Tôhoku Math. J.
40 (1988), 331-341.

AN APPLICATION OF PSEUDO-INVERSES —THE ANALYTIC CHARACTERISTIC OF T-REGULAR POINTS FOR CLOSED OPERATORS

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(Received January 9, 1987)

Abstract. In this paper we shall prove that $\lambda \in \rho_{SF}(T)$ is T-regular if and only if there exists a family of holomorphic vector functions $\{f_i(\mu)\}_{i\in I}$ defined in some neighborhood Δ of λ such that $\{f_i(\mu)\}_{i\in I}$ forms a basis of $\ker(T-\mu)$ for $\mu \in \Delta$, where T is a densely defined closed operator acting in a complex Hilbert space, $\rho_{SF}(T)$ is the semi-Fredholm domain of T.

In 1976, Apostol [1] gave the definition of T-regular points for linear bounded operators acting in a complex Hilbert space: λ is called a T-regular point if the function

$$\mu \to P_{\ker(T-\mu)}$$

is continuous at λ. Two years later Cowen and Douglas [3] proved that if $\lambda \in \rho_F(T)$ and λ is a point of stability, namely $\dim\ker(T-\mu)$ is constant in some neighborhood of λ, then there exist holomorphic vector functions $\{f_i(\mu)\}_{i=1}^n$ defined in some neighborhood Δ of λ such that $\{f_i(\mu)\}_{i=1}^n$ forms a basis for $\ker(T-\mu)$ for $\mu \in \Delta$. Since in $\rho_F(T)$ each T-regular point is a point of stability,

(*) the continuity in the definition of T-regularity implies the analyticity in a sense.

But for $\lambda \in \rho_{SF}(T)$ the T-regularity does not imply that λ is a point of stability. This suggests the following question:

For $\lambda \in \rho_{SF}(T)$ does the statement (*) hold?

In this paper we shall give an affirmative answer to this question for closed operators using pseudo-inverses as our main tool. To this aim we should first investigate the structure of $\rho_{SF}(T)$ for closed operators, because the main results concerning $\rho_{SF}(T)$ in [1] are established on the basis of boundedness of operators.

The whole paper is divided into four sections. In Section 1, we obtain several lemmas which were proved for bounded operators in [1]. Section 2 is devoted to the structure of the semi-Fredholm domain $\rho_{SF}(T)$. The results are parallel to those in [1] and some of them are deeper and more precise. In Section 3, we shall give the analytic characteristic of T-regular points. Finally in Section 4 we shall give some applications of the analytic characteristic.

1. Notation and preliminaries. Let H be a Hilbert space over the complex field C and $B(H)$ $(L(H))$ be the set of all bounded (closed) linear operators acting in H. For $T \in L(H)$, we denote by $D(H)$ and $\ker T$ the domain and kernel of T, respectively. If T is densely defined, then let T^* be the conjugate of T. For a subset $M \subset H$, TM will denote the set $\{Tx, x \in M \cap D(T)\}$. TH is called the range of T and denoted by $R(T)$. Put

$$\operatorname{nul} T = \dim \ker T\ , \quad \operatorname{def} T = \operatorname{codim} R(T)\ .$$

If $R(T)$ is closed and at least one of $\operatorname{nul} T$ and $\operatorname{def} T$ is finite, then we call T a semi-Fredholm operator. Let

$$\rho_{SF}(T) = \{\lambda \in C\colon T - \lambda \text{ is semi-Fredholm}\}$$

and call it the semi-Fredholm domain of T. $\lambda \in C$ is called a T-singular point if λ is not T-regular. Put

$$\rho^r_{SF}(T) = \{\lambda \in \rho_{SF}(T)\colon \lambda \text{ is } T\text{-regular}\}\ .$$
$$\rho^s_{SF}(T) = \{\lambda \in \rho_{SF}(T)\colon \lambda \text{ is } T\text{-singular}\}\ .$$

Let ρ_r, ρ_l, σ_p denote the right resolvent set, left resolvent set, point spectrum of T, respectively. $\mathrm{Lat}(T)$ will denote the set of all invariant subspaces under T. For $M \in \mathrm{Lat}(T)$, T_M will denote the restriction of T to M. For a subset $M \subset H$, let $\mathrm{lm}\{M\}$ be the linear subspace spanned by M, $\mathrm{clm}\{M\}$ be the closure of $\mathrm{lm}\{M\}$ and $M^\perp$ be the annihilator of M. For a closed linear subspace M of H, P_M will denote the orthogonal projection onto M and put $T_M = (P_M T)_M$ which is called the compression of T on M.

