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VECTOR BUNDLES OVER QUATERNIONIC KÄHLER MANIFOLDS

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Introduction. On vector bundles over oriented 4-dimensional Riemannian manifolds, the notion of self-dual and anti-self-dual connections plays an important role in the geometry of 4-dimensional Yang-Mills theory (see Atiyah, Hitchin and Singer [A-H-S]).

On the other hand, in his differential-geometric study of stable holomorphic vector bundles, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds. Let E be a vector bundle over a quaternionic Kähler manifold M, and $p: Z \to M$ the corresponding twistor space defined by Salamon [S1]. Now the purpose of the present paper is to give a quaternionic Kähler analogue of self-dual and anti-self-dual connections, and then to construct a natural correspondence between E's with such connections and the set of Einstein-Hermitian vector bundles over Z.

Let $\boldsymbol{H}$ be the skew field of quaternions. Then the $Sp(n)\cdot Sp(1)$-module $\wedge^2\boldsymbol{H}^n$ is a direct sum $N_2' \oplus N_2'' \oplus L_2$ of its irreducible submodules N_2', N_2'', L_2, where N_2' (resp. L_2) is the submodule of the elements fixed by $Sp(n)$ (resp. $Sp(1)$) and for $n = 1$, we have $N_2'' = \{0\}$. Hence, the vector bundle $\wedge^2 T^*M$ is written as a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its holonomy-invariant subbundles in such a way that A_2', A_2'', B_2 correspond respectively to N_2', N_2'', L_2. Now, a connection for E is called an *A_2'-connection* (resp. *B_2-connection*) if the corresponding curvature is an $\mathrm{End}(E)$-valued A_2'-form (resp. B_2-form). Then we have:

THEOREM (0.1). *All A_2'-connections and also all B_2-connections are Yang-Mills connections.*

Furthermore, for E with a B_2-connection we can associate an E-valued elliptic complex (cf. (3.2)) similar to those of Salamon [S2]. Such complexes allow us to analyze the space of infinitesimal deformations of B_2-connections (see Theorem (3.5)).

For our quaternionic Kähler manifold M, a pair (E, D_E) of a vector bundle E over M and a B_2-connection D_E on E is called a *Hermitian pair* on M if D_E is a Hermitian connection on E. On the other hand, a pair (F, D_F) of a holomorphic vector bundle over Z and a Hermitian (1, 0)-

connection D_F on F is called an *excellent pair* on Z if the following conditions are satisfied:

(a) F with the corresponding Hermitian metric h_F restricts to a flat bundle on each fibre of $p: Z \to M$. (Hence the real structure $\tau: Z \to Z$ (cf. Nitta and Takeuchi [N-T]) naturally lifts to a bundle automorphism $\tau': F \to F$.)

(b) Let $\sigma: F \to F^*$ be the bundle map defined by $F_z \ni f \mapsto \sigma(f) \in F^*_{\tau(z)}$ $(z \in Z)$, where $\sigma(f)(g) := h_F(g, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then σ is an anti-holomorphic bundle automorphism. We then have the following generalization of a result of Penrose's type (cf. Atiyah, Hitchin and Singer [A-H-S]; see also Salamon [S2], Berard-Bergery and Ochiai [B-O]):

THEOREM (0.2). *Let $\mathscr{H}$ (resp. $\tilde{\mathscr{H}}$) be the set of all Hermitian pairs (resp. all excellent pairs) on M (resp. Z). Then*

$$\mathscr{H} \ni (E, D_E) \mapsto (p^*E, p^*D_E) \in \tilde{\mathscr{H}}$$

defines a bijective correspondence between $\mathscr{H}$ and $\tilde{\mathscr{H}}$.

In particular, if M has positive scalar curvature, then every excellent pair (F, D_F) on Z is a Ricci-flat Einstein-Hermitian vector bundle.

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1. Notation, convention and preliminaries. In this section, we give a quick review of the basic facts on quaternionic Kähler manifolds (for more details see Salamon [S1], Nitta and Takeuchi [N-T]).

(1.1) Let $H^{(m)}$ denote the standard $Sp(m)$-module $\boldsymbol{H}^m$ $(=\boldsymbol{C}^{2m})$ of complex dimension $2m$, where $\boldsymbol{H} = \boldsymbol{R} + i\boldsymbol{R} + j\boldsymbol{R} + k\boldsymbol{R}$ $(=\boldsymbol{C} + j\boldsymbol{C})$. $Sp(m) = \{S \in GL(m, \boldsymbol{H}) \mid S \cdot {}^t\bar{S} = I\}$ is imbedded in $GL(2m, \boldsymbol{C})$ by

$$Sp(m) \ni A + jB \mapsto \begin{pmatrix} A, & -\bar{B} \\ B, & \bar{A} \end{pmatrix} \in GL(2m, \boldsymbol{C})$$

where $A, B \in GL(m, \boldsymbol{C})$. Then the multiplication on $\boldsymbol{H}^m$ by j from the right naturally induces a $Sp(m)$-equivariant anti-linear map $j^{(m)}: H^{(m)} \to H^{(m)}$ with $(j^{(m)})^2 = -\mathrm{id}$. We now define a non-degenerate skew-symmetric bilinear form $\omega^{(m)}$ on $\boldsymbol{H}^m$ by

$$\omega^{(m)}(h, h') := -\langle h, j^{(m)}h' \rangle \quad (h, h' \in \boldsymbol{H}^m)\ ,$$

where $\langle\ ,\ \rangle$ is the standard Hermitian inner product on $\boldsymbol{C}^{2m}$ $(=\boldsymbol{H}^m)$. This $\omega^{(m)}$ can be regarded as an $Sp(m)$-invariant bilinear form on $H^{(m)}$ such that

$$(1.1.1)\qquad \omega^{(m)}(j^{(m)}h, j^{(m)}h') = (\omega^{(m)}(h, h'))^- \quad (h, h' \in H^{(m)}) .$$

Let $Sp(n)\cdot Sp(1) = Sp(n)\times Sp(1)/\boldsymbol{Z}_2$. Then $H^{(n)} \otimes_C H^{(1)}$ is naturally a $Sp(n)\cdot Sp(1)$-module of complex dimension $4n$ with a real structure $H^{(n)} \otimes_C H^{(1)} \ni a \mapsto \bar{a} \in H^{(n)} \otimes_C H^{(1)}$ defined by

$$(1.1.2)\qquad (h \otimes h')^- := j^{(m)}h \otimes j^{(1)}h' \quad (h \in H^{(n)}, h' \in H^{(1)}) .$$

We consider the corresponding real form $(H^{(n)} \otimes_C H^{(1)})_R$ of $H^{(n)} \otimes_C H^{(1)}$. Then the symmetric bilinear form $\omega^{(n)} \otimes \omega^{(1)} \in S^2((H^{(n)})^* \otimes (H^{(1)})^*)$ induces an inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $(H^{(n)} \otimes_C H^{(1)})_R$.

