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CHARACTERIZATIONS OF CERTAIN WEAKLY PSEUDOCONVEX DOMAINS $E(k, \alpha)$ IN C^n

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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Introduction. Let D be a domain in C^n and $\mathrm{Aut}(D)$ the group of all biholomorphic transformations of D onto itself. Let p be a point of ∂D, the boundary of D. Throughout this paper, we say that *the condition* $(*)$ *is fulfilled for* (D, p) if

$(*)$ *there exist a compact set* K *in* D, *a sequence* $\{k_\nu\}$ *in* K *and a sequence* $\{\varphi_\nu\}$ *in* $\mathrm{Aut}(D)$ *such that* $\lim_{\nu\to\infty}\varphi_\nu(k_\nu) = p$.

Moreover, a point $p \in \partial D$ is said to be *a strictly pseudoconvex boundary point of* D if there exist an open neighborhood U of p and a C^2-smooth strictly plurisubharmonic function $\rho: U \to \boldsymbol{R}$ such that $D \cap U = \{z \in U \mid \rho(z) < 0\}$ and $d\rho(z) \neq 0$ for all $z \in \partial D \cap U$.

In 1977, it was shown by Wong [14] that *if* D *is a bounded strictly pseudoconvex domain in* C^n *with* C^∞*-smooth boundary and* $\mathrm{Aut}(D)$ *is non-compact, then* D *is biholomorphically equivalent to the open unit ball* B^n *in* C^n. It was later extended by Rosay to the following:

THEOREM R (Rosay [12]). *Let* D *be a bounded domain in* C^n *with a strictly pseudoconvex boundary point* $p \in \partial D$. *Assume that the condition* $(*)$ *is fulfilled for* (D, p). *Then* D *is biholomorphically equivalent to* B^n.

Here it seems natural to ask what happens when the point p is a weakly pseudoconvex boundary point of D. In a recent work of Greene and Krantz [3] the weakly pseudoconvex domain

$$E(m) = \left\{z \in C^n \mid -1 + \sum_{i=1}^{n-1}|z_i|^2 + |z_n|^{2m} < 0\right\}, \quad 0 < m \in \boldsymbol{Z}$$

in C^n is studied exclusively in connection with this problem and the following characterization of it is obtained as their main result:

THEOREM G-K (Greene and Krantz [3]). *Let* D *be a bounded domain in* C^n *with* C^{n+1}*-smooth boundary such that* $p = (1, 0, \cdots, 0) \in \partial D$. *Assume that there are neighborhoods* U, V *of* p *in* C^n *such that, up to a local* *biholomorphism, $U\cap\partial D$ and $V\cap\partial E(m)$ coincide. Assume further that the condition $(*)$ is fulfilled for (D, p). Then D is biholomorphically equivalent to the domain $E(m)$.*

Their proof is very interesting, but contains a difficult and complicated lemma [3; Lemma 4.3], which was shown by the uniform estimates for the $\bar{\partial}$-equation on D. A glance at the proof of Theorem G-K tells us that the global C^{n+1}-smoothness assumption on ∂D cannot be avoided with their technique. However, in view of Theorem R it would be naturally expected that the same conclusion is also true if only D has a C^2-smooth boundary near the point p. The main purpose of this paper is to clear up this matter. In fact, employing the same technique as in our previous papers [6], [7] instead of using the $\bar{\partial}$-equation on D, we can avoid their hard part and obtain more general results without any smoothness assumption on ∂D.

In order to state our results, we here introduce the following notation: For every integer $k=1, \cdots, n$ and every real number $\alpha>0$, we set

$$\rho(k, \alpha; z) = -1 + \sum_{i=1}^{k}|z_i|^2 + \Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha}$$

and

$$E(k, \alpha) = \{z\in \boldsymbol{C}^n \mid \rho(k, \alpha; z)<0\}\ .$$

So $E(m)=E(n-1, m)$; and if $k=n$ or $\alpha=1$, then $E(k, \alpha)$ is nothing but the open unit ball B^n. Moreover, note that $\partial E(k, \alpha)$ is not smooth in general. (Consider, for example, the domain $E(1, 1/4)=\{(z_1, z_2)\in \boldsymbol{C}^2 \mid -1+|z_1|^2+|z_2|^{1/2}<0\}$ in $\boldsymbol{C}^2$.) In this notation, we can prove the following:

THEOREM I. *Let D be a bounded domain in $\boldsymbol{C}^n$ satisfying the following conditions:*

(i) $p=(1, 0, \cdots, 0)\in\partial D$;

(ii) *there is an open neighborhood U of p such that $D\cap U = E(k, \alpha)\cap U$;*

(iii) *the condition $(*)$ is fulfilled for (D, p).*

Then D is biholomorphically equivalent to the domain $E(k, \alpha)$.

In the theorem of Greene and Krantz [3], we may assume without loss of generality that there exists an open neighborhood U of $p=(1, 0, \cdots, 0)$ such that $D\cap U = E(m)\cap U$ (see the proof of [3, Theorem 1.1]). Moreover, any smoothness of ∂D is not assumed in our theorem. Therefore Theorem I is a natural generalization of Theorem G-K.

Clearly the condition (ii) of Theorem I imposes crucial restrictions on the boundary of D, and so we want to remove it. This cannot be achieved in full generality at this moment. But, under some additional condition on the convergence $\varphi_\nu(k_\nu) \to p$ we can prove the following theorem. (For the definition of R-lim, see Section 1.)

THEOREM II. *Let D be a bounded domain in $\boldsymbol{C}^n$ with $p = (1, 0, \cdots, 0) \in \partial D$. Assume that there exist an open neighborhood U of p and a continuous function $\rho: U \to \boldsymbol{R}$ such that:*

(i) $D \cap U = \{z \in U \mid \rho(z) < 0\}$;

(ii) $\rho(z) = \rho(k, \alpha; z) + R(z)$, $z \in U$ *with*

$$R(z) = o\Big(|z_1 - 1|^2 + \sum_{i=2}^{k} |z_i|^2 + \Big(\sum_{j=k+1}^{n} |z_j|^2\Big)^\alpha\Big)$$

in a neighborhood of p; and assume further that:

(iii) *There exist a compact set K in D, a sequence $\{k_\nu\}$ in K and a sequence $\{\varphi_\nu\}$ in* Aut(D) *such that*

$$\text{R-}\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = p\ ,$$

Then D is biholomorphically equivalent to the domain $E(k, \alpha)$.

Taking account of the case of strictly pseudoconvex boundary points, it is reasonable that $R(z)$ has the estimate as in (ii). Moreover, it should be remarked that, in some sense, the assumption (iii) is not so strong. Indeed, in the model case $D = E(k, \alpha)$ with $\alpha \neq 1$, we have the following: For any convergent sequence $\varphi_\nu(k_\nu) \to p$, there exists a sequence $\{\tilde{\varphi}_\nu\}$ in Aut(D) such that R-$\lim_{\nu\to\infty} \tilde{\varphi}_\nu(k_\nu) = p$ (see Example 2 in Section 1).

Next we assume that a complex manifold M can be exhausted by biholomorphic images of a complex manifold D, that is, for any compact subset K of M there exists a biholomorphic mapping f_K from D into M such that $K \subset f_K(D)$. Then, how can we describe M using the data of D? In connection with this, Fridman [2] showed that *if a complete hyperbolic manifold M of complex dimension n in the sense of Kobayashi* [5] *can be exhausted by biholomorphic images of a bounded strictly pseudoconvex domain D in $\boldsymbol{C}^n$ with C^3-smooth boundary, then M is biholomorphically equivalent either to D or to the open unit ball B^n.* The following theorem tells us that the analogue is still valid for the weakly pseudoconvex domain $E(k, \alpha)$ with arbitrary $\alpha > 0$.

THEOREM III. *Let M be a hyperbolic manifold of complex dimension n in the sense of Kobayashi* [5]. *Assume that M can be exhausted by biholomorphic images of the weakly pseudoconvex domain $E(k, \alpha)$. Then*

M is biholomorphically equivalent either to $E(k, \alpha)$ or to B^n.

Our proofs of the theorems above are based on the normal family arguments developed in our previous papers [6], [7] and Pinčuk [10], [11]. Although there are some overlaps with those papers, we carry out the proofs in detail for the sake of completeness and self-containedness. After some preliminaries in Section 1, Theorems I, II and III will be proven in Sections 2, 3 and 4, respectively. In the final Section 5, we mention the analogues of Theorems I and II in the case where D is a not necessarily bounded hyperbolic domain in C^n.

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1. Preliminaries. For later purpose, we shall recall some definitions and study the structure of the model space $E(k, \alpha)$ with arbitrary $\alpha > 0$.

Let M and N be complex manifolds and $\mathrm{Hol}(N, M)$ the family of all holomorphic mappings from N into M. A sequence $\{f_\nu\}$ in $\mathrm{Hol}(N, M)$ is said to be *compactly divergent on* N if, for any compact sets L, K in N, M, respectively, there exists an integer ν_0 such that $f_\nu(L) \cap K = \emptyset$ for all $\nu \geqq \nu_0$. After Wu [15], we shall define the tautness of complex manifolds as follows:

DEFINITION 1. A complex manifold M is said to be *taut* if $\mathrm{Hol}(N, M)$ is a normal family for any complex manifold N, i.e., any sequence in $\mathrm{Hol}(N, M)$ contains a subsequence which is either uniformly convergent on every compact subset of N or compactly divergent on N.

Let d_M, d_N be the Kobayashi pseudodistances of M, N, respectively [5]. The following distance-decreasing property will play an important role in the proofs of our theorems: *Let $f: N \to M$ be a holomorphic mappping. Then*

$$(1.1) \qquad d_M(f(p), f(q)) \leqq d_N(p, q) \quad \textit{for all } p, q \in N .$$

Consequently, every biholomorphic mapping f from N onto M is an isometry with respect to d_N and d_M; and if N is a complex submanifold of M, then $d_M(p, q) \leqq d_N(p, q)$ for all $p, q \in N$.