DEFINITION 1.1. Each $T \in L(H)$ induces a one to one operator from $(\ker T)^\perp$ onto TH. We define S to be the operator which on TH is the inverse of that induced operator and which is zero on $(TH)^\perp$. We call S the pseudo-inverse of T.

Clearly we have

$$ST = I - P \quad \text{on} \quad D(T)\ , \tag{1}$$

$$TS = I - Q \quad \text{on} \quad TH \oplus (TH)^\perp \tag{2}$$

where $P = P_{\ker T}$, $Q = P_{R(T)^\perp}$.

REMARK 1.2. S is bounded if and only if $R(T)$ is closed (see [4, Theorem 3.1.2]). If $TH = H$, then S is the right inverse of T; if $\ker T = \{0\}$, then S is the left inverse of T. If T is densely defined, then S^* is the pseudo-inverse of T^*.

LEMMA 1.3. *Let $T \in L(H)$ have closed range and S be the pseudo-inverse of T. Let $Y \in \mathrm{Lat}(T)$, $TY = Y$ and $\ker T \subset Y$. Then $Y \in \mathrm{Lat}(S)$ and the restriction S_Y of S to Y is the right inverse of the restriction T_Y of T to Y.*

PROOF. Corresponding to the decomposition $H = Y \oplus Y^{\perp}$, (1) can be written as

$$\begin{pmatrix} S_Y & * \\ S_{21} & S_{Y^{\perp}} \end{pmatrix} \begin{pmatrix} T_Y & * \\ 0 & T_{Y^{\perp}} \end{pmatrix} = \begin{pmatrix} I_Y - P & 0 \\ 0 & I_{Y^{\perp}} \end{pmatrix},$$

where $P = P_{\ker T}$ and I_Y, $I_{Y^{\perp}}$ are the identities on Y, $Y^{\perp}$ respectively. Since $TY = Y$, we see that $S_{21} = 0$ and hence $Y \in \mathrm{Lat}(S)$. Since $Y = TY \subset TH$, we have $(TH)^{\perp} \subset Y^{\perp}$ and hence (2) can be written as

$$\begin{pmatrix} T_Y & * \\ 0 & T_{Y^{\perp}} \end{pmatrix} \begin{pmatrix} S_Y & * \\ 0 & S_{Y^{\perp}} \end{pmatrix} = \begin{pmatrix} I_Y & 0 \\ 0 & I_{Y^{\perp}} - Q \end{pmatrix},$$

where $Q = P_{R(T)^{\perp}}$. Therefore

$$S_Y T_Y = I_Y - P\,,$$
$$T_Y S_Y = I_Y\,.$$

These two equalities show that S_Y is the right inverse of T.

LEMMA 1.4. *Let $T \in L(H)$ have closed range, $Y \in \mathrm{Lat}(T)$ and $\ker T \subset Y$. Then $R(T_Y)$ is closed.*

PROOF. Suppose $R(T_Y)$ is not closed. Then there exists a sequence $\{x_n\} \subset Y$ such that

(3) $$\|Tx_n\| \to 0\,, \qquad \mathrm{dist}(x_n, \ker T_Y) \geqq 1\,.$$

Since $\ker T_Y = \ker T$, (3) shows that $R(T)$ is not closed (see [5, p. 231]).

The following is a key lemma.

LEMMA 1.5. *Let $T \in L(H)$ be densely defined, G be a subset of C and $Y = \mathrm{clm}\{\ker(T - \lambda)\}_{\lambda \in G - \{0\}}$. Then*

(i) *$Y \in \mathrm{Lat}(T)$;*

(ii) *if TY is closed, then $0 \in \rho_r(T_Y) \cap \rho_l(T_{Y^{\perp}})$.*

PROOF. Put $Y_0 = \mathrm{lm}\{\ker(T - \lambda)\}_{\lambda \in G - \{0\}}$. It is trivial that