(1.2) Recall that a $4n$-dimensional Riemannian manifold (M, g_M) is called a quaternionic Kähler manifold, if its linear holonomy group is contained in $Sp(n)\cdot Sp(1)$ $(\subset SO(4n))$ with the additional condition for $n = 1$ that g_M is a self-dual Einstein metric. *Throughout this paper, we fix once for all a quaternionic Kähler manifold* (M, g_M). By the well-known reduction theorem (see, for instance, Kobayashi and Nomizu [K-N]), the frame bundle of the tangent bundle TM is reduced to a principal $Sp(n)\cdot Sp(1)$-bundle P. Then TM can be regarded as the vector bundle

$$(1.2.1)\qquad P\times_{Sp(n)\cdot Sp(1)}(H^{(n)} \otimes_C H^{(1)})_R$$

associated to the $Sp(n)\cdot Sp(1)$-module $(H^{(n)} \otimes_C H^{(1)})_R$. The inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $(H^{(n)} \otimes_C H^{(1)})_R$ induces a Riemannian metric g on TM, which coincides with g_M up to constant multiple. Without loss of generality, we may assume $g = g_M$.

(1.3) Let $Sp(n)$ act trivially on $\boldsymbol{C}^2$. Then the standard $Sp(1)$-action on $\boldsymbol{C}^2$ naturally induces an $Sp(n)\times Sp(1)$-action (resp. $Sp(n)\cdot Sp(1)$-action) on $\boldsymbol{C}^2$ (resp. $\boldsymbol{P}^1\boldsymbol{C}$). Associated to these actions, we have:

$$\hat{p}\colon V(:=P\times_{Sp(n)\times Sp(1)}\boldsymbol{C}^2) \to M$$
$$(\text{resp. } p\colon Z(:=P\times_{Sp(n)\cdot Sp(1)}\boldsymbol{P}^1\boldsymbol{C}) \to M) ,$$

which is a "locally defined" vector bundle (resp. a globally defined fibre bundle). Here, the bundle Z is nothing but $\boldsymbol{P}(V) := V - \{\text{zero section}\}/\boldsymbol{C}^*$, and is called the twistor space of M (see Salamon [S1; p. 147]). Then Z is a complex manifold with a natural real structure τ as follows:

(1.3.1) By the connection on V induced from that of P, we have a decomposition of $T(V - \{\text{zero section}\})$ into the subbundles S^h and S^v corresponding respectively to horizontal and vertical distributions. Let y be an arbitrary point of $V - \{\text{zero section}\}$, and put $x := \hat{p}(y)$. Via the projection $\hat{p}$, the fibre $(S^h)_y$ of S^h over y is regarded as the tangent space T_xM at x. Then by the identification of $H^{(n)} \otimes_C H^{(1)}$ with $(T_xM)^C$ (cf. (1.2.1)), the space $H^{(n)} \otimes Cy$ defines a C-linear subspace of $(T_xM)^C$, denoted also by $H^{(n)} \otimes Cy$. Furthermore, let $(H^{(n)} \otimes Cy)'$ be the subspace of $(T_x^*M)^C$ corresponding to $H^{(n)} \otimes Cy$ via the natural isomorphism $(T_x^*M)^C \cong (T_xM)^C$ induced by g_M. Now we define the complex structure of T_yV by specifying the subspace $\wedge_y^{1,0}$ of (1, 0)-forms in $(T_y^*V)^C$ as follows:

$$\wedge_y^{1,0} = (\wedge_y^{1,0})^h \oplus (\wedge_y^{1,0})^v ,$$

where $(\wedge_y^{1,0})^h := \hat{p}^*((H^{(n)} \otimes Cy)')$, and $(\wedge_y^{1,0})^v$ is the subspace of (1, 0)-forms in T_yC^2 by the identification of V_x with C^2. Then this induces a complex structure on Z.

(1.3.2) The map $j^{(1)}: H^{(1)} \to H^{(1)}$ naturally defines an antilinear bundle automorphism $\hat{\tau}: V \to V$, which induces a real structure τ on Z.

(1.3.3) Recall that M always has a constant scalar curvature (denoted by t). Let g_F be the Fubini-Study metric for P^1C $(=(C + jC - \{0\})/C^*)$. If $t \neq 0$, then for some nonzero real constant c_t,

$$g_Z := p^*g_M + c_t g_F$$

defines a pseudo-Kählerian metric on Z, i.e., the corresponding (1, 1)-form on Z is a nondegenerate d-closed (1, 1)-form.

2. A_2'-connections and B_2-connections. We shall here give fundamental properties of the A_2'-connections and B_2-connections defined in the Introduction.

(2.1) Let $(H^{(m)})^*$ be the dual $Sp(m)$-module of $H^{(m)}$. Then in view of $\wedge^2(H^{(1)})^* = C\omega^{(1)}$, we have

$$\wedge^2((H^{(n)})^* \otimes_C (H^{(1)})^*) = (\wedge^2(H^{(n)})^* \otimes_C S^2(H^{(1)})^*) \oplus (S^2(H^{(n)*}) \otimes_C C\omega^{(1)}) .$$

Furthermore, the $Sp(n)$-module $\wedge^2(H^{(n)})^*$ is written as a direct sum $C\omega^{(n)} + \wedge_0^2(H^{(n)})^*$ of its submodules, where $\wedge_0^2(H^{(n)})^*$ is the orthogonal complement of $C\omega^{(n)}$ in $\wedge^2(H^{(n)})^*$. Hence,

$$\wedge^2((H^{(n)})^* \otimes_C (H^{(1)})^*) = N_2'^C \oplus N_2''^C \oplus L_2^C , \tag{2.1.1}$$

where $N_2'^C := C\omega^{(n)} \otimes_C S^2(H^{(1)})^*$, $N_2''^C := \wedge_0^2(H^{(n)})^* \otimes_C S^2(H^{(1)})^*$ and $L_2^C := S^2(H^{(n)})^* \otimes_C C\omega^{(1)}$. Note that the $Sp(n)\cdot Sp(1)$-modules $N_2'^C$, $N_2''^C$, L_2^C respectively admit real forms N_2', N_2'', L_2 fixed by the real structure induced from the one in (1.1.2). We have the identification $H^{(n)} \otimes_C H^{(1)} \cong (H^{(n)})^* \otimes_C (H^{(1)})^*$ by the metric $\langle\langle\, ,\, \rangle\rangle$ (cf. (1.1)). Together with $H^{(n)} \otimes_C H^{(1)} \cong \boldsymbol{H}^n \otimes_R C$, the above (2.1.1) induces the decomposition of its real form:

$$\wedge^2 \boldsymbol{H}^n = N_2' \oplus N_2'' \oplus L_2 \,,$$

which is nothing but the decomposition in the Introduction now for our principal $Sp(n)\cdot Sp(1)$-bundle P, the bundle T^*M is regarded as the vector bundle associated to the $Sp(n)\cdot Sp(1)$-module $((H^{(n)})^* \otimes_C (H^{(1)*}))_R = \boldsymbol{H}^n$. Hence, $\wedge^2 T^*M$ is a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its subbundles A_2', A_2'', B_2 corresponding respectively to the $Sp(n)\cdot Sp(1)$-modules N_2', N_2'', L_2 (cf. Introduction).