Throughout this paper we use the following notation: For a point $z = (z_1, \cdots, z_n)$ of C^n and a mapping $f = (f_1, \cdots, f_n)$ from a set S into C^n, we set

$$z' = (z_1, \cdots, z_k) , \quad z'' = (z_{k+1}, \cdots, z_n) , \quad 'z = (z_1, \cdots, z_{n-1}) ,$$
$$'f = (f_1, \cdots, f_{n-1}) \quad \text{and} \quad |u|^2 = \sum_{i=1}^{l} |u_i|^2 \quad \text{for} \quad u = (u_1, \cdots, u_l) \in C^l .$$

Thus we can write the function $\rho(k, \alpha; z)$ and the domain $E(k, \alpha)$ in the form

$$\rho(k, \alpha; z) = -1 + |z'|^2 + |z''|^{2\alpha} ;$$
$$E(k, \alpha) = \{(z', z'') \in C^k \times C^{n-k} \,|\, |z'|^2 + |z''|^{2\alpha} < 1\} .$$

Recall that a domain D in C^n is called a Reinhardt domain if $((\exp\sqrt{-1}\theta_1)z_1, \cdots, (\exp\sqrt{-1}\theta_n)z_n) \in D$ whenever $(z_1, \cdots, z_n) \in D$ and $\theta_j \in R$, $j = 1, \cdots, n$. Moreover, we say that it is complete if $(z_1^\circ, \cdots, z_n^\circ) \in D$, $z = (z_1, \cdots, z_n) \in C^n$ and $|z_j| \leqq |z_j^\circ|$, $j = 1, \cdots, n$, implies $z \in D$. We now assert that $E(k, \alpha)$ *is a bounded pseudoconvex complete Reinhardt domain in* C^n *containing the origin* o. Hence, by a result of Pflug [9] it is complete hyperbolic in the sense of Kobayashi [5]. Since $E(k, \alpha)$ is obviously a bounded complete Reinhardt domain in C^n containing the origin, we have only to check that the domain

$$B = \{(x_1, \cdots, x_n) \in R^n \,|\, (\exp x_1, \cdots, \exp x_n) \in E(k, \alpha)\}$$

is geometrically convex in R^n [8; p. 120]. To do so, let us take arbitrary points $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ of B and arbitrary numbers $\lambda, \mu > 0$ such that $\lambda + \mu = 1$. Then, by using Hölder's inequality twice we obtain the following:

$$\sum_{i=1}^{k} \exp[2(\lambda x_i + \mu y_i)] + \left(\sum_{j=k+1}^{n} \exp[2(\lambda x_j + \mu y_j)]\right)^{\alpha}$$
$$\leqq \left(\sum_{i=1}^{k} \exp 2x_i\right)^{\lambda} \cdot \left(\sum_{i=1}^{k} \exp 2y_i\right)^{\mu} + \left[\left(\sum_{j=k+1}^{n} \exp 2x_j\right)^{\lambda} \cdot \left(\sum_{j=k+1}^{n} \exp 2y_j\right)^{\mu}\right]^{\alpha}$$
$$\leqq \left[\sum_{i=1}^{k} \exp 2x_i + \left(\sum_{j=k+1}^{n} \exp 2x_j\right)^{\alpha}\right]^{\lambda} \cdot \left[\sum_{i=1}^{k} \exp 2y_i + \left(\sum_{j=k+1}^{n} \exp 2y_j\right)^{\alpha}\right]^{\mu} < 1 ,$$

which shows $\lambda x + \mu y \in B$. Thus B is convex, as desired.

Next, setting $S = \{(0, z'') \in C^k \times C^{n-k} \,|\, |z''| = 1\} \subset \partial E(k, \alpha)$, we would like to show that $\partial E(k, \alpha)$ is real analytic and strictly pseudoconvex at every point contained in an open neighborhood W of S. It is easy to see that there is an open neighborhood W of S on which $\rho(k, \alpha; z)$ is real analytic and $d\rho(k, \alpha; z) \neq 0$ for all $z \in W$. Once $\partial E(k, \alpha)$ is shown to be strictly pseudoconvex at every point $(0, z'') \in S$, one can obtain a desired neighborhood W by the continuity of the Levi form. On the other hand, by direct calculation we obtain that

$$\sum_{i,j=1}^{n} [\partial^2 \rho(k, \alpha; z)/\partial z_i \partial \bar{z}_j] \xi_i \bar{\xi}_j$$
$$= |\xi'|^2 + \alpha |z''|^{2(\alpha-1)} |\xi''|^2 + \alpha(\alpha - 1) |z''|^{2(\alpha-2)} \left| \sum_{j=k+1}^{n} \bar{z}_j \xi_j \right|^2$$

for every $\xi = (\xi', \xi'') \in C^k \times C^{n-k}$ and every $z \in W$; and

$$\left\{\xi \in \boldsymbol{C}^n \,\middle|\, \sum_{i=1}^{n}[\partial\rho(k, \alpha; q)/\partial z_i]\xi_i = 0\right\} = \left\{\xi \in \boldsymbol{C}^n \,\middle|\, \sum_{j=k+1}^{n} \bar{z}_j\xi_j = 0\right\}$$

for every $q = (0, z'') \in S$. Hence $\partial E(k, \alpha)$ is actually strictly pseudoconvex at every point of S, as desired.

We study the biholomorphic automorphism group $\mathrm{Aut}(E(k, \alpha))$ of $E(k, \alpha)$. Denoting by $M(r, s)$ the set of all $r \times s$ complex matrices for positive integers r, s, we consider the closed Lie subgroup $SU(k, 1)$ of $GL(k + 1, \boldsymbol{C})$ consisting of all matrices

$$\gamma = \begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d \end{pmatrix}; \quad \begin{matrix} A \in M(k, k), & \mathfrak{b} \in M(k, 1) \\ \mathfrak{c} \in M(1, k), & d \in M(1, 1) \end{matrix} \tag{1.2}$$

satisfying the relations

$${}^t\bar{A}A - {}^t\bar{\mathfrak{c}}\mathfrak{c} = E_k, \quad {}^t\bar{\mathfrak{b}}\mathfrak{b} - |d|^2 = -1, \quad {}^t\bar{\mathfrak{b}}A = \bar{d}\mathfrak{c} \quad \text{and} \quad \det\gamma = 1,$$

where E_k is the unit matrix of degree k. For each $\gamma \in SU(k, 1)$ represented as in (1.2) and each $U \in U(n - k)$, the unitary group of degree $n - k$, we define the transformation $\Psi(\gamma, U)$ by

$$\Psi(\gamma, U)\colon \begin{cases} z' \mapsto (Az' + \mathfrak{b})/(\mathfrak{c}z' + d) \\ z'' \mapsto U\cdot z''/(\mathfrak{c}z' + d)^{1/\alpha} \end{cases} \tag{1.3}$$

for $(z', z'') \in \boldsymbol{C}^k \times \boldsymbol{C}^{n-k}$ (think of z', z'' as column vectors). Then, using the equality $|\mathfrak{c}z' + d|^2 - |Az' + \mathfrak{b}|^2 = 1 - |z'|^2$ for all $z' \in \boldsymbol{C}^k$, one can check that each $\Psi(\gamma, U)$ gives rise to a biholomorphic automorphism of $E(k, \alpha)$. In fact, according to Sunada [13] the identity component $\mathrm{Aut}_o(E(k, \alpha))$ of the Lie group $\mathrm{Aut}(E(k, \alpha))$ coincides with the group

$$G(k, \alpha) = \{\Psi(\gamma, U) \mid \gamma \in SU(k, 1), U \in U(n - k)\}$$

provided that $\alpha \neq 1$. More precisely, we here assert that $\mathrm{Aut}(E(k, \alpha)) = G(k, \alpha)$ in our case. To verify this assertion, observe that the $G(k, \alpha)$-orbit passing through the origin $o \in E(k, \alpha)$ is of lowest dimension in the set of all $G(k, \alpha)$-orbits, i.e., $\dim(G(k, \alpha)\cdot o) < \dim(G(k, \alpha)\cdot z)$ for any point $z \in E(k, \alpha) \setminus G(k, \alpha)\cdot o$. Hence

$$g\cdot G(k, \alpha)\cdot o = G(k, \alpha)\cdot o = \{(z', 0) \in \boldsymbol{C}^k \times \boldsymbol{C}^{n-k} \mid |z'| < 1\}$$

for each $g \in \mathrm{Aut}(E(k, \alpha))$. This combined with a well-known theorem of H. Cartan [8; p. 67] assures that every element g of $\mathrm{Aut}(E(k, \alpha))$ can be expressed as $g = \psi_g \cdot l_g$ for some $\psi_g \in G(k, \alpha)$ and $l_g \in GL(n; \boldsymbol{C})$. In particular, l_g can be written in the form

$$l_g(z', z'') = (Az' + Bz'', Dz''), \quad (z', z'') \in \boldsymbol{C}^k \times \boldsymbol{C}^{n-k},$$

where $A \in SU(k) = SL(k; \boldsymbol{C}) \cap U(k)$, $B \in M(k, n - k)$ and $D \in GL(n - k; \boldsymbol{C})$.

Then the fact $l_g(\partial E(k, \alpha)) = \partial E(k, \alpha)$ yields that

$$2\,\mathrm{Re}(Az', Bz'') + |Bz''|^2 + |Dz''|^{2\alpha} = |z''|^{2\alpha}, \quad (z', z'') \in \partial E(k, \alpha),$$

where $(\cdot, \cdot)$ denotes the standard Hermitian inner product on $\boldsymbol{C}^k$. Consequently, $B = 0$, $D \in U(n-k)$ and $l_g(z', z'') = (Az', Dz'')$ for $A \in SU(k)$, $D \in U(n-k)$. Finally, noting that both groups $SU(k)$ and $U(n-k)$ are naturally imbedded in $G(k, \alpha)$, we conclude that $l_g \in G(k, \alpha)$ and so $\mathrm{Aut}(E(k, \alpha)) = G(k, \alpha)$, as desired.