(4) $$Y_0 = TY_0 \subset TY \subset TH\,.$$

(i) Since $Y = \bar{Y}_{\circ}$, by (1) we can derive that

(5) $$\overline{SY} \supset \overline{STY} = \overline{(I-P)Y_{\circ}} \supset (I-P)Y \supset STY ,$$

where $P = P_{\ker T}$. If

(6) $$Y \supset T\overline{SY} ,$$

then by (5) we have

(7) $$Y \supset T\overline{SY} \supset TSTY = TY .$$

To obtain (6) we prove that

$$Y^{\perp} \subset (T\overline{SY})^{\perp} .$$

By [5, Theorem III.5.29] we need only to prove that

(8) $$Y^{\perp} \cap D(T^*) \subset (T\overline{SY})^{\perp} .$$

For $x \in Y^{\perp} \cap D(T^*)$, by (2) we observe that

$$0 = (x, y) = (x, TSy) = (T^*x, Sy) , \quad \text{for} \quad y \in Y \cap D(S) .$$

This shows that $T^*x \perp SY$ and hence $T^*x \perp \overline{SY}$. Thus (8) is proved and hence (7) holds.

(ii) Since TY is closed, (4) implies that

(9) $$Y \subset TY .$$

Combining (7) and (9) we obtain $Y = TY$, i.e., $\lambda = 0 \in \rho_r(T_Y)$. The verification of the rest is trivial.

2. Structure of $\rho_{SF}(T)$.

PROPOSITION 2.1. *Let $T \in L(H)$ have closed range, S be the pseudo-inverse of T and $P = P_{\ker T}$. Then*

$$\ker(T-\mu) \subset R((I-\mu S)^{-1}P) \quad \textit{for} \quad |\mu| < \|S\|^{-1} .$$

PROOF. It is easy to verify that

$$\ker(T-\mu) \subset \ker((I-\mu S)^{-1}S(T-\mu)) \subset \ker(I-(I-\mu S)^{-1}P) \subset R((I-\mu S)^{-1}P)$$

PROPOSITION 2.2. *Let $T \in L(H)$, $TH = H$ and $P = P_{\ker T}$. Let S be the pseudo-inverse of T. Then*

$$\ker(T-\mu) = R((I-\mu S)^{-1}P) \quad \textit{for} \quad |\mu| < \|S\|^{-1} .$$

PROOF. By Proposition 2.1 we need only to prove that

$$R((I-\mu S)^{-1}P) \subset \ker(T-\mu) .$$

Let $y = (I-\mu S)^{-1}Px$ for some $x \in H$. Then we have

$$y = \mu S y + P x \in D(T) .$$

By Remark 1.2 we obtain $Ty = \mu y$, i.e., $y \in \ker(T - \mu)$.

THEOREM 2.3. *For $T \in L(H)$, we have $\rho_r(T) \subset \rho_{SF}^r(T)$.*

PROOF. Since $\rho_r(T) \subset \rho_{SF}(T)$, it suffices to suppose $\lambda = 0 \in \rho_r(T)$ and prove that 0 is T-regular. Since $R(T)$ is closed, the pseudo-inverse S of T is bounded. Put $P = P_{\ker T}$ and $M_\mu = R((I - \mu S)^{-1} P)$ for all $|\mu| < \|S\|^{-1}$. By Proposition 2.2 and [2, p. 86] we have

$$\begin{aligned} \|P_{\ker(T-\mu)} - P\| = \max\{&\sup\{\mathrm{dist}(x, M_\mu); x \in \ker T, \|x\| = 1\} , \\ &\sup\{\mathrm{dist}(x, \ker T); x \in M_\mu, \|x\| = 1\}\} \to 0 \text{ as } \mu \to 0 . \end{aligned}$$

This shows that $0 \in \rho_{SF}^r(T)$.

Similarly to the proof of [1, Theorem 2.4], by using Lemma 1.5 and Theorem 2.3 we can prove the following.

THEOREM 2.4. *Let $T \in L(H)$ be densely defined. Then*

$$\rho_{SF}^r(T) = \{\mu \in \rho_{SF}(T) \colon \ker(T - \mu) \subset \mathrm{clm}\{\ker(T - \lambda)_{\lambda \neq \mu}\} .$$

COROLLARY 2.5. *For any densely defined closed operator, we have $\rho_{SF}^s(T) \subset \sigma_p(T)$.*

PROPOSITION 2.6. *Let $T \in L(H)$ be densely defined and have closed range. Put $P = P_{\ker T}$, $Q = P_{\ker T^*}$, $H_1 = R(T^*)$, $H_2 = R(T)$. Let S be the pseudo-inverse of T. Then for $|\mu| < \|S\|^{-1}$ we have*

(i) $(T - \mu) H_1 = \ker(Q(I - \mu S)^{-1})$;

(ii) $(T - \mu)^* H = \ker(P(I - \bar{\mu} S^*)^{-1})$.