(2.2) Fix an arbitrary point x of M. Note that each point z on the fibre Z_x defines an almost complex structure J_z on T_x^*M (cf. (1.3.1)). We then have the corresponding space $\wedge^{1,1}(T_x^*M, J_z)$ of (1, 1)-forms of (T_x^*M, J_z). Choose a point $y(\neq 0)$ of V such that its natural image (denoted by $[y]$) is z. In view of (1.3.1), the space $\wedge^{1,1}(T_x^*M, J_z)$ in $\wedge^2(T_x^*M)^C$ is associated to the C-linear subspace $(H^{(n)} \otimes_C Cy)' \wedge ((H^{(n)} \otimes_C Cy)')^-$ in the $Sp(n)\cdot Sp(1)$-module $(H^{(n)} \otimes_C H^{(1)})^* \wedge (H^{(n)} \otimes_C H^{(1)})^*$. Since $j^{(n)}$ preserves $H^{(n)}$, we have (cf. (1.1.2)):

$$\begin{aligned}(H^{(n)} \otimes_C Cy) \wedge ((H^{(n)} \otimes_C Cy)^-) &= (H^{(n)} \otimes_C Cy) \wedge (H^{(n)} \otimes_C Cj^{(1)}y)\\ &= (\wedge^2 H^{(n)} \otimes_C C(y \otimes j^{(1)}y + j^{(1)}y \otimes y)) \oplus (S^2 H^{(n)} \otimes_C C(y \wedge j^{(1)}y)) \,.\end{aligned}$$

The space $C(y \wedge j^{(1)}y)$ (where $y \wedge j^{(1)}y = (y \otimes j^{(1)}y - j^{(1)}y \otimes y)/2$) in $H^{(1)} \otimes_C H^{(1)}$ corresponds to $C\omega^{(1)}$ in $(H^{(1)})^* \otimes_C (H^{(1)})^*$ via the natural isomorphism $H^{(1)} \otimes_C H^{(1)} \cong (H^{(1)})^* \otimes_C (H^{(1)})^*$ induced by the nondegenerate bilinear form $\omega^{(1)}$. Furthermore,

$$\cap_y C(y \otimes j^{(1)}y + j^{(1)}y \otimes y) = \{0\} \,,$$

where $\cap_y$ always denotes the intersection taken over all y in $V_x - \{0\}$. Thus,

$$\cap_y (H^{(n)} \otimes Cy)' \wedge \overline{(H^{(n)} \otimes Cy)'} = S^2(H^{(n)})^* \otimes_C C\omega^{(1)} = L_2 \quad \text{(cf. Introduction)}\,,$$

and we obtain:

LEMMA (2.3). *The fibre $(B_2)_x$ of B_2 over x is given by*

$$(B_2)_x = \cap_y \wedge^{1,1}(T_x^*M, J_{[y]}) \,.$$

We next give a typical example of an A_2'-connection and also a B_2-connection.

EXAMPLE (2.4). If $n \geqq 2$, the induced connection on the locally defined vector bundle

$$V := P \times_{Sp(n)\times Sp(1)} H^{(1)} \quad (\text{resp. } W := P \times_{Sp(n)\times Sp(1)} H^{(n)})$$

is an A_2'-connection (resp. B_2-connection). See Salamon [S1; p. 150] for related computations of curvatures.

Recall that a connection ∇ is called a *Yang-Mills connection* if the corresponding curvature R^{∇} satisfies $d^{\nabla}*R^{\nabla}=0$. We shall finally show:

THEOREM (2.5). *All A_2'-connections and also all B_2-connections are Yang-Mills connections.*

COROLLARY (2.6). *The Riemannian connection on TM is a Yang-Mills connection.*

PROOF OF (2.6). By (1.2), (2.4) and (2.5), we obtain (2.6).

PROOF OF (2.5). Fix an arbitrary point x_0 of M. It then suffices to show $(d^{\nabla}*R^{\nabla})(x_0)=0$. We may take a local section s to P over a neighbourhood U of x_0 such that the corresponding differential at the point x_0 transforms the tangent space $T_{x_0}M$ to a horizontal space at $s(x_0)$ in the tangent space $T_{s(x_0)}P$. Let $(u^1, \cdots, u^{4n})$ be the local frame of $T^*M_{|U}$ associated to s. Then all covariant derivatives of u^i's $(1 \leqq i \leqq 4n)$ at the point x_0 is zero. Moreover in terms of the frame $(u^1, \cdots, u^{4n})$, we can identify $T^*M_{|U}$ with $U\times \boldsymbol{R}^{4n}$ $(U\times \boldsymbol{H}^n)$. Note that ∇ on E naturally induces a connection (denoted by the same ∇) on $\mathrm{End}(E)$.

(i) We first assume that ∇ is an A_2'-connection on E. Recall that the rank 3 subbundle A_2' of $\wedge^2 T^*M$ corresponds to the $Sp(n)\cdot Sp(1)$-submodule N_2' of $\wedge^2\boldsymbol{H}^n$, where N_2' is the irreducible submodule of the elements fixed by $Sp(n)$ (cf. Introduction). Let I, J and K be

$$I=\sum_{k=0}^{n-1}(u^{4k+1}\wedge u^{4k+2}+u^{4k+3}\wedge u^{4k+4}),$$

$$J=\sum_{k=0}^{n-1}(u^{4k+1}\wedge u^{4k+3}+u^{4k+4}\wedge u^{4k+2}),$$

$$K=\sum_{k=0}^{n-1}(u^{4k+1}\wedge u^{4k+4}+u^{4k+2}\wedge u^{4k+3}).$$

Then it is easy to check that $A'_{2|U}$ is spanned by the sections I, J and K. Therefore, the curvature form R^{∇} is written on U as

$$R^{\nabla}=a\otimes I+b\otimes J+c\otimes K,$$

where a, b and c are smooth sections to $\mathrm{End}(E)$ over U. Let $(u_1, \cdots, u_{4n})$ be the base for $TM_{|U}$ dual to $(u^1, \cdots, u^{4n})$ defined by $u^i(u_j)=\delta_{ij}$. Then by the first Bianchi identity,

$$\begin{aligned}0&=d^{\nabla}(R^{\nabla})(x_0)\\&=\sum_{i=1}^{4n}\{(\nabla_i a)u^i(x_0)\wedge I(x_0)+(\nabla_i b)u^i(x_0)\wedge J(x_0)+(\nabla_i c)u^i(x_0)\wedge K(x_0)\},\end{aligned}$$

where ∇_i denotes $\nabla_{u_i(x_0)}$. Consequently,

$$\nabla_i a = \nabla_i b = \nabla_i c = 0 , \quad \text{for} \quad 1 \leqq i \leqq 4n \quad \text{if} \quad n \geqq 2 .$$

Therefore, $(d^\nabla * R^\nabla)(x_0) = 0$.

(ii) We next assume that ∇ is a B_2-connection on E. Since the vector subbundle B_2 (of rank $n(2n+1)$) of $\wedge^2 T^*M$ corresponds to the irreducible $Sp(n)\cdot Sp(1)$-submodule L_2 of the elements in $\wedge^2 \boldsymbol{H}^n$ fixed by $Sp(1)$, the subbundle $B_{2|U}$ is spanned by

$$I_s,\ J_s,\ K_s,\ D_{pq},\ E_{pq},\ F_{pq},\ G_{pq}\ , \quad (0 \leqq s \leqq n-1,\ 0 \leqq p < q \leqq n-1) .$$

where

$$\begin{aligned}
I_s &= u^{4s+1} \wedge u^{4s+2} - u^{4s+3} \wedge u^{4s+4} , \\
J_s &= u^{4s+1} \wedge u^{4s+3} - u^{4s+4} \wedge u^{4s+2} , \\
K_s &= u^{4s+1} \wedge u^{4s+4} - u^{4s+2} \wedge u^{4s+3} , \\
D_{pq} &= u^{4p+1} \wedge u^{4q+1} + u^{4p+2} \wedge u^{4q+2} + u^{4p+3} \wedge u^{4q+3} + u^{4p+4} \wedge u^{4q+4} , \\
E_{pq} &= u^{4p+1} \wedge u^{4q+2} - u^{4p+2} \wedge u^{4q+1} - u^{4p+3} \wedge u^{4q+4} + u^{4p+4} \wedge u^{4q+3} , \\
F_{pq} &= u^{4p+1} \wedge u^{4q+3} + u^{4p+2} \wedge u^{4q+4} - u^{4p+3} \wedge u^{4q+1} - u^{4p+4} \wedge u^{4q+2} , \\
G_{pq} &= u^{4p+1} \wedge u^{4q+4} - u^{4p+2} \wedge u^{4q+3} + u^{4p+3} \wedge u^{4q+2} - u^{4p+4} \wedge u^{4q+1} .
\end{aligned}$$