Next we consider an arbitrary sequence $\{p^\nu\}_{\nu=1}^\infty$ in $E(k, \alpha)$ which converges to the point $p = (1, 0, \cdots, 0) \in \partial E(k, \alpha)$. Then there exists a sequence $\{\psi_\nu\}_{\nu=1}^\infty$ in $\mathrm{Aut}(E(k, \alpha))$ such that

$$\psi_\nu(p^\nu) = (0, \cdots, 0, \tilde{t}_\nu) \quad \text{with} \quad 0 \leqq \tilde{t}_\nu < 1 \tag{1.4}$$

for all $\nu = 1, 2, \cdots$. Indeed, since the product group $SU(k) \times U(n-k)$ is naturally identified with a subgroup of $\mathrm{Aut}(E(k, \alpha))$, we may assume that

$$p^\nu = (x_\nu, 0, \cdots, 0, y_\nu) \quad \text{with} \quad 0 \leqq x_\nu, y_\nu < 1 \tag{1.5}$$

for $\nu = 1, 2, \cdots$. Consider the one-parameter subgroup

$$\gamma(t) = \left(\begin{array}{c|c|c} \cosh t & 0 & \sinh t \\ \hline 0 & E_{k-1} & 0 \\ \hline \sinh t & 0 & \cosh t \end{array}\right), \quad t \in \boldsymbol{R} \tag{1.6}$$

of $SU(k, 1)$ and set $\psi_\nu = \Psi(\gamma(t_\nu), E_{n-k})$, $t_\nu = \tanh^{-1}(-x_\nu)$ for $\nu = 1, 2, \cdots$. Then it is easily seen that each $\psi_\nu(p^\nu)$ has the desired form as in (1.4).

Summarizing the above, we obtain the following:

LEMMA. *The domain $E(k, \alpha)$ has the following properties*:

(1) *$E(k, \alpha)$ is complete hyperbolic in the sense of Kobayashi* [5]. *In particular, it is a taut domain* [4].

(2) *The boundary $\partial E(k, \alpha)$ of $E(k, \alpha)$ is real analytic and strictly pseudoconvex near the point $q = (0, \cdots, 0, 1) \in \partial E(k, \alpha)$.*

(3) $\mathrm{Aut}(E(k, \alpha))$ *is a connected Lie group consisting of all biholomorphic transformations of $E(k, \alpha)$ as defined in* (1.3).

(4) *Let $\{p^\nu\}_{\nu=1}^\infty$ be a sequence in $E(k, \alpha)$ which converges to the point $p = (1, 0, \cdots, 0) \in \partial E(k, \alpha)$. Then there is a sequence $\{\psi_\nu\}_{\nu=1}^\infty$ in* $\mathrm{Aut}(E(k, \alpha))$ *such that $\psi_\nu(p^\nu) = (0, \cdots, 0, t_\nu)$ with $0 \leqq t_\nu < 1$ for all $\nu = 1, 2, \cdots$.*

Finally we shall define the R-limit. Let us fix a domain D in $\boldsymbol{C}^n$ such that $p = (1, 0, \cdots, 0) \in \partial D$ and the conditions (i), (ii) in Theorem II are satisfied for D. Without loss of generality, we may assume that the neighborhood U of p is a small open Euclidean ball with center at p satisfying the following inequalities:

$$2\,\mathrm{Re}(z_1-1)+A\Big[|z_1-1|^2+\sum_{i=2}^{k}|z_i|^2+\Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha}\Big]$$
$$\leqq \rho(z)\leqq 2\,\mathrm{Re}(z_1-1)+B\Big[|z_1-1|^2+\sum_{i=2}^{k}|z_i|^2+\Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha}\Big]$$

for every point $z\in U$, where A and B are arbitrarily given constants with $0<A<1<B$. Now, denoting by N the unit vector $(1, 0, \cdots, 0)$, we consider the half line $L(z)=\{z+tN \mid t\geqq 0\}$ in $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$ for each point $z\in D\cap U$. Then z has a unique farthest point $\zeta(z)$ in the set $\partial D\cap L(z)\cap U$, so that each point $z\in D\cap U$ can be written uniquely in the form $z=\zeta(z)+\lambda(z)N$, $\lambda(z)<0$. In particular, for a given sequence $\{p^\nu\}$ in D converging to p we have

$$(1.7)\qquad p^\nu=\zeta(p^\nu)+\lambda(p^\nu)N\,;\quad \zeta(p^\nu)=(\zeta_1(p^\nu),\cdots,\zeta_n(p^\nu))\in\partial D\cap U\,,\quad \lambda(p^\nu)<0$$

for all sufficiently large ν. Clearly $\zeta(p^\nu)\to p$ and $\lambda(p^\nu)\to 0$ as $\nu\to\infty$.

DEFINITION 2. In the notation above, we say that $\{p^\nu\}$ *converges restrictedly to* p, and write $\text{R-lim}_{\nu\to\infty}\,p^\nu=p$, if the sequence $\{\mathrm{Re}(\zeta_1(p^\nu)-1)/\lambda(p^\nu)\}$ is a bounded sequence in $\boldsymbol{R}$.

We shall present two examples of sequences $\{p^\nu\}$ in D which converge restrictedly to p. We set, for an arbitrary $\varepsilon>0$,

$$\Phi(z)=(\mathrm{Im}\,z_1)^2+\sum_{i=2}^{k}|z_i|^2+\Big(\sum_{j=k+1}^{n}|z_j|^2\Big)^{\alpha}\,,\quad z\in\boldsymbol{C}^n\,;$$
$$C(\varepsilon)=\{z\in\boldsymbol{C}^n \mid \mathrm{Re}\,z_1\leqq 1-\varepsilon\cdot[\Phi(z)]^{1/2}\}\,.$$

So, if $\alpha=1$, the region $C(\varepsilon)$ is nothing but a cone with vertex at p and axis in the direction of $-N$. The following example tells us that if $\{p^\nu\}$ converges to p non-tangentially in the usual sense, then it converges restrictedly in our sense.

EXAMPLE 1. Assume that ∂D is C^1-smooth near the point p and $\{p^\nu\}$ converges to p through the region $C(\varepsilon)$ for some $\varepsilon>0$. Then we have $\text{R-lim}_{\nu\to\infty}\,p^\nu=p$.

In fact, by our assumption, ∂D is a C^1-smooth real hypersurface near p and the vector N is perpendicular to ∂D at p with respect to the Euclidean structure on $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$. Thus we can write uniquely $p^\nu=\zeta^\nu+\lambda^\nu N$ with some $\zeta^\nu\in\partial D$ and $\lambda^\nu<0$ for all sufficiently large ν.

In order to check that the sequence $\{\mathrm{Re}(\zeta_1^\nu-1)/\lambda^\nu\}$ is bounded, we may assume (by passing to a subsequence if necessary) that $\mathrm{Re}(\zeta_1^\nu - 1) \neq 0$ for all $\nu = 1, 2, \cdots$. Since $\mathrm{R}(\zeta^\nu) = o((\mathrm{Re}(\zeta_1^\nu - 1))^2 + \Phi(\zeta^\nu))$ and

$$2\,\mathrm{Re}(\zeta_1^\nu - 1) + (\mathrm{Re}(\zeta_1^\nu - 1))^2 + \Phi(\zeta^\nu) + R(\zeta^\nu) = \rho(\zeta^\nu) = 0$$

for all large ν, it follows that $\lim_{\nu\to\infty} \Phi(\zeta^\nu)/\mathrm{Re}(\zeta_1^\nu - 1) = -2$. On the other hand, we know by assumption that

$$\mathrm{Re}(p_1^\nu - 1) \leqq -\varepsilon\cdot[\Phi(p^\nu)]^{1/2} = -\varepsilon\cdot[\Phi(\zeta^\nu)]^{1/2} < 0$$

for all sufficiently large ν. Thus

$$\begin{aligned}\lambda^\nu/\mathrm{Re}(\zeta_1^\nu - 1) &= [\mathrm{Re}(p_1^\nu - 1) - \mathrm{Re}(\zeta_1^\nu - 1)]/\mathrm{Re}(\zeta_1^\nu - 1)\\ &= |\mathrm{Re}(p_1^\nu - 1)/\mathrm{Re}(\zeta_1^\nu - 1)| - 1\\ &\geqq \varepsilon\cdot[\Phi(\zeta^\nu)]^{1/2}/|\mathrm{Re}(\zeta_1^\nu - 1)| - 1 \to +\infty\ .\end{aligned}$$

Obviously this implies that $\text{R-lim}_{\nu\to\infty} p^\nu = p$.

EXAMPLE 2. Let $\{k_\nu\}$ be a sequence of points contained in a compact subset of $E(k, \alpha)$, $\alpha \neq 1$, and let $\lim_{\nu\to\infty} \varphi_\nu(k_\nu) = (1, 0, \cdots, 0)$ for some sequence $\{\varphi_\nu\}$ in $\mathrm{Aut}(E(k, \alpha))$. Then there exists a new sequence $\{\tilde{\varphi}_\nu\}$ in $\mathrm{Aut}(E(k, \alpha))$ such that $\text{R-lim}_{\nu\to\infty} \tilde{\varphi}_\nu(k_\nu) = (1, 0, \cdots, 0)$.