PROOF. (i) Let $x \in \ker(Q(I - \mu S)^{-1})$. Then $(T - \mu S)^{-1} x \in H_2$ and hence $(I - \mu S)^{-1} x = T x_1$ for some $x_1 \in H \cap D(T)$. By (1) we have

$$x = (I - \mu S) T x_1 = (T - \mu) x_1 \in (T - \mu) H_1 .$$

Conversely, let $y = (T - \mu) x$ for some $x \in H_1 \cap D(T)$. Then

$$Q(I - \mu S)^{-1}(T - \mu) x = Q(I - \mu S)^{-1}(I - \mu S) T x = Q T x = 0 .$$

(ii) By considering T^*, we can prove (ii) similarly.

COROLLARY 2.7. *Let $T \in L(H)$ be densely defined and have closed range. Let S be the pseudo-inverse of T. Put $G = \{\mu \in C \colon |\mu| < \|S\|^{-1}\}$.*

(i) *If $0 \in \rho_r(T)$, then $G \subset \rho_r(T)$;*

(ii) *If $0 \in \rho_{SF}(T)$, then $G \subset \rho_{SF}(T)$.*

PROOF. (i) If $0 \in \rho_r(T)$, then $Q = P_{\ker T^*} = 0$ and hence $(T - \mu) H \supset (T - \mu) H_1 = H$ for $\mu \in G$, i.e., $G \subset \rho_r(T)$.

(ii) First we prove that $R(T-\mu)$ is closed for $\mu \in G$. Without loss of generality we may suppose $\mathrm{nul}\, T < \infty$. Note that

$$R(T-\mu) = (T-\mu)\ker T + (T-\mu)H_1 = \ker T + (T-\mu)H_1 .$$

By (i) of Proposition 2.6 and [5, Lemma III.1.9], we know that $R(T-\mu)$ is closed. Next we prove that $\mathrm{nul}(T-\mu) < \infty$ for $\mu \in G$. Indeed, by (ii) of Proposition 2.6 we have

$$\mathrm{nul}(T-\mu) + \mathrm{def}(T^* - \bar{\mu}) \leqq \mathrm{codim}(T^* - \bar{\mu})H_2 = \dim PH = \mathrm{nul}\, T < \infty .$$

Therefore $T-\mu$ is semi-Fredholm, i.e., $G \subset \rho_{SF}(T)$.

REMARK 2.8. Corollary 2.7 shows that $\rho_r(T)$ and $\rho_{SF}(T)$ are open sets.

THEOREM 2.9. *Let $T \in L(H)$ be densely defined, semi-Fredholm and let S be its pseudo-inverse. Put $G_0 = \{\mu \in C: 0 < |\mu| < \|S\|^{-1}\}$. Then*

(i) $\rho^r_{SF}(T^*) = \rho^r_{SF}(T)^-$;

(ii) $G_0 \subset \rho^r_{SF}(T)$.

PROOF. Put $Y = \mathrm{clm}\{\ker(T-\lambda)\}_{\lambda \in G_0}$.

(i) Without loss of generality we may suppose 0 is a T-regular point. By Lemmas 1.5 and 1.4 we have $0 \in \rho_r(T_Y) \cap \rho_l(T_Y)$. Clearly $(T_{Y^\perp})^* = T^*_{Y^\perp}$ and $\rho_l(T_{Y^\perp}) = \rho_r((T_Y)^*)^- = \rho_r(T^*_{Y^\perp})$. Therefore $0 \in \rho_r(T^*_{Y^\perp})^-$. By Corollary 2.7 and Theorem 2.3 there exists a neighborhood G of 0 such that

$$G \subset \rho_{SF}(T) \cap \rho_r(T^*_{Y^\perp}) \subset \rho^r_{SF}(T^*_{Y^\perp}) . \tag{10}$$

On the other hand, by [1, Lemma 2.1], we have

$$\ker(T-\lambda)^* \subset (\mathrm{clm}\{\ker(T-\lambda)\}_{\lambda \in C})^\perp \subset Y^\perp \quad \text{for all} \quad \lambda \in G ,$$

and hence

$$\ker(T-\lambda)^* = \ker(T^*_{Y^\perp} - \bar{\lambda}) \quad \text{for all} \quad \lambda \in G . \tag{11}$$

(10) and (11) imply that

$$\lim P_{\ker(T-\lambda)^*} = P_{\ker T^*} ,$$

i.e., $0 \in \rho^r_{SF}(T^*)$.