Let ∇ be a B_2-connection on E. Then over U, the curvature form R^∇ is written in the form

$$\begin{aligned}
R^\nabla = &\sum_{0 \leqq s \leqq n-1} (i_s \otimes I_s + j_s \otimes J_s + k_s \otimes K_s) \\
&+ \sum_{0 \leqq p < q \leqq n-1} (d_{pq} \otimes D_{pq} + e_{pq} \otimes E_{pq} + f_{pq} \otimes F_{pq} + g_{pq} \otimes G_{pq}) ,
\end{aligned}$$

where i_s, j_s, k_s, d_{pq}, e_{pq}, f_{pq} and g_{pq} are smooth sections to $\mathrm{End}(E)$ over U. In view of the first Bianchi identity $d^\nabla R^\nabla = 0$, we have

$$\begin{aligned}
-\nabla_{4s+3} i_s + \nabla_{4s+2} j_s + \nabla_{4s+1} k_s &= 0 , \\
\nabla_{4s+1} i_s - \nabla_{4s+4} j_s + \nabla_{4s+3} k_s &= 0 , \\
\nabla_{4s+4} i_s + \nabla_{4s+1} j_s - \nabla_{4s+2} k_s &= 0 , \\
\nabla_{4s+2} i_s + \nabla_{4s+3} j_s + \nabla_{4s+4} k_s &= 0 ,
\end{aligned}$$

for s with $0 \leqq s \leqq n-1$. Furthermore, if l is either p or q, the identity $d^\nabla R^\nabla = 0$ implies

$$\begin{aligned}
(-1)^{\varepsilon(l)} \nabla_{4l+1} d_{pq} - \nabla_{4l+2} e_{pq} - \nabla_{4l+3} f_{pq} - \nabla_{4l+4} g_{pq} &= 0 , \\
(-1)^{\varepsilon(l)} \nabla_{4l+1} d_{pq} - \nabla_{4l+3} e_{pq} + \nabla_{4l+2} f_{pq} + \nabla_{4l+1} g_{pq} &= 0 , \\
(-1)^{\varepsilon(l)} \nabla_{4l+2} d_{pq} + \nabla_{4l+1} e_{pq} - \nabla_{4l+4} f_{pq} + \nabla_{4l+3} g_{pq} &= 0 , \\
(-1)^{\varepsilon(l)} \nabla_{4l+3} d_{pq} + \nabla_{4l+4} e_{pq} + \nabla_{4l+1} f_{pq} - \nabla_{4l+2} g_{pq} &= 0 ,
\end{aligned}$$

for all p, q with $0 \leqq p < q \leqq n-1$, where $\varepsilon(p) := 0$ and $\varepsilon(q) := 1$.

Then a straightforward computation shows that $(d^{\nabla} * R^{\nabla})(x_0) = 0$, as required.

3. Deformations of B_2-connections. In this section, we shall give an elliptic complex whose first cohomology group canonically contains the space of infinitesimal deformations of B_2-connections on M (see Salamon [S2] for a similar complex).

(3.1) Let r be an integer with $r \geqq 2$. By setting $N_r^C := \wedge^r (H^{(n)})^* \otimes_C S^r(H^{(1)})^*$ (cf. (2.1)), we can express the $Sp(n)\cdot Sp(1)$-module $\wedge^r (H^{(n)} \otimes_C H^{(1)})^*$ as a direct sum $N_r^C \oplus L_r^C$, where L_r^C is the orthogonal complement of N_r^C in $\wedge^r (H^{(n)} \otimes_C H^{(1)})^*$. As in (2.1), the $Sp(n)\cdot Sp(1)$-modules N_r^C and L_r^C respectively admit real forms N_r and L_r fixed by the natural real structure (cf. (1.1.2)). Since T^*M is associated to the $Sp(n)\cdot Sp(1)$-module $(H^{(n)} \otimes_C H^{(1)})_R^*$ (see (1.2.1)), the vector bundle $\wedge^r T^*M$ is a direct sum $A_r \oplus B_r$ of its subbundles A_r, B_r corresponding respectively to N_r, L_r. Let π^r: $\wedge^r T^*M$ $(=A_r \oplus B_r) \to A_r$ be the projection to the first factor. Then we have:

THEOREM (3.2). *For a B_2-connection ∇ on E, the following is an elliptic complex*:

$$(3.2.1) \qquad 0 \to \mathscr{E}(E) \xrightarrow{\nabla} \mathscr{E}(E \otimes T^*M) \xrightarrow{d_1} \mathscr{E}(E \otimes A_2)$$
$$\xrightarrow{d_2} \mathscr{E}(E \otimes A_3) \xrightarrow{d_3} \cdots \xrightarrow{d_{2n-1}} \mathscr{E}(E \otimes A_{2n}) \to 0 \ ,$$

where $d_i := (\mathrm{id} \otimes \pi^{i+1}) \circ d^{\nabla}$ and for every vector bundle E' on M, we denote by $\mathscr{E}(E')$ the sheaf of germs of C^∞-sections of E'.

PROOF. (i) Fix a section $s \in \Gamma(M, E \otimes A_i)$ $(i \geqq 1)$ and define a section $t \in \Gamma(M, E \otimes B_{i+1})$ by

$$d^{\nabla} s = d_i s + t \ .$$

Then from $(d^{\nabla} \circ d^{\nabla})s = (d^{\nabla} \circ d_i)s + d^{\nabla} t$, we obtain

$$((\mathrm{id} \otimes \pi_{i+2}) \circ d^{\nabla} \circ d^{\nabla})s = (d_{i+1} \circ d_i)s + ((\mathrm{id} \otimes \pi_{i+2}) \circ d^{\nabla})t \ .$$

Since ∇ is a B_2-connection, the A_{i+2}-component of $(d^{\nabla} \circ d^{\nabla})s$ is zero, i.e.,

$$0 = (d_{i+1} \circ d_i)s + ((\mathrm{id} \otimes \pi_{i+2}) \circ d^{\nabla})t \ .$$

Write t as $t = \sum_k v_k \otimes b_k$ locally, where v_k, b_k is a local section of E, B_{i+1}, respectively. The $S^{i+1}(V^*)$-component of b_k is zero, and hence the $S^{i+2}(V^*)$-component of $\nabla(v_k) \wedge b_k$ is zero. Therefore,

$$((\mathrm{id} \otimes \pi_{i+2}) \circ d^{\nabla})t = \sum_k v_k \otimes db_k \ .$$

Since d is the composite of the Riemannian connection and the alternation operator, the $S^{i+2}(V^*)$-component of db_k is zero. Thus, $(d_{i+1}\circ d_i)s = 0$, as required.