Indeed, changing φ_ν into a suitable biholomorphic automorphism $\tilde{\varphi}_\nu = f_\nu \circ \varphi_\nu$, $f_\nu \in SU(k) \times U(n-k) \subset \mathrm{Aut}(E(k, \alpha))$ if necessary, we may assume as in (1.5) that

$$\varphi_\nu(k_\nu) = (x_\nu, 0, \cdots, 0, y_\nu) = \zeta^\nu + \lambda^\nu N$$

with $0 \leqq x_\nu, y_\nu < 1$, $\zeta^\nu = (\zeta_1^\nu, 0, \cdots, 0, \zeta_n^\nu) \in \partial E(k, \alpha)$, $\lambda^\nu < 0$ and $N = (1, 0, \cdots, 0)$. Here it can be seen that ζ^ν and λ^ν are uniquely determined by $\varphi_\nu(k_\nu)$. Now, we claim that $\text{R-lim}_{\nu\to\infty} \varphi_\nu(k_\nu) = (1, 0, \cdots, 0)$. To this end, note that $\{k_\nu\}$ lies in a compact subset of $E(k, \alpha)$ and recall the structure of $\mathrm{Aut}(E(k, \alpha))$. Then one can choose an r, $0 < r < 1$, in such a way that $\varphi_\nu(k_\nu) \in D(r)$ for all $\nu = 1, 2, \cdots$, where we have set

$$D(r) = \{(x, 0, \cdots, 0, y) \in \boldsymbol{R}^n \,|\, x^2 + (y/r)^{2\alpha} \leqq 1,\ 0 \leqq x, y\}\ .$$

Let us choose a unique point $q^\nu = \zeta^\nu + \mu^\nu N$, $\lambda^\nu \leqq \mu^\nu < 0$, such that

$$(\zeta_1^\nu + \mu^\nu)^2 + (\zeta_n^\nu/r)^{2\alpha} = 1 \quad \text{for each } \nu\ . \tag{1.8}$$

Then, substituting $(\zeta_n^\nu)^{2\alpha} = 1 - (\zeta_1^\nu)^2$ into (1.8) and rearranging the result, we obtain

$$(1 - \zeta_1^\nu)(1 + \zeta_1^\nu)/r^{2\alpha} = (1 - \zeta_1^\nu - \mu^\nu)(1 + \zeta_1^\nu + \mu^\nu)$$

for all ν. Consequently

$$\begin{aligned}(1 - \zeta_1^\nu)/\mu^\nu &= (1 + \zeta_1^\nu + \mu^\nu)/[1 + \zeta_1^\nu + \mu^\nu - (1 + \zeta_1^\nu)/r^{2\alpha}]\\ &\to r^{2\alpha}/(r^{2\alpha} - 1) \quad \text{as} \quad \nu \to \infty\ .\end{aligned}$$

Since $|(\zeta_1^\nu - 1)/\lambda^\nu| \leqq |(\zeta_1^\nu - 1)/\mu^\nu|$ for all ν, we conclude that$\{(\zeta_1^\nu - 1)/\lambda^\nu\}$ is a bounded sequence.

2. Proof of Theorem I. Passing to a subsequence if necessary, we may assume that $\{k_\nu\}$ converges to some point $k_o \in K$ and $\{\varphi_\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $\varphi: D \to \bar{D} \subset \boldsymbol{C}^n$. Let us define the holomorphic function Ψ_p on $\boldsymbol{C}^n$ by

$$\Psi_p(z) = \exp(z_1 - 1), \quad z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n,$$

where $p = (1, 0, \cdots, 0) \in \partial E(k, \alpha) \cap \partial D$. Then obviously Ψ_p is a holomorphic function for $E(k, \alpha) \cap U = D \cap U$ peaking at p in the sense that

$$\Psi_p(p) = 1 \quad \text{and} \quad |\Psi_p(z)| < 1 \quad \text{for all} \quad z \in \overline{D \cap U} \setminus \{p\}.$$

This combined with the maximum principle for the holomorphic function $\Psi_p \circ \varphi$ defined on an open neighborhood of k_o yields at once that $\varphi(z) = p$ for all $z \in D$. We can therefore assume that

$$\lim_{\nu \to \infty} \varphi_\nu(k_o) = p \quad \text{and} \quad p^\nu := \varphi_\nu(k_o) \in D \cap U = E(k, \alpha) \cap U$$

for $\nu = 1, 2, \cdots$. As in Greene and Krantz [3], we choose a sequence $\{\psi_\nu\}_{\nu=1}^\infty$ in $\mathrm{Aut}(E(k, \alpha))$ such that

$$q^\nu := \psi_\nu(p^\nu) = (0, \cdots, 0, t_\nu) \quad \text{with} \quad 0 \leqq t_\nu < 1 \tag{2.1}$$

for all $\nu = 1, 2, \cdots$. The existence of such a sequence of autmorphisms was already shown in Section 1. We have now two cases to consider.

Case 1. *$\{q^\nu\}_{\nu=1}^\infty$ has an accumulation point q in $E(k, \alpha)$.* We shall prove that D is biholomorphically equivalent to $E(k, \alpha)$ in this case. We may assume without loss of generality that

$$\lim_{\nu \to \infty} q^\nu = q \in E(k, \alpha).$$

Now let us fix a family of relatively compact subdomains D_j of D such that

$$D = \bigcup_{j=1}^\infty D_j \supset \cdots \supset D_{j+1} \supset D_j \supset \cdots \supset D_1 \ni k_o \tag{2.2}$$

and choose an integer $j \geqq 1$ arbitrarily. Since $\varphi_\nu(z) \to p$ uniformly on D_j, there exists an integer $\nu(j)$ such that

$$\varphi_\nu(D_j) \subset D \cap U = E(k, \alpha) \cap U \quad \text{for all} \quad \nu \geqq \nu(j).$$

So we can define biholomorphic mappings $f^\nu: D_j \to E(k, \alpha)$ by setting

$$f^\nu(z) = \psi_\nu(\varphi_\nu(z)), \quad z \in D_j \quad \text{for} \quad \nu \geqq \nu(j). \tag{2.3}$$

Since $E(k, \alpha)$ is taut and $f^\nu(k_o) \to q \in E(k, \alpha)$, we can assume by taking a subsequence if necessary that $\{f^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $f(j): D_j \to E(k, \alpha)$. By the usual diagonal argument, we may further assume that $\{f^\nu\}$ converges uniformly on D_j to the holomorphic mapping $f(j)$ for all $j = 1, 2, \cdots$. Accordingly, we can define a holomorphic mapping $f: D \to E(k, \alpha)$ by $f(z) = f(j)(z)$, $z \in D_j$ for $j = 1, 2, \cdots$.

Setting $E_\nu = \psi_\nu(E(k, \alpha) \cap U) = \psi_\nu(D \cap U)$ for $\nu = 1, 2, \cdots$, we consider the biholomorphic mappings $g^\nu: E_\nu \to D$ defined by

$$g^\nu(z) = \varphi_\nu^{-1}(\psi_\nu^{-1}(z)) , \quad z \in E_\nu \quad \text{for} \quad \nu = 1, 2, \cdots .$$

Then it is clear that

$$g^\nu \circ f^\nu = \mathrm{id}_{D_j} \quad \text{and} \quad f^\nu \circ g^\nu = \mathrm{id}_{f^\nu(D_j)} \tag{2.4}$$

for all $\nu \geqq \nu(j)$, $j = 1, 2, \cdots$. Let E' be an arbitrary subdomain of $E(k, \alpha)$ with compact closure. Then $\psi_\nu^{-1}(E') \subset E(k, \alpha) \cap U$ for all sufficiently large ν. Passing to a subsequence if necessary, we can therefore assume that $\{g^\nu\}$ converges uniformly on every compact subset of $E(k, \alpha)$ to a holomorphic mapping $g: E(k, \alpha) \to \bar{D} \subset \boldsymbol{C}^n$. Once $g(E(k, \alpha)) \subset D$ is shown, the equations (2.4) imply that $g \circ f = \mathrm{id}_D$ and $f \circ g = \mathrm{id}_{E(k,\alpha)}$; consequently, f gives a biholomorphic mapping from D onto $E(k, \alpha)$. Thus we have only to show that $g(E(k, \alpha)) \subset D$. To this end, take a subdomain E' of $E(k, \alpha)$ with compact closure such that $f(\bar{D}_1)$, $f^\nu(\bar{D}_1) \subset E'$ for all $\nu \geqq \nu_0$, where D_1 is the domain appearing in (2.2) and ν_0 is a large integer. Then, for any point $z \in D_1$ there is a sequence $\{z_i\}_{i=1}^\infty$ in E' such that $g^{\nu_i}(z_i) = z$ for all i and $z_i \to z_0$ for some point $z_0 \in \bar{E}'$. Hence $z = \lim_{i\to\infty} g^{\nu_i}(z_i) = g(z_0) \in g(E(k, \alpha))$, and accordingly, $D_1 \subset g(E(k, \alpha))$. On the other hand, being the local uniform limit of regular holomorphic mappings $\{g^\nu\}$, the mapping g is either regular on $E(k, \alpha)$ or the Jacobian determinant of g vanishes identically on $E(k, \alpha)$. But, $g(E(k, \alpha))$ contains a non-empty open set in $\boldsymbol{C}^n$, as we have already seen above. Hence we conclude that $g: E(k, \alpha) \to \boldsymbol{C}^n$ is regular on $E(k, \alpha)$ and so $g(E(k, \alpha)) \subset D$ by [1; Lemma 0] or [8; p. 79], completing the proof in Case 1.

