Tôhoku Math. J.
40 (1988), 465-471.

HARMONIC FOLIATIONS ON A COMPACT RIEMANNIAN MANIFOLD OF NON-NEGATIVE CONSTANT CURVATURE

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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(Received May 1, 1987)

Introduction. Let M be a compact oriented manifold and $\mathscr{F}$ a Riemannian and harmonic foliation with respect to a bundle-like metric. Kamber and Tondeur [3] proved the fundamental formula for a special variation of $\mathscr{F}$, and making use of it they showed in [4] that the index of a Riemannian and harmonic foliation on the sphere S^n $(n > 2)$ for which the standard metric is bundle-like is not smaller than $q + 1$, where q is the codimension of $\mathscr{F}$.

The purpose of this paper is to prove that any harmonic foliation on a compact Riemannian manifold of non-negative constant curvature for which the normal plane field is minimal (see § 1 for the definition) is totally geodesic. As a corollary we can state that any Riemannian and harmonic foliation on the sphere S^n $(n > 2)$ for which the standard metric is bundle-like is totally geodesic. Moreover, Escobales [1] has classified recently all totally geodesic foliations on the spheres for which the standard metrics are bundle-like. This means that harmonic foliations on the spheres for which the standard metrics are bundle-like have been completely classified.

On the other hand, a theorem of Ferus [2] gives an estimate for the codimension of a totally geodesic foliation of the sphere S^n. Thus we can apply these results to the foregoing theory of Kamber and Tondeur to sharpen their result.

The authors wish to thank the referee for his useful advice.

1. Preliminaries. We shall be in the C^∞-category. Let (M, g) be an n-dimensional Riemannian manifold, and $\mathscr{F}$ a foliation of codimension q on M. Then there arise two tensor fields associated with a foliated Riemannian manifold $(M, g, \mathscr{F})$ as follows. Denote by $V(M)$ the space of vector fields on M, and by ∇ the Riemannian connection on M. For any $X \in V(M)$ we decompose it as

$$X = X' + X'' ,$$

where X' (resp. X'') is tangent (resp. normal) to $\mathscr{F}$. Actually, choosing a suitable Riemannian metric on the tangent bundle $T(M)$ of M, we may decompose $T(M)$ as the direct product $\mathscr{F} \oplus \mathscr{F}^{\perp}$, where $\mathscr{F}^{\perp}$ is called a *normal plane field*. Then we define two tensors A and h of type (1, 2) on M by

$$A(X,\ Y) = -(\nabla_{Y''}X'')' ,\qquad h(X,\ Y) = (\nabla_{Y'}X')'' ,\quad X, Y \in V(M) . \tag{1.1}$$

The restriction of h to each leaf of $\mathscr{F}$ is what is called the *second fundamental form* of the leaf. From now on we express them with respect to a locally defined orthonormal frame field, and derive some basic formulas among them and their derivatives. As for the range of indices we use the following convention unless otherwise stated:

$$A, B, C, \cdots = 1, \cdots, n ;$$
$$i, j, k, \cdots = 1, \cdots, p ;$$
$$\alpha, \beta, \gamma, \cdots = p+1, \cdots, n ,$$

where $p = n - q$ denotes the dimension of $\mathscr{F}$. The summation $\sum$ is taken over all repeated indices. Let $\{e_1, \cdots, e_n\}$ be a local field of orthonormal frames on M such that $e_1, \cdots, e_p$ are always tangent to $\mathscr{F}$. Denote its dual forms by $\omega_1, \cdots, \omega_n$. The connection forms ω_{AB} with respect to ω_A are defined by the equations

$$\omega_{BA} + \omega_{AB} = 0 ,\qquad d\omega_A + \sum \omega_{AB} \wedge \omega_B = 0 . \tag{1.2}$$

The Riemannian connection ∇ on M is given by

$$\nabla_{e_A} e_B = \sum \omega_{CB}(e_A) e_C . \tag{1.3}$$

It follows from (1.1) and (1.3) that

$$h(e_i,\ e_j) = \sum \omega_{\alpha i}(e_j) e_\alpha ,\qquad A(e_\alpha,\ e_\beta) = \sum \omega_{\alpha j}(e_\beta) e_j . \tag{1.4}$$

Thus the only components $h^A{}_{BC}$ (resp. $A^B{}_{CD}$) of h (resp. A) which may not vanish are

$$h^\alpha{}_{ij} = \omega_{\alpha i}(e_j) \quad (\text{resp. } A^i{}_{\alpha\beta} = \omega_{\alpha i}(e_\beta)) . \tag{1.5}$$

Moreover the connection forms $\omega_{\alpha i}$ are given by

$$\omega_{\alpha i} = \sum h^\alpha{}_{ij}\omega_j + \sum A^i{}_{\alpha\beta}\omega_\beta . \tag{1.6}$$

The foliation $\mathscr{F}$ is said to be *harmonic* or *minimal* (resp. *totally geodesic*) if $\sum h^\alpha{}_{ii} = 0$ (resp. $h^\alpha{}_{ij} = 0$).