(ii) Secondly, we shall show that (3.1.1) is an elliptic complex. Then we need to calculate the symbol $\sigma(d_i, u)$ ($u \in T_x^*M - \{0\}$). Fix a point of M and an element s of $E_x \otimes A_{ix}$. All computations below are taken at the point x.

$$\sigma(d_i, u)s := (d/dt)(e^{-tq}d_i(e^{tq}s))|_{t=0} = (\mathrm{id} \otimes \pi_{i+1})(u \wedge s) ,$$

where q is a locally defined function such that $dq_x = u$. We next show that the following sequence is exact for every u:

$$E \otimes A_{i-1} \xrightarrow{\sigma(d_{i-1}, u)} E \otimes A_i \xrightarrow{\sigma(d_i, u)} E \otimes A_{i+1} . \tag{3.2.2}$$

Without loss of generality, we may assume

$$u = e_1 \otimes h_1 + (e_1 \otimes h_1)^{-} (= e_1 \otimes h_1 + e_2 \otimes h_2) ,$$

where $\langle e_1, \cdots, e_{2n}\rangle$ (resp. $\langle h_1, h_2\rangle$) is a symplectic basis of $W^* \cong W$ (resp. $V^* \cong V$), i.e., an orthonormal basis and $j^{(n)}e_{2j+1} = e_{2j+2}$ (resp. $j^{(1)}h_1 = h_2$). Let $s \in E \otimes A_i$ be such that $\sigma(d_{i+1}, u)s = 0$. Note that $S^i V^* = \mathrm{Span}(h_1^k \cdot h_2^{i-k};\ 0 \leqq k \leqq i)$, where $h_1^k \cdot h_2^{i-k}$ denotes the symmetric component of $h_1^k \otimes h_2^{i-k}$. Hence, there are local sections $s_0, \cdots, s_i$ of $E \otimes \wedge^i W^*$ such that

$$s = \sum_{k=0}^{i} s_k \otimes h_1^k \cdot h_2^{i-k} .$$

We can now write $\sigma(d_{i+1}, s) = 0$ as follows:

$$\begin{aligned} 0 &= (\mathrm{id} \otimes \pi_{i+1})(u \wedge s) = (\mathrm{id} \otimes \pi_{i+1})((e_1 \otimes h_1 + e_2 \otimes h_2) \wedge \textstyle\sum s_k \otimes h_1^k \cdot h_2^{i-k}) \\ &= \sum_{k=0}^{i}((e_1 \wedge s_k) \otimes h_1^{k+1} \cdot h_2^{i-k} + (e_2 \wedge s_k) \otimes h_1^k \cdot h_2^{i+1-k}) . \end{aligned}$$

Since the coefficient of the right-hand side in $h_1^k \cdot h_2^{i+1-k}$ is zero, we have:

$$\begin{aligned} &(0) && e_2 \wedge s_0 = 0 , \\ &(1) && e_1 \wedge s_0 + e_2 \wedge s_1 = 0 , \\ &&& \vdots \\ &(\mathrm{i}) && e_1 \wedge s_{i-1} + e_2 \wedge s_i = 0 , \\ &(\mathrm{i}+1) && e_1 \wedge s_i = 0 . \end{aligned}$$

By (0), there exists $r_0 \in \wedge^{i-1} W^*$ such that $s_0 = e_2 \wedge r_0$. Plugging this into (1), we obtain $e_2 \wedge (-e_1 \wedge r_0 + s_1) = 0$. Hence there exists $r_1 \in \wedge^{i-1} W^*$ such that $s_1 = e_1 \wedge r_0 + e_2 \wedge r_1$. Repeating this process inductively, we obtain $r_k \in \wedge^{i-1} W^*$ such that $s_k = e_1 \wedge r_{k-1} + e_2 \wedge r_k$, $1 \leqq k \leqq i$. Now by (i + 1), the identity $e_1 \wedge e_2 \wedge r_i = 0$ holds. It then follows that there exists $r'_i \in \wedge^{i-2} W^*$ such that $e_2 \wedge r_i = e_1 \wedge e_2 \wedge r'_i$. Since $e_2 \wedge (r_{i-1} + e_2 \wedge r'_i) = e_2 \wedge r_{i-1}$, we may replace r_{i-1} by $r_{i-1} + e_2 \wedge r'_i$. Therefore,

$$\begin{aligned} s_0 &= \qquad\qquad e_2 \wedge r_0 \,, \\ s_1 &= e_1 \wedge r_0 + e_2 \wedge r_1 \,, \\ &\vdots \\ s_i &= e_1 \wedge r_{i-1} \end{aligned}$$

Thus,

$$s = \sum_{k=0}^{i} s_k \otimes h_1^k \cdot h_2^{i-k} = \sigma(d_{i-1}, u)\Big(\sum_{k=0}^{i-1} r_k \otimes h_1^k \cdot h_2^{i-1-k}\Big) \,,$$

i.e., the sequence (3.2.2) is exact, as required.

DEFINITION (3.3). Let $\mathscr{C}$ be the set of all B_2-connections on E with holonomy groups contained in a compact semisimple Lie group G. Assume that $\mathscr{C} \neq \emptyset$ and let $\nabla \in \mathscr{C}$. Then the frame bundle Q of E can be regarded as a principal G-bundle. Put $G_Q := Q \times_\theta G$ and $\mathfrak{g}_Q := Q \times_{\mathrm{Ad}} \mathfrak{g}$, where θ is the group conjugation and $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is the adjoint representation of G. Now, a C^∞-section to G_Q over M is called a *gauge transformation* of Q. Let $\mathscr{G}$ be the set of all gauge transformations of Q. Then $\mathscr{G}$ naturally acts on $\mathscr{C}$ (see Atiyah-Hitchin-Singer [A-H-S]). We call $\mathscr{M}(:=\mathscr{C}/\mathscr{G})$ the *moduli space* of the B_2-connections on E with holonomy groups in G.

(3.4) Let $\nabla \in \mathscr{C}$ be irreducible in the sense that $\mathfrak{g}_Q$ admits no non-zero parallel section over M. Fix a smooth one-parameter family ∇^t ($|t| < \varepsilon$) of connections in $\mathscr{C}$ such that $\nabla^0 = \nabla$. Put $S = (d/dt)\nabla^t|_{t=0}$. We write the curvature form R^{∇^t} of ∇^t as

$$R^{\nabla^t} = R^{\nabla} + t d^{\nabla'} S + \text{higher order terms in } t \,,$$

where ∇' is the connection on $\mathfrak{g}_Q$ naturally induced by ∇. Since R^{∇^t} is a $\mathfrak{g}_Q$-valued B_2-form, the corresponding derivative $d^{\nabla'}S$ at $t = 0$ also satisfies

$$((\mathrm{id} \otimes \pi^2) \circ d^{\nabla'})S = 0 \,.$$

Let f^t ($|t| < \varepsilon$) be a one-parameter family of gauge transformations such that $f^0 = \mathrm{id}$. Then,

$$\frac{d}{dt}(f^t(\nabla))_{|t=0} = \nabla'(\dot{f}) \,,$$

where $\dot{f} := (d/dt)(f^t)|_{t=0}$. Since $f^t(\nabla) \in \mathscr{C}$ for all t, the same argument as above shows that the $\mathfrak{g}_Q$-valued 1-form $\nabla'(\dot{f})$ satisfies

$$((\mathrm{id}\otimes\pi^2)\circ d^{\nabla'})(\nabla'(\dot f)) = 0 .$$

For each $A\in\Gamma(\mathfrak{g}_Q)$, there exists a one-parameter family $f^t = \exp(tA)$ such that $(d/dt)f^t|_{t=0} = A$. Then together with (3.2), we immediately obtain the following:

THEOREM (3.5). *Assume that $\mathscr{C}\neq\emptyset$ and let $\nabla\in\mathscr{C}$ be irreducible. Then the space of infinitesimal (essential) deformations at ∇ of connections in $\mathscr{C}$, that is, the tangent space of $\mathscr{M}$ at ∇ is a linear subspace of the first cohomology group of the elliptic complex*

$$\begin{aligned}0\to\mathscr{E}(\mathfrak{g}_Q)&\xrightarrow{\nabla'}\mathscr{E}(\mathfrak{g}_Q\otimes T^*M)\xrightarrow{d_1'}\mathscr{E}(\mathfrak{g}_Q\otimes A_2)\\&\xrightarrow{d_2'}\mathscr{E}(\mathfrak{g}_Q\otimes A_3)\xrightarrow{d_3'}\cdots\xrightarrow{d_{2n-1}'}\mathscr{E}(\mathfrak{g}_Q\otimes A_{2n})\to 0\ ,\end{aligned}$$

where $d_i' := (\mathrm{id}\otimes\pi^{i+1})\circ d^{\nabla'}$.

4. Einstein-Hermitian connections associated with B_2-connections. In this section we shall prove Theorem (0.2) (see the Introduction) which clarifies the relationship between B_2-connections and the corresponding Einstein-Hermitian connections.

PROOF OF (0.2). (i) Let (E, D_E) be a Hermitian pair. Then by the definition of B_2-connections, the curvature form corresponding to the connection D_E is an End(E)-valued B_2-form, and by Lemma (2.3) the curvature form corresponding to the connection p^*D_E on p^*E is an End(p^*E)-valued (1, 1)-form. Hence the connection p^*D_E induces naturally an integrable complex structure on p^*E as follows: Put $l := \mathrm{rank}(E)$ and denote by $q\colon p^*E\to Z$ the natural projection. Let $(s_1, \cdots, s_l)$ (resp. $(y^1, \cdots, y^l)$) be a local unitary frame for p^*E (resp. the dual frame corresponding to $(s_1, \cdots, s_l)$). Then the vector subbundle $\wedge^{1,0}T^*(p^*E)$ of type (1, 0) in the complexification $T^*(p^*E)^C$ of the cotangent bundle $T^*(p^*E)$ is defined as the direct sum of the pull-back $q^*(\wedge^{1,0}T^*Z)$ and the space spanned by $\{dy^j + \sum_{i=1}^l y^i q^*\theta_{ji},\ 1\leqq j\leqq l\}$, where (θ_{ij}) is the connection matrix for p^*D_E with respect to the frame $(s_1, \cdots, s_l)$ (i.e., $(p^*D_E)s_j = \sum_{i=1}^l s_i\theta_{ij}$). Now, we may take the frame $(s_1, \cdots, s_l)$ as the pull-back $(p^*t_1, \cdots, p^*t_l)$ of a local unitary frame $(t_1, \cdots, t_l)$ on E. Then the 1-forms θ_{ij}, $1\leqq i,\ j\leqq l$, are written as $p^*\psi_{ij}$, where (ψ_{ij}) denotes the connection matrix for D_E with respect to the frame $(t_1, \cdots, t_l)$. Let $q'\colon (p^*E)^*\to Z$ be the projection naturally induced from $q\colon p^*E\to Z$. Since the real structure $\tau\colon Z\to Z$ is antiholomorphic (cf. Nitta and Takeuchi [N-T]), and since the mapping $q'\circ\sigma\colon p^*E\to Z$ is equal to $\tau\circ q$, the mapping $\sigma\colon p^*E\to(p^*E)^*$ is clearly an antiholomorphic bundle automorphism by the definition of the complex structures on p^*E and $(p^*E)^*$.

(ii) We next fix an arbitrary excellent pair (F, D_F) on Z. Then by the condition (a) in the definition of excellent pair (see the Introduction), we can choose an open cover $\{U_\lambda\}$ of M, and a local unitary frame $(f_1^\lambda, \cdots, f_r^\lambda)$ ($r = \text{rank of } F$) of $F_{|p^{-1}(U_\lambda)}$ such that each restriction $(f_{1|p^{-1}(x)}^\lambda, \cdots, f_{r|p^{-1}(x)}^\lambda)$ over $p^{-1}(x)$ ($x \in U_\lambda$) forms a holomorphic frame for $F_{|p^{-1}(x)}$. When $U_\lambda \cap U_\mu \neq \varnothing$, the transition matrix for F in terms of the frames $(f_1^\lambda, \cdots, f_r^\lambda)$, $(f_1^\mu, \cdots, f_r^\mu)$ is holomorphic (and hence constant) along each fibre $p^{-1}(x)$ ($x \in U_\lambda \cap U_\mu$). Hence there exists a Hermitian vector bundle E on M such that, including metrics, we have $p^*E = F$. In particular, we obtain a local unitary frame $(f_1'^\lambda, \cdots, f_r'^\lambda)$ for $E_{|U_\lambda}$ such that $(p^*f_1'^\lambda, \cdots, p^*f_r'^\lambda)$ coincides with the previous $(f_1^\lambda, \cdots, f_r^\lambda)$ over $p^{-1}(U_\lambda)$. Fix an arbitrary λ. If there is no fear of confusion, we shall omit the suffix λ and denote U_λ, $(f_1^\lambda, \cdots, f_r^\lambda)$, $\cdots$ simply by U, $(f_1, \cdots, f_r)$, $\cdots$, respectively. Let (ω_{ij}) be the connection matrix of D_F with respect to the frame $(f_1, \cdots, f_r)$, i.e., $D_F f_j = \sum_{i=1}^r f_i \omega_{ij}$. Furthermore, we choose a local symplectic basis $(e_1, \cdots, e_{2n})$ (resp. (h_1, h_2)) for $W^*_{|U}$ (resp. $V^*_{|U}$) (see Section 3). Now, since D_F is a Hermitian connection, we have:

$$\omega_{ij} + \overline{\omega_{ji}} = 0\,, \quad \text{for} \quad 1 \leqq i,\ j \leqq r\,. \tag{1}$$

Then the construction of D_E is reduced to showing that there exist 1-forms ω'_{ij} ($1 \leqq i,\ j \leqq r$) on U satisfying $\omega_{ij} = p^*\omega'_{ij}$. In fact, once we can find such 1-forms ω'_{ij}, they define a Hermitian connection on E, such that the corresponding curvature form is pulled back by p to an End(F)-valued (1, 1)-form on Z, which together with Lemma (2.3) implies that our connection on E is a B_2-connection. Recall that, for each $x \in U$, the frame $(f_{1|p^{-1}(x)}, \cdots, f_{r|p^{-1}(x)})$ for $F_{|p^{-1}(x)}$ is trivial. Hence,

$$\omega_{ij}(v) = 0\,, \quad 1 \leqq i,\ j \leqq r\,, \tag{2}$$

for every vector v tangent to $p^{-1}(x)$ ($\cong \boldsymbol{P}^1\boldsymbol{C}$). Since $(e_1 \otimes h_1, e_1 \otimes h_2, \cdots, e_{2n} \otimes h_1, e_{2n} \otimes h_2)$ is a frame for $T^*M^C{}_{|U} = W^*{}_{|U} \otimes V^*{}_{|U}$, there exist by (2) C^∞-functions a_{ij}^k, b_{ij}^k ($1 \leqq i,\ j \leqq r$, $1 \leqq k \leqq 2n$) on $p^{-1}(U)$ such that

$$\omega_{ij} = \sum_{k=1}^{2n} (a_{ij}^k p^*(e_k \otimes h_1) + b_{ij}^k p^*(e_k \otimes h_2))\,, \quad 1 \leqq i,\ j \leqq r\,. \tag{3}$$