(ii) First we consider the case in which $0 \in \rho^r_{SF}(T)$. The regularity of $\lambda = 0$ implies that $\ker T \subset Y$. Then $0 \in \rho_r(T_Y)$ follows from Lemmas 1.5 and 1.4. By Lemma 1.3, S_Y is the pseudo-inverse of T_Y. Put $G_Y = \{\mu \in C: |\mu| < \|S_Y\|^{-1}\}$. Clearly

$$G_0 \subset G_Y .$$

By Corollary 2.7 and Theorem 2.3 we have

$$G_Y \subset \rho_r(T_Y) \subset \rho^r_{SF}(T_Y) .$$

By the definition of Y, we have

$$\ker(T - \mu) = \ker(T_Y - \mu) \quad \text{for all} \quad \mu \in G_\circ ,$$

and hence

$$\lim_{\lambda \to \mu} P_{\ker(T-\lambda)} = P_{\ker(T-\mu)} \quad \text{for all} \quad \mu \in G_\circ ,$$

i.e., $G_\circ \subset \rho^r_{SF}(T)$.

Next suppose $0 \in \rho^s_{SF}(T)$. By (i) we may suppose $\mathrm{nul}\, T < \infty$. By [5, Lemma IV.5.29] we know that TY is closed. Thus $0 \in \rho_r(T_Y)$ follows from Lemma 1.5. Then following the above argument we obtain $G_\circ \subset \rho^r_{SF}(T)$.

By (ii) of Theorem 2.9 we know that for densely defined operator $T \in L(H)$, $\rho^r_{SF}(T)$ is an open set and $\rho^s_{SF}(T)$ has no accumulation points in $\rho_{SF}(T)$.

3. Analytic characteristic. Throughout this section we assume that $T \in L(H)$ is densely defined. Put as in [1]

$$\begin{aligned}
H_r(T) &= \mathrm{clm}\{\ker(T - \lambda); \lambda \in \rho^r_{SF}(T)\} , \\
H_l(T) &= \mathrm{clm}\{\ker(T - \lambda)^*; \lambda \in \rho^r_{SF}(T)\} , \\
H_\circ(T) &= H \ominus (H_r(T) \oplus H_l(T)) , \\
T_r &= T_{H_r(T)} , \quad T_\circ = T_{H_\circ(T)} , \quad T_l = T_{H_l(T)} .
\end{aligned}$$

LEMMA 3.1. *Corresponding to the decomposition $H = H_r \oplus H_\circ \oplus H_l$, T has the form*

$$T = \begin{pmatrix} T_r & * & * \\ 0 & T_\circ & * \\ 0 & 0 & T_l \end{pmatrix} . \tag{12}$$

PROOF. Regardless of whether $\lambda = 0$ is T-regular or not we observe that $H_r(T) = \mathrm{clm}\{\ker(T - \lambda); \lambda \in \rho^r_{SF}(T), \lambda \neq 0\}$. Thus (12) follows from (i) of Lemma 1.5.

Clearly

$$H_r(T) = H_l(T^*) , \quad H_l(T) = H_r(T^*) , \quad (T_r)^* = (T^*)_l , \quad (T_l)^* = (T^*)_r .$$

THEOREM 3.2. *Let $T \in L(H)$ be densely defined. Then*

(i) $\rho_{SF}(T) \subset \rho_r(T_r) \cap \rho_l(T_l)$;

(ii) $\rho^r_{SF}(T) \subset \rho(T_\circ)$.

PROOF. (i) Let $\lambda \in \rho_{SF}(T)$ and put $Y = H_r(T)$. Without loss of generality we may suppose $\mathrm{nul}(T - \lambda) < \infty$. Hence $(T - \lambda)Y$ is closed. By

Lemma 1.5 we see that $Y \in \mathrm{Lat}(T-\lambda)$ and $\lambda \in \rho_r(T_r)$. Since $\bar{\lambda} \in \rho_{SF}(T^*)$, we have $\lambda \in \rho_r(T_r^*)^- = \rho_r((T_l)^*) = \rho_l(T_l)$.