Case 2. *$\{q^\nu\}_{\nu=1}^\infty$ has no accumulation point in $E(k, \alpha)$.* In this case we show that both domains D and $E(k, \alpha)$ are biholomorphically equivalent to the open unit ball B^n. We may assume that

$$\lim_{\nu\to\infty} q^\nu = (0, \cdots, 0, 1) =: q \in \partial E(k, \alpha) .$$

Since q is a strictly pseudoconvex boundary point of $E(k, \alpha)$ by the lemma in Section 1, there exist a small open neighborhood W of q and a C^2-strictly plurisubharmonic function $\rho: W \to \boldsymbol{R}$ such that

$$W \subset \{(z', z'') \in \boldsymbol{C}^k \times \boldsymbol{C}^{n-k} \mid |z'| \leqq 1/2\} ; \tag{2.6}$$

$$E(k, \alpha) \cap W = \{z \in W \mid \rho(z) < 0\} \quad \text{and} \quad d\rho(z) \neq 0,\ z \in W ; \tag{2.7}$$

$$(\partial\rho(q)/\partial z_1, \cdots, \partial\rho(q)/\partial z_{n-1}, \partial\rho(q)/\partial z_n) = (0, \cdots, 0, 1) . \tag{2.8}$$

To simplify the notation, we set

$$a_{ij} = (1/2)\cdot\partial^2\rho(q)/\partial z_i\partial z_j , \quad b_{i\bar{j}} = \partial^2\rho(q)/\partial z_i\partial\bar{z}_j$$

for $1 \leqq i,\ j \leqq n$ and consider the coordinate changes as follows:

$$H_1\colon u_j = z_j \quad (1 \leqq j \leqq n-1) , \quad u_n = z_n - 1 ;$$

$$H_2\colon v_j = u_j \quad (1 \leqq j \leqq n-1) , \quad v_n = u_n + \sum_{i,j=1}^{n} a_{ij}u_iu_j .$$

Clearly, H_1 is a globally defined change of coordinates and H_2 is a well-defined change of coordinates in a sufficiently small neighborhood of $u = o$. In the new coordinates $v = (v_1, \cdots, v_n)$, we have by Taylor's formula

$$\rho(v) = 2\operatorname{Re} v_n + \sum_{i,j=1}^{n} b_{i\bar{j}}v_i\bar{v}_j + o(|v|^2)$$

in a neighborhood of the origin,

$$q = (0, \cdots, 0) \quad \text{and} \quad q^\nu = (0, \cdots, 0, \delta_\nu)$$

with $\delta_\nu = (t_\nu - 1)[1 + a_{nn}(t_\nu - 1)]$ for $\nu = 1, 2, \cdots$, where t_ν are the numbers given by (2.1). Hence

$$\lim_{\nu\to\infty}(\delta_\nu, \delta_\nu/|\delta_\nu|) = (0, -1) . \tag{2.9}$$

In particular, we may assume that $0 < |\delta_\nu| < 1$ for all $\nu = 1, 2, \cdots$. Since $(b_{i\bar{j}})_{1\leqq i,j\leqq n-1}$ is a positive definite Hermitian matrix of degree $n-1$, it is diagonalizable. Thus, after a suitable change of coordinates $(v_1, \cdots, v_{n-1})$ in $\boldsymbol{C}^{n-1}$, we can obtain a new coordinate system $w = (w_1, \cdots, w_n)$, $w_n = v_n$, with respect to which ρ can be written in the form

$$\rho(w) = 2\operatorname{Re} w_n + |'w|^2 + A(w) \tag{2.10}$$

in a small neighborhood of the origin, where $'w = (w_1, \cdots, w_{n-1})$ as in Section 1 and

$$A(w) = 2\operatorname{Re}\Big(\sum_{j=1}^{n} c_jw_j\bar{w}_n\Big) + o(|w|^2)$$

with some constants $c_1, \cdots, c_n \in \boldsymbol{C}$. In particular, there are a continuous function $r(x)$ and a constant $C > 0$ such that

$$r(x) \to 0 \quad \text{as} \quad x \to 0 ; \tag{2.11}$$

$$|A(w)| \leqq C|w||w_n| + r(|w|^2)|w|^2 \quad \text{near} \quad w = o . \tag{2.12}$$

Let $\{D_j\}_{j=1}^{\infty}$ be the increasing family of relatively compact subdomains of D defined in (2.2). Then, as in (2.3) we can define a family of biholomorphic mappings $f^{\nu} = \psi_{r_\nu} \circ (\varphi_{\nu|D_j})$ for $\nu \geqq \nu(j)$, $j = 1, 2, \cdots$ which converges uniformly on compact subsets to a holomorphic mapping $f\colon D \to \overline{E(k, \alpha)} \subset C^n$ with $f(k_o) = q \in \partial E(k, \alpha)$. Taking now the plurisubharmonic function $\rho \circ f$ defined on an open neighborhood of k_o instead of the holomorphic function $\Psi_p \circ \varphi$ in Case 1, we can see that $f(z) = q$ for all $z \in D$. Let us fix an integer $j \geqq 1$ arbitrarily. Then, since $f^{\nu}(z) \to q$ uniformly on D_j, there exists an integer ν_j such that

$$f^{\nu}(D_j) \subset E(k, \alpha) \cap W \quad \text{for all} \quad \nu \geqq \nu_j .$$

We define mappings $L_\nu\colon C^n \to C^n$ and $F^{\nu}\colon D_j \to C^n$ by setting

$$L_\nu(w) = ('w/\sqrt{|\delta_\nu|}, \ -w_n/\delta_\nu) , \quad w = ('w, w_n) \in C^n ; \tag{2.13}$$

$$F^{\nu}(z) = L_\nu(f^{\nu}(z)) , \quad z \in D_j \tag{2.14}$$

for all $\nu \geqq \nu_j$, where δ_ν are the numbers appearing in (2.9). Then L_ν are non-singular linear transformations of C^n and F^{ν} are biholomorphic mappings D_j into C^n. Moreover, it is easily seen by construction that

$$F^{\nu}(k_o) = (0, \cdots, 0, -1) \quad \text{and} \quad F^{\nu}(D_j) \subset W_\nu \tag{2.15}$$

for all $\nu \geqq \nu_j$, where

$$W_\nu = L_\nu(E(k, \alpha) \cap W) = \{w \in C^n \,|\, L_\nu^{-1}(w) \in W, \ \rho \circ L_\nu^{-1}(w) < 0\} \tag{2.16}$$

for $\nu = 1, 2, \cdots$. Now we would like to show that some subsequence of $\{F^{\nu}\}$ converges uniformly on every compact set in D to a holomorphic mapping $F\colon D \to C^n$. To see this, we set

$$\rho^{\nu}(w) = [\rho \circ L_\nu^{-1}(w)]/|\delta_\nu| \quad \text{and} \quad A^{\nu}(w) = [A \circ L_\nu^{-1}(w)]/|\delta_\nu|$$

for $\nu = 1, 2, \cdots$. It follows then from (2.10), (2.12) that

$$\rho^{\nu}(w) = 2\,\mathrm{Re}(-\delta_\nu w_n/|\delta_\nu|) + |'w|^2 + A^{\nu}(w) ; \tag{2.17}$$

$$|A^{\nu}(w)| \leqq [C\sqrt{|\delta_\nu|} + r(|L_\nu^{-1}(w)|^2)] \cdot |w|^2 \tag{2.18}$$

in a neighborhood of the origin. Now, for the sake of simplicity we put

$$w^{\nu} = F^{\nu}(z) \quad \text{for each point} \quad z \in D_j .$$

Since $L_\nu^{-1}(w^{\nu}) = f^{\nu}(z) \to q = o$ uniformly on D_j, it follows from (2.11) and (2.18) that $|A^{\nu}(w^{\nu})|/|w^{\nu}|^2 \to 0$ uniformly on D_j. This combined with the inequality $\rho^{\nu}(w^{\nu}) < 0$ for $\nu \geqq \nu_j$ yields that

$$|w_n^{\nu}|^2 + 2\,\mathrm{Re}(\delta_\nu w_n^{\nu}/|\delta_\nu|) > |w^{\nu}|^2 + A^{\nu}(w^{\nu}) \geqq |w^{\nu}|^2/2 \geqq 0 \tag{2.19}$$

for all $\nu \geqq \nu_0$ and all $z \in D_j$, where ν_0 is a large integer depending on D_j. Here we may assume by (2.9) that $|1 + (\delta_\nu/|\delta_\nu|)| < 1/3$ for $\nu \geqq \nu_0$. Thus $\{F_n^\nu\}_{\nu \geqq \nu_0}$ forms a normal family, because F_n^ν for every $\nu \geqq \nu_0$ can now be regarded as a holomorphic mapping from D_j into the taut domain $C \setminus \{1/2, 1\}$. Moreover $F_n^\nu(\{k_0\}) \cap \{-1\} \neq \varnothing$ for all ν by (2.15). Hence we may assume that $\{F_n^\nu\}_{\nu \geqq \nu_0}$ converges uniformly on compact subsets to a holomorphic function on D_j. By (2.19) this means that $\{F^\nu\}_{\nu \geqq \nu_0}$ is uniformly bounded on every compact subset of D_j, and consequently some subsequence of $\{F^\nu\}_{\nu \geqq \nu_0}$ converges uniformly on compact subsets to a holomorphic mapping from D_j into C^n. Hence, passing again to a subsequence if necessary, we may assume that $\{F^\nu\}$ itself converges uniformly on every compact set in D to a holomorphic mapping $F: D \to C^n$.

Here we consider the following domain $\mathscr{B}$ and the mapping C:

$$\mathscr{B} = \{w \in C^n \mid 2 \operatorname{Re} w_n + |'w|^2 < 0\} ; \tag{2.20}$$

$$C: ('w, w_n) \mapsto (\sqrt{2}\, 'w/(w_n - 1), (w_n + 1)/(w_n - 1)) . \tag{2.21}$$

It is easily seen that there is an open neighborhood X of $\bar{\mathscr{B}}$ such that C gives rise to a biholomorphic mapping from X into C^n and $C(\mathscr{B}) = B^n$. In particular, $\mathscr{B}$ is a strictly pseudoconvex domain with real analytic boundary. Now we wish to show that $F(D) \subset \mathscr{B}$. For this let us fix a point $z \in D$ arbitrarily. Then, since $w^\nu = F^\nu(z) \to F(z)$ and $L_\nu^{-1}(w^\nu) = f^\nu(z) \to q = o$ as $\nu \to \infty$, we obtain from (2.9), (2.17) and (2.18) that

$$2 \operatorname{Re} F_n(z) + |'F(z)|^2 = \lim_{\nu \to \infty} \rho^\nu(w^\nu) \leqq 0 ,$$

which says that $F(D) \subset \bar{\mathscr{B}}$. But, thanks to the strict pseudoconvexity of $\mathscr{B}$, the image $F(D)$ can meet the boundary $\partial\mathscr{B}$ only when F is a constant mapping from D into $\partial\mathscr{B}$. Consequently, $F(D) \subset \mathscr{B}$, since by (2.15) $F(D)$ contains the point $(0, \cdots, 0, -1)$ of $\mathscr{B}$.