For every form η on $Z_{|U}$, we denote by $\hat{\eta}$ the pull-back of η to $(V - \{\text{zero section}\})_{|U}$. Then by (3), we have:

$$\begin{aligned} \hat{R}_{ij} &= d\hat{\omega}_{ij} + \sum_{t=1}^{r} \hat{\omega}_{it} \wedge \hat{\omega}_{tj} \\ &= \sum_{k=1}^{2n} d(\hat{a}_{ij}^k \hat{p}^*(e_k \otimes h_1)) + d(\hat{b}_{ij}^k \hat{p}^*(e_k \otimes h_2)) + \sum_{t=1}^{r} \hat{\omega}_{it} \wedge \hat{\omega}_{tj}\,. \end{aligned}$$

Fix an arbitrary point x on U. Choosing an appropriate $(e_1, \cdots, e_{2n})$ (resp.

(h_1, h_2)), we may assume that $(\nabla^{V^*}e_k)(x)=0$, $k=1, 2, \cdots, 2n$ (resp. $(\nabla^{W^*}h_i)(x)=0$, $i = 1, 2$), where ∇^{V^*} (resp. ∇^{W^*}) denotes the connection of V^* (resp. W^*) canonically induced by that of P (cf. Example (2.4)). Then, on $\hat{p}^{-1}(x)$,

$$\hat{R}_{ij} = \sum_{k=1}^{2n}\{d(\hat{a}^k_{ij}) \wedge \hat{p}^*(e_k \otimes h_1) + d(\hat{b}^k_{ij}) \wedge \hat{p}^*(e_k \otimes h_2)\} + \sum_{t=1}^{r} \hat{\omega}_{it} \wedge \hat{\omega}_{tj} .$$

Recall that the complex structure on the twistor space Z ($=(V - \{\text{zero section}\})/C^*$) is induced by the complex structure on $V - \{\text{zero section}\}$ (see Section 1). Since $\hat{R}_{ij}$ is of type (1, 1), we have:

$$(4)\qquad \sum_{k=1}^{2n}\{\partial(\hat{a}^k_{ij}) \wedge (\hat{p}^*(e_k \otimes h_1))^{(1,0)} + \partial(\hat{b}^k_{ij}) \wedge (\hat{p}^*(e_k \otimes h_2))^{(1,0)}\} + \sum_{t=1}^{r} \hat{\omega}^{(1,0)}_{it} \wedge \hat{\omega}^{(1,0)}_{tj} = 0 \quad \text{on} \quad p^{-1}(x) ;$$

$$(5)\qquad \sum_{k=1}^{2n}\{\bar{\partial}(\hat{a}^k_{ij}) \wedge (\hat{p}^*(e_k \otimes h_1))^{(0,1)} + \bar{\partial}(\hat{b}^k_{ij}) \wedge (\hat{p}^*(e_k \otimes h_2))^{(0,1)}\} + \sum_{t=1}^{r} \hat{\omega}^{(0,1)}_{it} \wedge \hat{\omega}^{(0,1)}_{tj} = 0 \quad \text{on} \quad p^{-1}(x) ,$$

where for every 1-forms ζ on $(V - \{\text{zero section}\})_{|U}$, $\zeta^{(1,0)}$ (resp. $\zeta^{(0,1)}$) always denotes the (1, 0)-component (resp. (0, 1)-component) of ζ. Let (z^1, z^2) be the local triviality for $V_{|U}$ corresponding to (h_1, h_2). Then, by the definition of the complex structure of ($V - \{\text{zero section}\}$), we obtain from (4) and (5) the following:

$$(4')\qquad \sum_{k=1}^{2n} \Big\{\Big(\frac{\partial}{\partial z^1}\hat{a}^k_{ij}dz^1 + \frac{\partial}{\partial z^2}\hat{a}^k_{ij}dz^2\Big) \wedge \bar{z}^1(z^1\hat{p}^*(e_k \otimes h_1) + z^2\hat{p}^*(e_k \otimes h_2)) + \Big(\frac{\partial}{\partial z^1}\hat{b}^k_{ij}dz^1 + \frac{\partial}{\partial z^2}\hat{b}^k_{ij}dz^2\Big) \wedge \bar{z}^2(z^1\hat{p}^*(e_k \otimes h_1) + z^2\hat{p}^*(e_k \otimes h_2))\Big\} = 0 \quad \text{on} \quad \hat{p}^{-1}(x) ;$$

$$(5')\qquad \sum_{k=1}^{2n} \Big\{\Big(\frac{\partial}{\partial \bar{z}^1}\hat{a}^k_{ij}d\bar{z}^1 + \frac{\partial}{\partial \bar{z}^2}\hat{a}^k_{ij}d\bar{z}^2\Big) \wedge (-z^2)(\bar{z}^1\hat{p}^*(e_k \otimes h_2) - \bar{z}^2\hat{p}^*(e_k \otimes h_1)) + \Big(\frac{\partial}{\partial \bar{z}^1}\hat{b}^k_{ij}d\bar{z}^1 + \frac{\partial}{\partial \bar{z}^2}\hat{b}^k_{ij}d\bar{z}^2\Big) \wedge z^1(\bar{z}^1\hat{p}^*(e_k \otimes h_2) - \bar{z}^2\hat{p}^*(e_k \otimes h_1))\Big\} = 0 \quad \text{on} \quad \hat{p}^{-1}(x) .$$

Since both $z^1_{|p^{-1}(x)}$ and $z^2_{|p^{-1}(x)}$ are holomorphic on $\hat{p}^{-1}(x) \cong C^2 - \{0\}$, we have

$$\frac{\partial}{\partial \bar{z}^i}(z^1\bar{\hat{a}}^k_{ij} + z^2\bar{\hat{b}}^k_{ij}) = \frac{\partial}{\partial \bar{z}^i}(-z^2\hat{a}^k_{ij} + z^1\hat{b}^k_{ij}) = 0 \quad (i = 1, 2) ,$$

on $p^{-1}(x)$, i.e., both $f_1(z^1, z^2) := z^1\bar{\hat{a}}^k_{ij} + z^2\bar{\hat{b}}^k_{ij}$ and $f_2(z^1, z^2) := -z^2\hat{a}^k_{ij} + z^1\hat{b}^k_{ij}$ are holomorphic on $C^2 - \{0\}$. By Hartogs' theorem, both f_1 and f_2 extend further to holomorphic functions on C^2. Since $f_i(cz^1, cz^2) = cf_i(z^1, z^2)$ for all $z = (z^1, z^2) \in C^2$ and $c \in C^*$ $(i = 1, 2)$, there exist constants α^k_{ij}, β^k_{ij}, γ^k_{ij}, $\delta^k_{ij} \in C$ independent of z such that

$$(6)\qquad z^1\bar{\hat{a}}^k_{ij} + z^2\bar{\hat{b}}^k_{ij} = z^1\bar{\alpha}^k_{ij} + z^2\bar{\beta}^k_{ij}\ ,$$

$$(7)\qquad -z^2\hat{a}^k_{ij} + z^1\hat{b}^k_{ij} = -z^2\gamma^k_{ij} + z^1\delta^k_{ij}\ , \quad (1 \leqq k \leqq 2n)\ .$$

