ii} = 0) \,,$$
$$d\eta_i = \sum_{j=1}^{3} \eta_{ij} \wedge \eta_j \quad (\text{first structural equation}) \,,$$
$$d\eta_{ij} = \sum_{k=1}^{3} \eta_{ik} \wedge \eta_{kj} \quad (\text{second structural equation}) \,,$$
$$\text{where} \quad 1 \leqq i \,, \quad j \leqq 3 \,.$$

$\eta_3 = 0$ on M^2, so we have $\eta_{13} \wedge \eta_1 + \eta_{23} \wedge \eta_2 = 0$. Thus, by Cartan's lemma we get

$$\eta_{13} = a\eta_1 + b\eta_2 \,, \qquad (1)$$
$$\eta_{23} = b\eta_1 + c\eta_2 \,.$$

Then the mean and Gaussian curvatures of M^2 are

$$H=\frac{1}{2}(a+c)\,,$$
$$K=ac-b^2\quad\text{(Gauss Equation) (GE)}\,.$$

We also have

$$\begin{aligned}&d\eta_{12}=-K\eta_1\wedge\eta_2\\&\left.\begin{aligned}d\eta_{13}&=\eta_{12}\wedge\eta_{23}=-bd\eta_2+cd\eta_1\\d\eta_{23}&=\eta_{21}\wedge\eta_{13}=ad\eta_2-bd\eta_1\end{aligned}\right\}\quad\text{(Codazzi Equations) (CE)}\,.\end{aligned}$$

A given Riemannian surface can be realized in E^3 if the CE and GE are satisfied.

We now let

$$e_1=(\cos\sigma)e_1'+(\sin\sigma)e_2'\,,$$
$$e_2=-(\sin\sigma)e_1'+(\cos\sigma)e_2'$$

be the principal frame of M^2. For this frame the function $\tilde{b}$ defined by (1) is zero and $\tilde{a}$, $\tilde{c}$ are the principal curvatures.

In the sequel, we consider M^2 with no umbilic points. We may then assume for the principal curvatures $\tilde{a}$ and $\tilde{c}$ that $\tilde{a}>\tilde{c}$ and we put

$$J=\frac{\tilde{a}-\tilde{c}}{2}>0\,.$$

Working in a manner similar to the one in [3], after putting $\phi=2\sigma$ and using the above, we can show that the CE are equivalent to

$$dH=H_1\eta_1+H_2\eta_2\quad\text{(thus defining } H_1,\ H_2)\,,\tag{2}$$

$$d\phi=-(\sin\phi)\Big(\frac{H_1}{J}\eta_1-\frac{H_2}{J}\eta_2\Big)+(\cos\phi)\Big(\frac{H_2}{J}\eta_1+\frac{H_1}{J}\eta_2\Big)\tag{3}$$
$$-*d\log J-2\eta_{12}\,,$$

where $*$ is the Hodge operator whose action on the 1-forms is described by

$$*\eta_1=\eta_2\,,\quad *\eta_2=-\eta_1\quad(*^2=-1)\,.$$

3. Some facts about helicoidal surfaces. Every helicoidal surface can be parametrized by (s,t), where

t: time along orbits from a fixed $t=t_o$,

s: arc-length of curves orthogonal to orbits.

Then the curves $t=$ constant are carried along the orbits by the helicoidal motion. They remain orthogonal to the orbits and foliate the surface. So an orthonormal frame $\{e_1',e_2'\}$ is determined along these coordinate curves. The corresponding coframe may be written as

$$\eta_1 = ds\,,$$
$$\eta_2 = q(s)dt \quad (q \text{ depends only on } s)\,.$$

Thus,

$$\eta_{12} = \frac{q'(s)}{q(s)}\eta_2 = \mu(s)\eta_2\,.$$

Hence the η_1-curves are geodesics and the η_2-curves (orbits) have geodesic curvature equal to

$$\mu(s) = \frac{d}{ds}\log(|q(s)|)\,.$$

Along each orbit a, c, μ, ϕ are constant, hence they depend on s only. So, in this case we get $H_2 = 0$. Also if we put $dJ = J_1\eta_1 + J_2\eta_2$, we get $J_2 = 0$ and $d\log J = (J_1/J)\eta_1$. Hence (3) becomes

$$d\phi = -(\sin\phi)\Big(\frac{H_1}{J}\eta_1\Big) + (\cos\phi)\Big(\frac{H_1}{J}\eta_2\Big) - \frac{J_1}{J}\eta_2 - 2\mu\eta_2\,.$$

Since $d\phi = (d\phi/ds)ds$, this implies

$$\frac{d\phi}{ds} = -(\sin\phi)\frac{\frac{dH}{ds}}{J}\,, \tag{4}$$

$$\mu = \frac{1}{2}\left((\cos\phi)\frac{\frac{dH}{ds}}{J} - \frac{\frac{dJ}{ds}}{J}\right). \tag{5}$$

Finally, by direct computation or by well known facts about curves on surfaces and depending on which vector (e_1' or e_2') is the major principal axis, we get:

The space curvature of orbits is either

$$\{\mu^2 + (\tilde{a}(\cos^2\sigma) + \tilde{c}(\sin^2\sigma))^2\}^{1/2} \quad \text{or} \quad \{\mu^2 + (\tilde{a}(\sin^2\sigma) + \tilde{c}(\cos^2\sigma))^2\}^{1/2}\,, \tag{6}$$

and the space torsion of orbits is $\pm(\tilde{a} - \tilde{c})(\sin\sigma)(\cos\sigma)$.