(ii) Let $\lambda \in \rho_{SF}^r(T)$. Then (i) implies that $\lambda \in \rho_l(T_{H_r(T)}) \subset \rho_l(T_\circ)$. Similarly, we can prove that $\lambda \in \rho_l(T_\circ^*)^- = \rho_r(T_\circ)$. Thus $\lambda \in \rho(T_\circ)$.

THEOREM 3.3. *Let $0 \in \rho_{SF}^r(T)$ and let S be the pseudo-inverse of T. Then corresponding to the decomposition $H = H_r \oplus H_\circ \oplus H_l$, S has the form*

$$S = \begin{pmatrix} S_r & * & * \\ 0 & S_\circ & * \\ 0 & 0 & S_l \end{pmatrix}. \tag{13}$$

PROOF. $\ker T \subset H$, since $0 \in \rho_{SF}^r(T)$. Hence (1) can be written as

$$\begin{pmatrix} S_r & * & * \\ S_{21} & S_\circ & * \\ S_{31} & S_{32} & S_l \end{pmatrix}\begin{pmatrix} T_r & * & * \\ 0 & T_\circ & * \\ 0 & 0 & T_l \end{pmatrix} = \begin{pmatrix} I-P & 0 & 0 \\ 0 & I_\circ & 0 \\ 0 & 0 & I_l \end{pmatrix} \quad \text{on} \quad H_r \oplus H_\circ \oplus H_l\,,$$

where $P = P_{\ker T}$. It is easy to see that $S_{21} = S_{31} = S_{32} = 0$ by $0 \in \rho_r(T_Y)$ and $0 \in \rho(T_\circ)$. Thus (13) is proved.

REMARK 3.4. If $\lambda = 0 \in \rho_{SF}^s(T)$, we can only obtain $S_{31} = 0$, i.e.,

$$S = \begin{pmatrix} S_r & * & * \\ S_{21} & S_\circ & * \\ 0 & S_{32} & S_l \end{pmatrix}.$$

THEOREM 3.5. *Let $\lambda = 0 \in \rho_{SF}^r(T)$, $P = P_{\ker T}$, $Q = P_{\ker T^*}$ and let S be the pseudo-inverse of T. Put $G = \{\mu: |\mu| < \|S\|^{-1}\}$. Then for $\mu \in G$,*

(i) $\ker(T-\mu) = R((I-\mu S)^{-1}P)$;

(ii) $\ker(T-\mu)^* = R((I-\bar{\mu}S^*)^{-1}Q)$.

PROOF. Since $0 \in \rho_{SF}^r(T)$ by (i) of Theorem 3.2 and Lemma 1.3 we have $Y \in \mathrm{Lat}(S)$. Since $|\mu|\,\|S_r\| \leqq |\mu|\,\|S\| < 1$, by (1) of Corollary 2.7 we see that $G \subset \rho_r(T_r)$. By (13) we can derive

$$\begin{aligned} (I-\mu S)^{-1}P &= (I+\mu S+\cdots+\mu^n S^n+\cdots)P \\ &= (I+\mu S_r+\cdots+\mu^n S_r^n+\cdots)P \\ &= (I-\mu S_r)^{-1}P \quad \text{for} \quad \mu \in G\,. \end{aligned} \tag{14}$$

By (1) of Theorem 2.9 and (14) we see that

$$\ker(T-\mu) = \ker(T_r-\mu) = R((I-\mu S_r)^{-1}P) = R((I-\mu S)^{-1}P) \quad \text{for all} \quad \mu \in G\,.$$

Thus (i) is proved.

By passing to T^* we can prove (ii) similarly.

THEOREM 3.6. *Let* $\lambda = 0 \in \rho^s_{SF}(T)$, $P = P_{\ker T}$, $Q = P_{\ker T^*}$ *and let* S *be the pseudo-inverse of* T. *Put* $G_\circ = \{\mu: 0 < |\mu| < \|S\|^{-1}\}$. *Then for each* $\mu \in G_\circ$,

(i) $\ker(T - \mu) = R((I - \mu S)^{-1}P) \cap H_r$;

(ii) $\ker(T - \mu)^* = R((I - \bar{\mu}S^*)^{-1}Q) \cap H_l$.