After Kitahara [5] and Reinhart [9], we define the second fundamental form B of the normal plane field $\mathscr{F}^{\perp}$ by

(1.7) $$B(X,Y) = \{A(X,Y) + A(Y,X)\}/2\,, \quad X, Y \in V(M)\,.$$

The normal plane field $\mathscr{F}^{\perp}$ is said to be *minimal* (resp. *totally geodesic*) if $\operatorname{Tr} B = \sum A^j{}_{\alpha\alpha} e_j = 0$ (resp. $B = 0$).

The curvature form $\Omega = (\Omega_{AB})$ of M is defined by

(1.8) $$\Omega_{AB} = d\omega_{AB} + \sum \omega_{AC} \wedge \omega_{CB}\,.$$

We put

(1.9) $$\Omega_{AB} = -\sum (R_{ABCD}/2)\omega_C \wedge \omega_D\,, \quad R_{ABCD} + R_{ABDC} = 0\,.$$

Then the components R_{ABCD} of Ω satisfy

(1.10) $$R_{ABCD} = -R_{BACD} = R_{CDAB}\,.$$

Since the distribution $\omega_\alpha = 0$ is integrable by definition, we have

(1.11) $$h^\alpha{}_{ij} = h^\alpha{}_{ji}\,.$$

The distribution $\omega_i = 0$ is integrable if and only if

(1.12) $$A^i{}_{\alpha\beta} = A^i{}_{\beta\alpha}\,.$$

On the contrary, the Riemannian metric g is bundle-like (see Molino [6] or Reinhart [8]) if and only if

(1.13) $$A^i{}_{\alpha\beta} = -A^i{}_{\beta\alpha}\,.$$

Thus, the Riemannian metric is bundle-like if and only if $B = 0$, and then the normal plane field $\mathscr{F}^{\perp}$ is minimal.

Now, for a tensor filed $T = (T^{A_1\cdots A_r}{}_{B_1\cdots B_s})$ on M, we define the covariant derivative $(T^{A_1\cdots A_r}{}_{B_1\cdots B_s C})$ by

(1.14) $$\sum T^{A_1\cdots A_r}{}_{B_1\cdots B_s C}\omega_C = dT^{A_1\cdots A_r}{}_{B_1\cdots B_s} - \sum_{a=1}^{r} T^{A_1\cdots A_{a-1} C A_{a+1}\cdots A_r}{}_{B_1\cdots B_s}\omega_{CA_a} - \sum_{b=1}^{s} T^{A_1\cdots A_r}{}_{B_1\cdots B_{b-1} C B_{b+1}\cdots B}\omega_{CB_b}\,.$$

Then the exterior derivative of (1.6) gives

(1.15) $$h^\alpha{}_{ijk} - h^\alpha{}_{ikj} = R_{\alpha ijk}\,,$$

(1.16) $$h^\alpha{}_{ij\beta} - A^i{}_{\alpha\beta j} - \sum h^\alpha{}_{ik} h^\beta{}_{kj} - \sum A^i{}_{\alpha\gamma} A^j{}_{\gamma\beta} = R_{\alpha ij\beta}\,,$$

(1.17) $$A^i{}_{\alpha\beta\gamma} - A^i{}_{\alpha\gamma\beta} + \sum h^\alpha{}_{ij}(A^j{}_{\beta\gamma} - A^j{}_{\gamma\beta}) = -R_{\alpha i\gamma\beta}\,.$$

Moreover, from the definition of $(h^A{}_{BCD})$ and (1.6) it follows that we have