4. Proof of the Theorem. The Theorem follows from the next two propositions. We consider a helicoidal surface parametrized by the parameters (s, t) of Section 3. We have:

PROPOSITION 1. *For a helicoidal surface the mapping $(s, t) \to (s, -t)$ is a nontrivial isometry which preserves the mean curvature.*

PROOF. By what is exhibited in Section 3, this mapping is an isometry which preserves the mean curvature. An isometry is trivial, in general, if and only if it preserves the mean curvatures and the principal directions. Thus, in this case, the above mapping is trivial if and only if σ is a multiple of $\pi/2$. Then by (6) we get that the orbits are plane curves, which is impossible for a helicoidal surface. q.e.d.

PROPOSITION 2. *A helicoidal surface is a Bonnet surface of the second type if and only if the following relation is satisfied:*

$$\frac{d}{ds}\left(\frac{\frac{dH}{ds}}{J}\right) - \left(\frac{\frac{dH}{ds}}{J}\right)^2 (\cos\phi(s)) + \frac{\frac{dH}{ds}}{J}\frac{d\log(|q(s)|)}{ds} = 0 ,$$

with $H = H(s)$ nonconstant.

Since this relation may be viewed as an ordinary differential equation for the real-valued function which determines the helicoidal surface under the helicoidal motion, the existence of such a surface is guaranteed by the local existence theorem for solutions of an ordinary differential equation.

REFEREE'S REMARK. The equation in Proposition 2 is equivalent to:

$$J^{-1}\left(\frac{dH}{ds}\right)\cdot(\sin\phi(s))\cdot q(s) = \text{constant}$$

$$\text{with} \quad H = H(s) \quad \text{nonconstant} .$$

Using (4), the equation can be integrated.

PROOF. We consider $\sigma = \sigma(s)$ as in Section 2 and the principal coframe

$$\omega_1 = (\cos\sigma(s))ds + (\sin\sigma(s))q(s)dt ,$$
$$\omega_2 = -(\sin\sigma(s))ds + (\cos\sigma(s))q(s)dt .$$

We define H_1, H_2 by putting $dH = H_1\omega_1 + H_2\omega_2$. We set

$$\alpha_1 = (H_1/J)\omega_1 - (H_2/J)\omega_2 , \quad \alpha_2 = (H_2/J)\omega_1 + (H_1/J)\omega_2 .$$

Chern has shown in [3] that the criterion for the existence of the Bonnet surfaces of the second type is:

$$d\alpha_1 = 0 \quad \text{and} \quad d\alpha_2 = \alpha_1 \wedge \alpha_2 .$$

Substituting the formulas for ω_1, ω_2 into those of α_1, α_2, and using (4) in Section 3, it is a matter of direct computation to verify that: $d\alpha_1 = 0$ and $d\alpha_2 = \alpha_1 \wedge \alpha_2$ simultaneously if and only if the relation claimed is satisfied. q.e.d.

We conclude this work with the following geometric characterization of the helicoidal surfaces of the first type:

PROPOSITION 3. *A helicoidal surface has constant mean curvature if and only if its principal directions make an angle constant with the orbits.*

PROOF. Immediate consequence of (4) and (6) in Section 3. q.e.d.

REFERENCES

[1] O. BONNET, Mémoire sur la theorie des surfaces applicables, J. Éc. Polyt. 42 (1867) 72–92.
[2] É. CARTAN, Sur les couples de surfaces applicables avec conservation des courbures, principales, Bull. Sci. Math. 66 (1942), 1–30, or Oeuvres Complètes, Partie III, vol. 2, 1591–1620.
[3] S.-S. CHERN, Deformation of surfaces preserving principal curvatures, Diff. Geom. and Complex Analysis, H. E. Rauch Memorial Volume, Springer-Verlag (1985), 155–163.
[4] M. P. DO CARMO AND M. DAJCZER, Helicoidal surfaces with constant mean curvature, Tôhoku Math. J., 34 (1982), 425–435.
[5] H. B. LAWSON, JR. AND RENATO DE A. TRIBUZY, On the mean curvature function for compact surfaces, J. of Diff. Geom. 16 (1981), 179–183.
[6] I. M. ROUSSOS, Mean-Curvature-Preserving-Isometries of Surfaces in Ordinary Space, Ph. D. Thesis, University of Minnesota, March 1986.

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