Next we prove that $F: D \to \mathscr{B}$ is, in fact, a biholomorphic mapping from D onto $\mathscr{B}$. Observe first that $L_\nu^{-1}(W_\nu) = E(k, \alpha) \cap W$ for all ν and $\psi_\nu^{-1}(E(k, \alpha) \cap W) \to \{p\}$ by the choice of W as in (2.6). Hence there is an integer ν_0 such that

$$\psi_\nu^{-1}(L_\nu^{-1}(W_\nu)) \subset E(k, \alpha) \cap U = D \cap U \quad \text{for all} \quad \nu \geqq \nu_0$$

and so we can define holomorphic mappings $G^\nu: W_\nu \to D$ by setting

$$G^\nu = \varphi_\nu^{-1} \circ \psi_\nu^{-1} \circ L_\nu^{-1} \quad \text{for} \quad \nu \geqq \nu_0 .$$

Clearly we have $G^\nu \circ F^\nu = \mathrm{id}_{D_j}$ and $F^\nu \circ G^\nu = \mathrm{id}_{F^\nu(D_j)}$ for all $\nu \geqq \max(\nu(j), \nu_0)$, $j = 1, 2, \cdots$. On the other hand, for an arbitrarily given subdomain $\mathscr{B}'$ of $\mathscr{B}$ with compact closure in $\mathscr{B}$ one can choose an integer $\nu(\mathscr{B}')$ in such a way that $\mathscr{B}' \subset W_\nu$ for all $\nu \geqq \nu(\mathscr{B}')$, because $\rho^\nu(w) \to 2\,\mathrm{Re}\, w_n + |'w|^2 < 0$ uniformly on $\mathscr{B}'$ by (2.9), (2.17) and (2.18). Therefore, passing to a subsequence if necessary, we may assume that $\{G^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $G: \mathscr{B} \to \bar{D} \subset \boldsymbol{C}^n$. With exactly the same method as in Case 1 one can now check that $G(\mathscr{B}) \subset D$ and F defines a biholomorphic mapping from D onto the domain $\mathscr{B} \cong B^n$.

Finally, assuming the correctness of Theorem II, we shall complete the proof by showing that D and $E(k, \alpha)$ are both biholomorphically equivalent to B^n. For this purpose, let us choose a sequence of positive numbers x_ν in such a way that

$$x_\nu \uparrow 1 \quad \text{and} \quad p^\nu := (x_\nu, 0, \cdots, 0) \in D \cap U$$

for $\nu = 0, 1, 2, \cdots$. Since D is now biholomorphically equivalent to B^n, there exists a sequence $\{\sigma_\nu\}_{\nu=1}^\infty$ in $\mathrm{Aut}(D)$ such that $\sigma_\nu(p^0) = p^\nu$ for $\nu = 1, 2, \cdots$. In particular, we have $\text{R-lim}_{\nu\to\infty} \sigma_\nu(p^0) = (1, 0, \cdots, 0)$. Moreover $D \cap U = E(k, \alpha) \cap U = \{z \in U \mid \rho(k, \alpha; z) < 0\}$ by assumption. As an immediate consequence of Theorem II, D is biholomorphically equivalent to $E(k, \alpha)$. q.e.d.

3. Proof of Theorem II. By the change of coordinates $u_1 = z_1 - 1$, $u_j = z_j$ $(2 \leqq j \leqq n)$, we have $p = (0, \cdots, 0)$ and ρ can be written in the form

$$\rho(u) = 2\,\mathrm{Re}\, u_1 + |u'|^2 + |u''|^{2\alpha} + R(u)\,, \quad R(u) = o(|u'|^2 + |u''|^{2\alpha})$$

in a neighborhood of the origin $u = o$. For any given constants A, B with $0 < A < 1 < B$, we can therefore assume that

$$(3.1) \qquad 2\,\mathrm{Re}\, u_1 + A(|u'|^2 + |u''|^{2\alpha}) \leqq \rho(u) \leqq 2\,\mathrm{Re}\, u_1 + B(|u'|^2 + |u''|^{2\alpha})$$

on U by shrinking U if necessary. So the holomorphic function $\Psi_p(u) = \exp u_1$ on $\boldsymbol{C}^n$ is peaking for $D \cap U$ at $p = o$. Hence, by the same reasoning as in Case 1 of the proof of Theorem I we may assume without loss of generality that

$$(3.2) \qquad \varphi_\nu(z) \to p \quad \text{uniformly on compact subsets of } D\,;$$

$$(3.3) \qquad \underset{\nu\to\infty}{\text{R-lim}}\, \varphi_\nu(k_\nu) = p \quad \text{and} \quad p^\nu := \varphi_\nu(k_\nu) \in D \cap U\,, \quad \nu = 1, 2, \cdots\,.$$

Therefore, writing

$$(3.4) \qquad p^\nu = \zeta^\nu + \lambda^\nu N \quad \text{with some} \quad \zeta^\nu \in \partial D \cap U\,, \quad \lambda^\nu < 0$$

uniquely as in (1.7) and taking a subsequence if necessary, we obtain by the assumption (iii) that

$$\lim_{\nu\to\infty} \operatorname{Re} \zeta_1^\nu/|\lambda^\nu| = d_o \quad \text{for some finite number} \quad d_o \leqq 0 .$$

For the sake of simplicity, we set

$$r_\nu = |\lambda^\nu|^{1/2} , \quad s_\nu = |\lambda^\nu|^{1/(2\alpha)} \quad \text{for} \quad \nu = 1, 2, \cdots .$$

The proof is now divided into two cases as follows:

Case 1. $d_o = 0$. In this case, it follows at once from (3.1) that

$$(3.5) \qquad (\operatorname{Re} \zeta_1^\nu/|\lambda^\nu|, \zeta_i^\nu/r_\nu, \zeta_j^\nu/s_\nu, R(\zeta^\nu)/|\lambda^\nu|) \to (0, 0, 0, 0)$$

as $\nu \to \infty$ for each i, j with $1 \leqq i \leqq k < j \leqq n$. Let us choose a sequence of relatively compact subdomains D_j of D such that

$$D = \bigcup_{j=1}^{\infty} D_j \supset \cdots \supset D_{j+1} \supset D_j \supset \cdots \supset D_1 \supset K ,$$

where K is the compact subset of D as in the theorem, and fix an integer $j \geqq 1$ arbitrarily. Since $\varphi_\nu(u) \to p$ uniformly on D_j, there exists an integer $\nu(j)$ such that

$$(3.6) \qquad \varphi_\nu(D_j) \subset D \cap U \quad \text{for all} \quad \nu \geqq \nu(j) .$$

Now define mappings h_ν, L_ν and F^ν by

$$h_\nu(u) = (u_1 - \zeta_1^\nu, \cdots, u_n - \zeta_n^\nu) , \quad u \in \boldsymbol{C}^n ;$$
$$L_\nu(w) = (-w_1/\lambda^\nu, w_2/r_\nu, \cdots, w_k/r_\nu, w_{k+1}/s_\nu, \cdots, w_n/s_\nu) , \quad w \in \boldsymbol{C}^n ;$$
$$F^\nu(u) = L_\nu \circ h_\nu \circ \varphi_\nu(u) , \quad u \in D_j$$

for all $\nu \geqq \nu(j)$. Then both h_ν and L_ν are biholomorphic transformations of $\boldsymbol{C}^n$, while F^ν are biholomorphic mapping from D_j into $\boldsymbol{C}^n$. It is clear that

$$(3.7) \qquad F^\nu(k_\nu) = (-1, 0, \cdots, 0) \quad \text{and} \quad F^\nu(D_j) \subset W_\nu$$

for all $\nu \geqq \nu(j)$, where

$$(3.8) \qquad W_\nu = \{w \in \boldsymbol{C}^n \,|\, (L_\nu \circ h_\nu)^{-1}(w) \in U, \ \rho \circ (L_\nu \circ h_\nu)^{-1}(w) < 0\}$$

for $\nu = 1, 2, \cdots$. Now we claim that some subsequence of $\{F^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $F: D \to \boldsymbol{C}^n$. For this, we set

$$\rho^\nu(w) = \rho \circ (L_\nu \circ h_\nu)^{-1}(w) , \quad R^\nu(w) = R \circ (L_\nu \circ h_\nu)^{-1}(w)$$

for $\nu = 1, 2, \cdots$ and

$$w^\nu = F^\nu(u) , \quad u \in D_j \quad \text{for} \quad \nu \geqq \nu(j) .$$

Then, since $(L_\nu \circ h_\nu)^{-1}(F^\nu(D_j)) = \varphi_\nu(D_j) \subset D \cap U$ for $\nu \geqq \nu(j)$, we obtain by (3.1), (3.7) and (3.8) that

$$0 > \rho^\nu(w^\nu) \geqq 2\,\mathrm{Re}(-\lambda^\nu w_1^\nu + \zeta_1^\nu) + A \cdot \left[|-\lambda^\nu w_1^\nu + \zeta_1^\nu|^2 + \sum_{i=2}^{k} |r_\nu w_i^\nu + \zeta_i^\nu|^2 + \left(\sum_{j=k+1}^{n} |s_\nu w_j^\nu + \zeta_j^\nu|^2 \right)^\alpha \right]$$