(1.18) $$h^l{}_{ijk} = -\sum h^\alpha{}_{ij} h^\alpha{}_{lk}\,,$$

(1.19) $$h^l{}_{ij\alpha} = -\sum h^\beta{}_{ij} A^l{}_{\beta\alpha}\,,$$

(1.20) $$h^\alpha{}_{i\beta j} = -\sum h^\alpha{}_{ik} h^\beta{}_{kj}\,,$$

$$h^\alpha{}_{i\beta\gamma} = -\sum h^\alpha{}_{ki} A^k{}_{\beta\gamma}\,, \tag{1.21}$$

$$h^\alpha{}_{\beta ij} = -\sum h^\alpha{}_{ik} h^\beta{}_{kj}\,, \tag{1.22}$$

$$h^\alpha{}_{\beta i\gamma} = -\sum h^\alpha{}_{ik} A^k{}_{\beta\gamma}\,. \tag{1.23}$$

The Ricci formulas on the second covariant derivatives of h are given by the exterior derivative of the definition of the components $h^A{}_{BCD}$. For later use we write down these equations:

$$h^\alpha{}_{ijkl} - h^\alpha{}_{ijlk} = \sum h^\beta{}_{ij} R_{\alpha\beta kl} + \sum h^\alpha{}_{mj} R_{imkl} + \sum h^\alpha{}_{im} R_{jmkl}\,, \tag{1.24}$$

$$h^\alpha{}_{ijk\beta} - h^\alpha{}_{ij\beta k} = \sum h^\gamma{}_{ij} R_{\alpha\gamma k\beta} + \sum h^\alpha{}_{lj} R_{ilk\beta} + \sum h^\alpha{}_{il} R_{jlk\beta}\,, \tag{1.25}$$

$$h^\alpha{}_{ij\beta\gamma} - h^\alpha{}_{ij\gamma\beta} = \sum h^\delta{}_{ij} R_{\alpha\delta\beta\gamma} + \sum h^\alpha{}_{kj} R_{ik\beta\gamma} + \sum h^\alpha{}_{ik} R_{jk\beta\gamma}\,, \tag{1.26}$$

$$A^i{}_{\alpha\beta jk} - A^i{}_{\alpha\beta kj} = \sum A^l{}_{\alpha\beta} R_{iljk} + \sum A^i{}_{\gamma\beta} R_{\alpha\gamma jk} + \sum A^i{}_{\alpha\gamma} R_{\beta\gamma jk}\,, \tag{1.27}$$

$$A^i{}_{\alpha\beta j\gamma} - A^i{}_{\alpha\beta\gamma j} = \sum A^k{}_{\alpha\beta} R_{ikj\gamma} + \sum A^i{}_{\delta\beta} R_{\alpha\delta j\gamma} + \sum A^i{}_{\alpha\delta} R_{\beta\delta j\gamma}\,, \tag{1.28}$$

$$A^i{}_{\alpha\beta\gamma\delta} - A^i{}_{\alpha\beta\delta\gamma} = \sum A^k{}_{\alpha\beta} R_{ik\gamma\delta} + \sum A^i{}_{\varepsilon\beta} R_{\alpha\varepsilon\gamma\delta} + \sum A^i{}_{\alpha\varepsilon} R_{\beta\varepsilon\gamma\delta}\,. \tag{1.29}$$

2. Proof of Theorem. Let $(M, g, \mathscr{F})$ be a foliated Riemannian manifold. We keep the notation in § 1. The global vector field $v = \sum v_A e_A$ on M is defined by

$$v_k = \sum h^\alpha{}_{ij} h^\alpha{}_{ijk}\,, \quad v_\alpha = 0\,.$$

The divergence δv of v is first calculated.

LEMMA 2.1.

$$\begin{aligned}\delta v = {} & \sum v_i A^i{}_{\alpha\alpha} + \sum h^\alpha{}_{ijk} h^\alpha{}_{ijk} + \sum h^\alpha{}_{ij} R_{\alpha ijkk} + \sum h^\alpha{}_{ij} R_{\alpha kikj} \\ & + \sum h^\alpha{}_{ij} h^\beta{}_{kk} h^\alpha{}_{ij\beta} + \sum h^\alpha{}_{ij} h^\alpha{}_{kkij} + \sum (h^\beta{}_{ik} R_{\alpha\beta jk} + h^\alpha{}_{lk} R_{iljk} + h^\alpha{}_{il} R_{kljk}) h^\alpha{}_{ij} \\ & + \sum h^\alpha{}_{ij} h^\alpha{}_{lk} h^\beta{}_{ij} h^\beta{}_{lk} + 2 \sum h^\alpha{}_{ij} h^\beta{}_{ik} h^\alpha{}_{jl} h^\beta{}_{lk}\,.\end{aligned}$$