Let $\Gamma(Z, F^*)$ (resp. $\Gamma(Z, F^* \otimes T^*Z^C)$) be the space of global C^∞-sections over Z to F^* (resp. $F^* \otimes T^*Z^C$). Let $\psi: \Gamma(Z, F^*) \to \Gamma(Z, F^* \otimes T^*Z^C)$ be the C-linear map sending each $s \in \Gamma(Z, F^*)$ to an element $\psi(s)$ of $\Gamma(Z, F^* \otimes T^*Z^C)$ defined by

$$\psi(s)(X) := \sigma((D_F)_{\overline{\tau_*(X)}}(\sigma^{-1}s)) \in F^*_z\ ,$$

for $X \in T_zZ^C$ $(z \in Z)$.

Then by the condition (b) in the Introduction, this ψ defines a Hermitian (1, 0)-connection on the holomorphic vector bundle F^*. The corresponding connection matrix with respect to the frame $(\sigma f_1, \cdots, \sigma f_r)$ for $F^*_{|p^{-1}(U)}$ is written as $(\tau^*\overline{\omega_{ij}})$. By the definition of σ, it is easy to check that the frame $(\sigma f_1, \cdots, \sigma f_r)$ is dual to our previous $(f_1, \cdots, f_r)$. Hence the uniqueness of the (1, 0)-connection on the Hermitian vector bundle F^* implies the equality $(\tau^*\omega_{ij})^- = \omega^*_{ij}$, where $\omega^*_{ij} := -\omega_{ji}$. In view of (1), we have $\tau^*\omega_{ij} = \omega_{ij}$ and $\hat{\tau}^*\hat{\omega}_{ij} = \hat{\omega}_{ij}$. By (3) and $\hat{p}\circ\hat{\tau} = \hat{p}$, we obtain:

$$(8)\qquad \hat{\tau}^*\hat{a}^k_{ij} = \hat{a}^k_{ij} \quad \text{and} \quad \hat{\tau}^*\hat{b}^k_{ij} = \hat{b}^k_{ij} \quad (1 \leqq k \leqq 2n)\ .$$

Therefore,

$$-\bar{z}^2\hat{\tau}^*\bar{\hat{a}}^k_{ij} + \bar{z}^1\hat{\tau}^*\bar{\hat{b}}^k_{ij} = -\bar{z}^2\bar{\alpha}^k_{ij} + \bar{z}^1\bar{\beta}^k_{ij} \quad (1 \leqq k \leqq 2n)\ .$$

Moreover by (6),

$$(9)\qquad -z^2\hat{a}^k_{ij} + z^1\hat{b}^k_{ij} = -z^2\alpha^k_{ij} + z^1\beta^k_{ij} \quad (1 \leqq k \leqq 2n)\ .$$

Hence by (7) and (9), we obtain:

$$(10)\qquad \alpha^k_{ij} = \gamma^k_{ij} \quad \text{and} \quad \beta^k_{ij} = \delta^k_{ij} \quad (1 \leqq k \leqq 2n)\ .$$

Now, in view of (6), (7) and (10), we see that

$$\begin{pmatrix} \bar{z}^1, & \bar{z}^2 \\ -z^2, & z^1 \end{pmatrix}\begin{pmatrix} \hat{a}^k_{ij} - \alpha^k_{ij} \\ \hat{b}^k_{ij} - \beta^k_{ij} \end{pmatrix} = 0 \quad (1 \leqq k \leqq 2n)\ ,$$

where $(z^1, z^2) \in C^2 - \{0\}$ $(=\hat{p}^{-1}(x))$. Thus, $\hat{a}^k_{ij} = \alpha^k_{ij}$ and $\hat{b}^k_{ij} = \beta^k_{ij}$ $(1 \leqq k \leqq 2n)$, i.e., both a^k_{ij} and b^k_{ij} are constant along $p^{-1}(x)$, as required.

REMARK (4.1). In some sense, our Theorem (0.2) completely clarifies the following result by Salamon [S2] (see Berard Bergery and Ochiai [B-O] for another generalization):

For a Hermitian pair (E, D_E) *on* M, *the pull-back* (p^*E, p^*D_E) *to* Z *is a Hermitian holomorphic vector bundle over* Z.

COROLLARY (4.2). *Let* (F, D_F) *be an excellent pair on* Z. *If the quaternionic Kähler manifold* M *has positive scalar curvature, then* F *with* D_F *is a Ricci-flat Einstein Hermitian vector bundle over* Z.

PROOF. Consider the twistor space $p: Z \to M$. Then the horizontal component of the Kähler form on Z is a p^*A_2'-form (cf. (1.2), (1.3)). Recall that the curvature of D_F is an End(F)-valued p^*B_2-form. Hence the Hermitian vector bundle F with D_F is Ricci-flat.

REMARK (4.3). We have the decomposition of $TZ = T^h \oplus T^v$, where T^h (resp. T^v) is the horizontal (resp. vertical) distribution in terms of the connection on Z induced by that of P. Since the complex structure on TZ is a direct sum of complex structures on T^h and T^v, the holomorphic part $TZ^{(1,0)}$ admits the corresponding decomposition $TZ^{(1,0)} = T^{h(1,0)} \oplus T^{v(1,0)}$, where $T^{h(1,0)}$ (resp. $T^{v(1,0)}$) denotes $T^{hC} \cap TZ^{(1,0)}$ (resp. $T^{vC} \cap TZ^{(1,0)}$). Recently, Zandi [Z] obtained the following:

The vector bundle $(T^{h(1,0)}, D^h)$ *is an Einstein-Hermitian vector bundle, where* D^h *is the connection on* $T^{h(1,0)}$ *obtained as the restriction of the Riemannian connection on* TZ *to* $T^{h(1,0)}$.

This result can be regarded as a straightforward consequence of our (4.2). We denote by L a locally defined (line) subbundle of p^*W (cf. (2.4)) such that, along each fibre $p^{-1}(x) = \boldsymbol{P}^1\boldsymbol{C}$ $(x \in M)$, it restricts to a universal bundle over $\boldsymbol{P}^1\boldsymbol{C}$. Let ∇^V (resp. ∇^W) denote the connection of V (resp. W) canonically induced by that of P and ∇^L the restriction of $p^*\nabla^W$ to L. Then the vector bundle $(T^{h(1,0)}, D^h)$ is nothing but $(p^*W \otimes L^*, p^*\nabla^W \otimes (\nabla^L)^*)$, where $(L^*, (\nabla^L)^*)$ is dual to (L, ∇^L) (see Salamon [S1]). Since L^* is a locally defined line bundle and since ∇^W is a B_2-connection on W, Corollary (4.2) clearly implies Zandi's result.

Added in proof. After the completion of this paper, the author received a preprint by M. M. Capria and S. M. Salamon entitled "Yang-Mills fields on quaternionic Kähler spaces", which gives (i) a result slightly stronger than (2.6) and (ii) a statement similar to (3.2).

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