and so

$$0 > 2\,\mathrm{Re}(w_1^\nu + \zeta_1^\nu/|\lambda^\nu|) + A \cdot \left[\sum_{i=2}^{k} |w_i^\nu + \zeta_i^\nu/r_\nu|^2 + \left(\sum_{j=k+1}^{n} |w_j^\nu + \zeta_j^\nu/s_\nu|^2 \right)^\alpha \right]$$

for all $\nu \geqq \nu(j)$. Hence, if we define a domain $W(k, \alpha, A)$ in $\boldsymbol{C}^n$ and holomorphic mappings $\Phi^\nu : D_j \to \boldsymbol{C}^n$, $\nu \geqq \nu(j)$, by setting

$$W(k, \alpha, A) = \left\{ w \in \boldsymbol{C}^n \,|\, 2\,\mathrm{Re}\, w_1 + A \cdot \left[\sum_{i=2}^{k} |w_i|^2 + \left(\sum_{j=k+1}^{n} |w_j|^2 \right)^\alpha \right] < 0 \right\} ; \tag{3.9}$$

$$\Phi^\nu = (F_1^\nu + \mathrm{Re}\, \zeta_1^\nu/|\lambda^\nu|,\ F_2^\nu + \zeta_2^\nu/r_\nu,\ \cdots,\ F_k^\nu + \zeta_k^\nu/r_\nu,\ F_{k+1}^\nu + \zeta_{k+1}^\nu/s_\nu,\ \cdots,\ F_n^\nu + \zeta_n^\nu/s_\nu) , \tag{3.10}$$

then every Φ^ν gives rise to a holomorphic mapping from D_j into $W(k, \alpha, A)$. On the other hand, it is easily seen that $W(k, \alpha, A)$ is biholomorphically equivalent to the domain $E(k, \alpha)$ via the correspondence $C_A : (w_1, \cdots, w_n) \mapsto (z_1, \cdots, z_n)$ given by

$$C_A : \begin{cases} z_1 = (w_1 + 1)/(w_1 - 1) \\ z_i = (2A)^{1/2} \cdot w_i/(w_1 - 1) , \quad i = 2, \cdots, k \\ z_j = (2A)^{1/(2\alpha)} \cdot w_j/(w_1 - 1)^{1/\alpha} , \quad j = k+1, \cdots, n . \end{cases} \tag{3.11}$$

Hence $W(k, \alpha, A)$ is taut by the lemma in Section 1 and $\{\Phi^\nu\}$ forms a normal family. Moreover, it follows from (3.5) and (3.7) that

$$\Phi^\nu(k_\nu) = (-1 + \mathrm{Re}\, \zeta_1^\nu/|\lambda^\nu|,\ \zeta_2^\nu/r_\nu,\ \cdots,\ \zeta_k^\nu/r_\nu,\ \zeta_{k+1}^\nu/s_\nu,\ \cdots,\ \zeta_n^\nu/s_\nu) \to (-1, 0, \cdots, 0) \in W(k, \alpha, A) \quad \text{as} \quad \nu \to \infty ,$$

that is, $\{\Phi^\nu\}$ is not compactly divergent on D_j. Therefore we may assume that $\{\Phi^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $\Phi : D_j \to W(k, \alpha, A)$. Here it is obvious from (3.5) and (3.10) that $\lim_{\nu\to\infty} F^\nu = \Phi$ uniformly on compact subsets of D_j. By the usual diagonal argument, we may further assume that $\{F^\nu\}$ itself converges uniformly on every compact subset of D to a holomorphic mapping $F : D \to \boldsymbol{C}^n$.

We wish to prove that the image $F(D)$ is contained in the domain $W(k, \alpha) := W(k, \alpha, 1)$ defined in (3.9) with $A = 1$. To this end, recall that $R(u) = o(|u'|^2 + |u''|^{2\alpha})$. So there is a continuous function $r(x)$ such that

$$r(x) \to 0 \quad \text{as} \quad x \to 0 ; \tag{3.12}$$

(3.13) $$|R(u)| \leqq r(|u'|^2 + |u''|^{2\alpha})\cdot[|u'|^2 + |u''|^{2\alpha}] \quad \text{near the origin}.$$

Since $(L_\nu \circ h_\nu)^{-1}(w) \to o$ uniformly on compact sets, these combined with (3.5) yield that

$$|R^\nu(w)/\lambda^\nu| \leqq r(x_\nu)\cdot y_\nu \to 0 \quad \text{as} \quad \nu \to \infty$$

uniformly on every compact subset of C^n, where we have set

$$x_\nu = |[(L_\nu \circ h_\nu)^{-1}(w)]'|^2 + |[(L_\nu \circ h_\nu)^{-1}(w)]''|^{2\alpha};$$

$$y_\nu = |r_\nu w_1 + \zeta_1^\nu/r_\nu|^2 + \sum_{i=2}^{k}|w_i + \zeta_i^\nu/r_\nu|^2 + |w'' + (\zeta^\nu)''/s_\nu|^{2\alpha}.$$

Now take a point $u \in D$ arbitrarily and set again $w^\nu = F^\nu(u)$. Then $w^\nu \to F(u)$ as $\nu \to \infty$ and it follows from (3.7), (3.8) that

(3.14) $$0 > \rho^\nu(w^\nu)/|\lambda^\nu| = 2\,\mathrm{Re}(w_1^\nu + \zeta_1^\nu/|\lambda^\nu|) + |r_\nu w_1^\nu + \zeta_1^\nu/r_\nu|^2 + \sum_{i=2}^{k}|w_i^\nu + \zeta_i^\nu/r_\nu|^2 + \left(\sum_{j=k+1}^{n}|w_j^\nu + \zeta_j^\nu/s_\nu|^2\right)^\alpha + R^\nu(w^\nu)/|\lambda^\nu|$$

for all sufficiently large ν, and so letting ν tend to infinity, we have

$$0 \geqq 2\,\mathrm{Re}\,F_1(u) + \sum_{i=2}^{k}|F_i(u)|^2 + \left(\sum_{j=k+1}^{n}|F_j(u)|^2\right)^\alpha.$$

Clearly this means $F(u) \in \overline{W(k, \alpha)}$ and accordingly $F(D) \subset \overline{W(k, \alpha)}$.

Next step is to show that $F(D) \subset W(k, \alpha)$. Observe first that the interior of the closure $\overline{W(k, \alpha)}$ coincides with $W(k, \alpha)$ in our case. Hence the problem reduces to showing that $F: D \to C^n$ is an open mapping. We define biholomorphic mappings $G^\nu: W_\nu \to D$, $\nu = 1, 2, \cdots$, by

$$G^\nu(w) = \varphi_\nu^{-1} \circ h_\nu^{-1} \circ L_\nu^{-1}(w), \quad w \in W_\nu,$$

where W_ν are the domains given by (3.8). Clearly we have

(3.15) $$G^\nu \circ F^\nu|_{D_j} = \mathrm{id}_{D_j} \quad \text{and} \quad F^\nu \circ G^\nu|_{F^\nu(D_j)} = \mathrm{id}_{F^\nu(D_j)}$$

for all $\nu \geqq \nu(j)$, $j = 1, 2, \cdots$. Let W' be an arbitrary subdomain of $W(k, \alpha)$ with compact closure. Then we obtain by (3.5) and (3.14) that

$$\rho^\nu(w)/|\lambda^\nu| \to 2\,\mathrm{Re}\,w_1 + \sum_{i=2}^{k}|w_i|^2 + \left(\sum_{j=k+1}^{n}|w_j|^2\right)^\alpha < 0$$

uniformly on W'. Thus there exists an integer $\nu(W')$ such that

(3.16) $$W' \subset W_\nu \quad \text{for all} \quad \nu \geqq \nu(W').$$

Now, by the compactness of K we may assume that $k_\nu \to k_o \in K$. Then $F(k_o) = \lim_{\nu\to\infty} F^\nu(k_\nu) = (-1, 0, \cdots, 0) \in W(k, \alpha)$. Choose open neighborhoods W', D' of the points $(-1, 0, \cdots, 0)$, k_o with compact closures in $W(k, \alpha)$,

D, respectively, in such a way that $F(\bar{D}')\subset W'$. There exists an integer $\nu(D', W')$ so large that

$$F^{\nu}(D')\subset W' \quad \text{for all} \quad \nu \geqq \nu(D', W') . \tag{3.17}$$

Once it is shown that $F: D \to C^n$ is injective on D', $F(D)$ contains the non-empty open set $F(D')$, accordingly, we may conclude by the same reasoning as in the proof of Theorem I that $F(D)\subset W(k, \alpha)$. Now assume that $F(u_1) = F(u_2) = w$ for some $u_1, u_2 \in D'$. It follows then from (1.1) and (3.15) $\sim$ (3.17) that

$$\begin{aligned} d_{W'}(F^{\nu}(u_1), F^{\nu}(u_2)) &= d_{G^{\nu}(W')}(G^{\nu}(F^{\nu}(u_1)), G^{\nu}(F^{\nu}(u_2))) \\ &= d_{G^{\nu}(W')}(u_1, u_2) \geqq d_D(u_1, u_2) \end{aligned}$$

for all $\nu \geqq \max(\nu(W'), \nu(D', W'))$, and so letting $\nu \to \infty$ we have $u_1 = u_2$, as desired.

Finally we assert that $F: D \to W(k, \alpha)$ is a biholomorphic mapping from D onto $W(k, \alpha)$. Indeed, thanks to the fact (3.16) we may assume without loss of generality that $\{G^{\nu}\}$ converges uniformly on every compact set in $W(k, \alpha)$ to a holomorphic mapping $G: W(k, \alpha) \to \bar{D}\subset C^n$. Then, repeating exactly the same argument as in the proof of Theorem I, we can verify that $G(W(k, \alpha))\subset D$ and F defines a biholomorphic mapping from D onto $W(k, \alpha)$. Since the domain $W(k, \alpha)$ is biholomorphically equivalent to $E(k, \alpha)$ via the correspondence C_1 defined by (3.11), we have completed the proof in the first case.