PROOF. From the definition of (v_{AB}), we have

$$\sum v_{\alpha A}\omega_A = dv_\alpha - \sum v_A \omega_{A\alpha} = -\sum v_i \omega_{i\alpha}\,,$$

which implies

$$\sum v_{\alpha\alpha} = \sum v_i A^i{}_{\alpha\alpha}\,. \tag{2.1}$$

Moreover we have

$$\begin{aligned}\sum v_{kA}\omega_A &= dv_k - \sum v_A \omega_{Ak} = d(\sum h^\alpha{}_{ij} h^\alpha{}_{ijk}) - \sum v_i \omega_{ik} \\ &= \sum h^\alpha{}_{ijk}(h^\alpha{}_{ijA}\omega_A - h^\beta{}_{ij}\omega_{\alpha\beta} + h^\alpha{}_{lj}\omega_{li} + h^\alpha{}_{il}\omega_{lj}) \\ &\quad + \sum h^\alpha{}_{ij}(h^\alpha{}_{ijkA}\omega_A - h^l{}_{ijk}\omega_{\alpha l} + h^\alpha{}_{ljk}\omega_{li} + h^\alpha{}_{ilk}\omega_{lj} \\ &\quad + h^\alpha{}_{ijl}\omega_{lk} - h^\beta{}_{ijk}\omega_{\alpha\beta} + h^\alpha{}_{\beta jk}\omega_{\beta i} + h^\alpha{}_{i\beta k}\omega_{\beta j} + h^\alpha{}_{ij\beta}\omega_{\beta k}) \\ &\quad - \sum h^\alpha{}_{jl} h^\alpha{}_{jli}\omega_{ik} \\ &= \sum h^\alpha{}_{ijk} h^\alpha{}_{ijA}\omega_A + \sum h^\alpha{}_{ij}(h^\alpha{}_{ijkA}\omega_A - h^l{}_{ijk}\omega_{\alpha l} + 2h^\alpha{}_{i\beta k}\omega_{\beta j} + h^\alpha{}_{ij\beta}\omega_{\beta k})\,,\end{aligned}$$

which together with (1.18) and (1.22) gives

$$\text{(2.2)}\qquad \sum v_{ii} = \sum h^{\alpha}{}_{ijk}h^{\alpha}{}_{ijk} + \sum h^{\alpha}{}_{ij}h^{\alpha}{}_{ijkk} + \sum h^{\alpha}{}_{ij}h^{\beta}{}_{ij}h^{\alpha}{}_{kl}h^{\beta}{}_{kl} + 2\sum h^{\alpha}{}_{ij}h^{\alpha}{}_{jk}h^{\beta}{}_{kl}h^{\beta}{}_{li} + \sum h^{\alpha}{}_{ij}h^{\beta}{}_{kk}h^{\alpha}{}_{ij\beta} .$$

On the other hand, we have

$$\begin{aligned} h^{\alpha}{}_{ijkk} &= R_{\alpha ijkk} + h^{\alpha}{}_{ikjk} && \text{(by (1.15))} \\ &= R_{\alpha ijkk} + \sum h^{\beta}{}_{ik}R_{\alpha\beta jk} + \sum h^{\alpha}{}_{lk}R_{iljk} + \sum h^{\alpha}{}_{il}R_{kljk} + h^{\alpha}{}_{ikkj} && \text{(by (1.24))} \\ &= R_{\alpha ijkk} + \sum h^{\beta}{}_{ik}R_{\alpha\beta jk} + \sum h^{\alpha}{}_{lk}R_{iljk} + \sum h^{\alpha}{}_{il}R_{kljk} + R_{\alpha kikj} + h^{\alpha}{}_{kkij} . && \text{(by (1.15))} \end{aligned}$$

This, (2.1) and (2.2) complete the proof.

LEMMA 2.2. *If the foliation is harmonic, then we have*

$$\text{(2.3)}\qquad \sum h^{\alpha}{}_{iiA} = 0 ,$$

$$\text{(2.4)}\qquad \sum h^{\alpha}{}_{iijk} = -2\sum h^{\alpha}{}_{il}h^{\beta}{}_{lj}h^{\beta}{}_{ik} .$$

PROOF. From the definition of $(h^{A}{}_{BCD})$ we have

$$\sum h^{\alpha}{}_{iiA}\omega_A = \sum dh^{\alpha}{}_{ii} + \sum h^{A}{}_{ii}\omega_{\alpha A} + \sum h^{\alpha}{}_{Ai}\omega_{iA} + \sum h^{\alpha}{}_{iA}\omega_{iA} = 0 ,$$

which proves (2.3). Similarly, we have

$$\begin{aligned} \sum h^{\alpha}{}_{iijA}\omega_A &= \sum dh^{\alpha}{}_{iij} + \sum h^{A}{}_{iij}\omega_{\alpha A} + \sum h^{\alpha}{}_{Aij}\omega_{iA} + \sum h^{\alpha}{}_{iAj}\omega_{iA} + \sum h^{\alpha}{}_{iiA}\omega_{jA} \\ &= \sum h^{k}{}_{iij}\omega_{\alpha k} + 2\sum h^{\alpha}{}_{\beta ij}\omega_{i\beta} . && \text{(by (2.3))} \end{aligned}$$

Hence we have from (1.18) and (1.20)

$$h^{\alpha}{}_{iijk} = \sum h^{l}{}_{iij}h^{\alpha}{}_{lk} - 2\sum h^{\alpha}{}_{\beta ij}h^{\beta}{}_{ik} = -2\sum h^{\alpha}{}_{il}h^{\beta}{}_{lj}h^{\beta}{}_{ik} . \qquad \text{q.e.d.}$$

Now we can prove the following:

THEOREM 2.3. *Let (M, g) be a compact Riemannian manifold of constant sectional curvature c $(\geqq 0)$. Let $\mathscr{F}$ be a harmonic foliation such that the normal plane field $\mathscr{F}^{\perp}$ is minimal. Then the foliation $\mathscr{F}$ is totally geodesic.*

PROOF. We may assume that M is orientable, because otherwise we may consider its double covering space instead. Then for the vector field v defined above we have

$$\int_M \delta v * 1 = 0 ,$$

where $*1$ denotes the volume element of M. Since M is of constant curvature c, we have

$$R_{ABCD} = c(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{DB}) ,$$

and so $R_{ABCDE}=0$. By assumption we have $\sum A^{i}{}_{\alpha\alpha}=0$. Then Lemma 2.1 and (2.4) imply

$$\int_M [\sum h^{\alpha}{}_{ijk}h^{\alpha}{}_{ijk} + cp\sum h^{\alpha}{}_{ij}h^{\alpha}{}_{ij} + \sum h^{\alpha}{}_{ij}h^{\beta}{}_{ij}h^{\alpha}{}_{kl}h^{\beta}{}_{kl} + 2\sum \mathrm{Tr}\,(H^{\alpha}H^{\alpha}H^{\beta}H^{\beta} - H^{\alpha}H^{\beta}H^{\alpha}H^{\beta})] * 1 = 0 , \tag{2.5}$$

where H^{α} denotes the $p\times p$ matrix $(h^{\alpha}{}_{ij})$. Since the matrix $H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha}$ is skew-symmetric, we find

$$\begin{aligned} 0 &\geqq \sum \mathrm{Tr}\,[(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})(H^{\alpha}H^{\beta}-H^{\beta}H^{\alpha})] \\ &= 2\sum \mathrm{Tr}\,(H^{\alpha}H^{\beta}H^{\alpha}H^{\beta} - H^{\alpha}H^{\alpha}H^{\beta}H^{\beta}) . \end{aligned}$$

Therefore each term in (2.5) is non-negative. In particular, we have $\sum h^{\alpha}{}_{ij}h^{\alpha}{}_{kl}=0$, and so $h^{\alpha}{}_{ij}=0$. q.e.d.

COROLLARY 1. *Let (M, g) be a compact Riemannian manifold of constant curvature c $(\geqq 0)$. Let $\mathscr{F}$ be a harmonic foliation such that the Riemannian metric is bundle-like. Then the foliation $\mathscr{F}$ is totally geodesic.*

In the case of $c=0$ in Corollary 1, it follows from (1.16) and the fact that $\mathscr{F}$ is totally geodesic that A vanishes identically (cf. Ranjan [7]). Thus we have:

COROLLARY 2. *Let (M, g) be a compact flat Riemannian manifold. Let $\mathscr{F}$ be a harmonic foliation such that $\mathscr{F}^{\perp}$ is minimal. Then $\mathscr{F}^{\perp}$ is integrable and tatally geodesic.*

REMARK. Theorem 2.3 does not hold if we replace the assumption "of constant curvature c $(\geqq 0)$" by "with positive Ricci curvature" (cf. Takagi and Yorozu [10], Theorem 3.4).

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