Case 2. $d_o \neq 0$. The following proof will be presented in outline, since the details of the steps can be filled in by consulting the corresponding passages in Case 1.

Passing to a subsequence if necessary, we may assume by (3.1) together with the estimate $R(u) = o(|u'|^2 + |u''|^{2\alpha})$ that

$$(\operatorname{Re} \zeta_1^{\nu}/|\lambda^{\nu}|, \zeta_i^{\nu}/r_{\nu}, \zeta_j^{\nu}/s_{\nu}, R(\zeta^{\nu})/|\lambda^{\nu}|) \to (d_o, d_i, d_j, 0) \tag{3.18}$$

for each i, j with $1 \leqq i \leqq k < j \leqq n$, where d_i, d_j are some finite complex numbers. Let us define holomorphic mappings F^{ν} and Φ^{ν} in the same manner as in Case 1. Then, repeating exactly the same arguments as in Case 1, we can show that some subsequence of $\{\Phi^{\nu}\}$ converges uniformly on compact subsets of D to a holomorphic mapping $\Phi: D \to W(k, \alpha, A)$, where $W(k, \alpha, A)$ is the domain in C^n defined by (3.9). Clearly this combined with (3.10), (3.18) guarantees that some subsequence of $\{F^{\nu}\}$ also converges uniformly on compact subsets to a holomorphic mapping F: $D \to C^n$. In exactly the same way as in Case 1, it can be shown that F defines a biholomorphic mapping from D onto the domain

$$W'(k, \alpha) = \Big\{ w \in C^n \,|\, 2\,\mathrm{Re}(w_1 + d_0 + |d_1|^2/2) + \sum_{i=2}^{k} |w_i + d_i|^2 + \Big(\sum_{j=k+1}^{n} |w_j + d_j|^2 \Big)^{\alpha} < 0 \Big\} ,$$

which is obviously biholomorphically equivalent to $W(k, \alpha)$ via a parallel translation in C^n. Therefore, we have shown that D is also biholomorphically equivalent to $E(k, \alpha)$ in Case 2. q.e.d.

4. **Proof of Theorem III.** To begin with, we fix a family $\{M_j\}_{j=1}^{\infty}$ of relatively compact subdomains of M such that

$$M = \bigcup_{j=1}^{\infty} M_j \supset \cdots \supset M_{j+1} \supset M_j \supset \cdots \supset M_1 \ni k_0 , \tag{4.1}$$

where k_0 is an arbitrarily fixed point of M. Since M can be exhausted by biholomorphic images of $E(k, \alpha)$, there exists a sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ of biholomorphic mappings from $E(k, \alpha)$ into M such that

$$M_\nu \subset \psi_\nu(E(k, \alpha)) , \quad \nu = 1, 2, \cdots .$$

We set

$$\varphi_\nu = \psi_\nu^{-1} \colon \psi_\nu(E(k, \alpha)) \to E(k, \alpha) , \quad \nu = 1, 2, \cdots .$$

Without loss of generality, we may assume that $\{\varphi_\nu\}$ converges uniformly on every compact set in M to a holomorphic mapping $\varphi \colon M \to \overline{E(k, \alpha)} \subset C^n$. Replacing ψ_ν, φ_ν by suitable holomorphic mappings of the form $\psi_\nu \circ \sigma_\nu^{-1}$, $\sigma_\nu \circ \varphi_\nu$ with some $\sigma_\nu \in \mathrm{Aut}(E(k, \alpha))$, if necessary, we may further assume that

$$q^\nu := \varphi_\nu(k_0) = (0, \cdots, 0, t_\nu) \quad \text{with} \quad 0 \leqq t_\nu < 1$$

for all $\nu = 1, 2, \cdots$. Again we have two cases to consider.

Case 1. *$\{q^\nu\}$ has an accumulation point q in $E(k, \alpha)$.* We claim that M is biholomorphically equivalent to $E(k, \alpha)$. We may assume that $q^\nu \to q$ and $\{\varphi^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $\varphi \colon M \to E(k, \alpha)$, since $E(k, \alpha)$ is taut and $\{\varphi_\nu(k_0)\}$ lies in a compact subset of $E(k, \alpha)$. Here we assert that $\varphi \colon M \to E(k, \alpha)$ is injective. Indeed, suppose that $\varphi(x_1) = \varphi(x_2) = z$ for $x_1, x_2 \in M$. It follows then from (1.1) that

$$\begin{aligned} d_{E(k,\alpha)}(\varphi_\nu(x_1), \varphi_\nu(x_2)) &= d_{\psi_\nu(E(k,\alpha))}(\psi_\nu(\varphi_\nu(x_1)), \psi_\nu(\varphi_\nu(x_2))) \\ &= d_{\psi_\nu(E(k,\alpha))}(x_1, x_2) \geqq d_M(x_1, x_2) \end{aligned}$$

for all sufficiently large ν. Consequently, we have $x_1 = x_2$, because M is hyperbolic and $d_{E(k,\alpha)}(\varphi_\nu(x_1), \varphi_\nu(x_2)) \to d_{E(k,\alpha)}(z, z) = 0$ as $\nu \to \infty$. Therefore, identifying M with the bounded domain $\varphi(M)\subset E(k, \alpha)$ and replacing the system $(\{f^\nu\}, \{g^\nu\}, D, \{D_j\})$ by $(\{\varphi_\nu\}, \{\psi_\nu\}, M, \{M_j\})$ in Case 1 of the proof of Theorem I, we can show that M is biholomorphically equivalent to $E(k, \alpha)$.

Case 2. $\{q^\nu\}_{\nu=1}^\infty$ *has no accumulation point in* $E(k, \alpha)$. In this case, we shall prove that M is biholomorphically equivalent to the open unit ball B^n. Without loss of generality, we may assume that:

$$\lim_{\nu\to\infty} q^\nu = (0, \cdots, 0, 1) =: q \in \partial E(k, \alpha) ; \tag{4.2}$$

$$\varphi_\nu(x) \to q \quad \text{uniformly on compact subsets of } M . \tag{4.3}$$

Hence there exists an integer ν_j such that

$$\varphi_\nu(M_j)\subset E(k, \alpha)\cap W \quad \text{for all} \quad \nu \geqq \nu_j ,$$

where M_j is an arbitrary subdomain of M appearing in the sequence (4.1) and W is the same neighborhood of q as that defined in Case 2 of the proof of Theorem I. Introducing a new coordinate system $w=(w_1, \cdots, w_n)$ in $\boldsymbol{C}^n$ as in Case 2 of the proof of Theorem I, we define biholomorphic mappings $L_\nu\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$ and $F^\nu\colon M_j \to \boldsymbol{C}^n$ for $\nu \geqq \nu_j$ by

$$L_\nu(w) = ('w/\sqrt{|\delta_\nu|}, -w_n/\delta_\nu) , \quad w = ('w, w_n) \in \boldsymbol{C}^n ;$$
$$F^\nu(x) = L_\nu(\varphi_\nu(x)) , \quad x \in M_j$$

as in (2.13) and (2.14). Then it can be shown that some subsequence of $\{F^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $F\colon M \to \mathscr{B}$, where $\mathscr{B}$ is the domain in $\boldsymbol{C}^n$ defined in (2.20). Indeed, considering the biholomorphic mappings

$$G^\nu(w) = \psi_\nu(L_\nu^{-1}(w)) , \quad w \in L_\nu(E(k, \alpha)\cap W) = W_\nu$$

for $\nu = 1, 2, \cdots$, one can check that F is a biholomorphic mapping from M into $\mathscr{B} \cong B^n$. In particular, M can be regarded as a bounded domain in $\boldsymbol{C}^n$. Therefore, repeating the same argument as in Case 2 of the proof of Theorem I, we conclude that M is biholomorphically equivalent to the domain $\mathscr{B} \cong B^n$. q.e.d.

5. Concluding remarks. Let D be a domain in $\boldsymbol{C}^n$ and p a point of $\bar{D}$. Then we say that D *is hyperbolically imbedded at* p if, for any neighborhood W of p in $\boldsymbol{C}^n$, there exists a neighborhood V of p in $\boldsymbol{C}^n$ such that

$$\bar{V}\subset W \quad \text{and} \quad d_D(D\cap(\boldsymbol{C}^n \setminus W), D\cap V) > 0 .$$

Note that, if D is a bounded domain in $\boldsymbol{C}^n$, then D is hyperbolically imbedded at every point p of $\bar{D}$.

REMARK 1. In Theorems I and II, the boundedness assumption on D can be replaced by the following weaker one: *D is a not necessarily bounded hyperbolic domain in C^n which is hyperbolically imbedded at $p=(1, 0, \cdots, 0) \in \partial D$.*

Indeed, by the existence of a local peaking function for D at p, one can extract in the same manner as in [7; Lemma 2] a subsequence of $\{\varphi_\nu\} \subset \mathrm{Aut}(D)$ which converges uniformly on compact subsets of D to the constant mapping $C_p(z)=p$, $z \in D$. Hence, the rests of the proofs of Theorems I and II will go through without any change.

REMARK 2. By a simple modification of the proof of Theorem II, one can see that the analogue of Theorem II is also valid for more general domains

$$E=\left\{(z_1, \cdots, z_s) \in \boldsymbol{C}^{n_1} \times \cdots \times \boldsymbol{C}^{n_s} \,\middle|\, |z_1|^2+\sum_{i=2}^{s}|z_i|^{2\alpha_i}<1\right\},$$

where $0 \leqq n_i \in \boldsymbol{Z}$, $0<\alpha_i \in \boldsymbol{R}$ for $i=2, \cdots, s$ and $1 \leqq n_1 \in \boldsymbol{Z}$.

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