PROOF. (i) By Proposition 2.1 and (ii) of Theorem 2.9 we have $\ker(T - \mu) \subset R((I - \mu S)^{-1}P) \cap H_r$. Conversely, let $y \in R((I - \mu S)^{-1}P) \cap H_r$. Then $y = (I - \mu S)^{-1}x$ for some $x \in \ker T$. Thus $(I - \mu S)y = x$ and hence $Ty = \mu TSy = \mu y$(via $y \in H_r \subset R(T)$). Therefore $R((I - \mu S)^{-1}P) \cap H_r \subset \ker(T - \mu)$.

The proof of (ii) is similar.

Combining Theorems 3.5 and 3.6 we obtain:

THEOREM 3.7. *Let* $\lambda = 0 \in \rho_{SF}(T)$ *and let* S *be the pseudo-inverse of* T. *Then* $\lambda \in \rho^r_{SF}(T)$ *if and only if there exist holomorphic* H*-valued functions* $\{f_i(\mu)\}_{i \in I}$ *defined on the neighborhood* $\Delta = \{|\mu| < \|S\|^{-1}\}$ *such that* $\{f_i(\mu)\}_{i \in I}$ *forms a basis for* $\ker(T - \mu)$ *for* $\mu \in \Delta$.

PROOF. Let $\{f_i\}_{i \in I}$ be an orthonormal basis of $\ker T$ and put $f_i(\mu) = (I - \mu S)^{-1}f_i$.

4. Some applications.

PROPOSITION 4.1. *Let* $T \in L(H)$ *be densely defined,* $x \in H$ *and let the set of the zeros of the function*

$$\lambda \to P_{\ker(T-\lambda)}x$$

have an accumulation point $\xi \in \rho^r_{SF}(T)$. *Let* G_ξ *denote the component of* $\rho^r_{SF}(T)$ *which contains* ξ. *Then we have*

$$P_{\ker(T-\lambda)}x = 0 \quad on \quad G_\xi \, .$$

PROOF. Let Δ be a neighborhood of ξ, on which there exist holomorphic H-valued functions $\{f_i(\lambda)\}_{i \in I}$ such that $\{f_i(\lambda)\}_{i \in I}$ forms a basis of $\ker(T - \lambda)$ for $\lambda \in \Delta$. Choose a sequence $\{\xi_n\} \subset \Delta$ such that $\xi_n \to \xi$ and $P_{\ker(T-\xi_n)}x = 0$. Thus we have that $(f_i(\xi_n), x) = 0$ for $i \in I$. Since $(f_i(\lambda), x)$ is a C-valued holomorphic function on Δ, we have that $(f_i(\lambda), x) \equiv 0$ on Δ for $i \in I$. Therefore $x \perp \ker(T - \lambda)$ for $\lambda \in \Delta$, i.e.,

$$P_{\ker(T-\lambda)}x = 0 \quad \text{on} \quad \Delta \, .$$

Let μ be any point in G_ξ and $\Gamma \subset G_\xi$ be an arc connecting ξ and μ.

For each $\lambda \in \Gamma$ there exists a neighborhood Δ_λ, on which there exist holomorphic H-valued functions described as above. By using the finite covering theorem and the above argument we can derive that in a finite number of steps

$$P_{\ker(T-\lambda)}x = 0 \quad \text{on} \quad \Delta_\mu .$$

Thus the proposition is proved.

REMARK 4.2. Since $\rho_r(T) \subset \rho^r_{SF}(T)$, we see that Lemma 1.6 of [1] is a corollary of Proposition 4.1.

PROPOSITION 4.3. *Let T be densely defined, have closed range and* $\ker T \subset Y = TY$. *Put* $G = \{\mu : |\mu| < \|S\|^{-1}\}$. *Then we have*

$$\ker(T - \mu) \subset Y \quad for \quad \mu \in G . \tag{15}$$

PROOF. Let S be the pseudo-inverse of T. Similary to the proof of Lemma 1.3, we obtain that

$$S = \begin{pmatrix} S_Y & * \\ 0 & S_{Y^\perp} \end{pmatrix} \quad \text{on} \quad Y \oplus Y^\perp . \tag{16}$$

Then (15) follows from (16) and Proposition 2.1.

PROPOSITION 4.4. *Let $T \in L(H)$ be densely defined and $0 \in \rho^r_{SF}(T)$. Let G be the component of $\rho^r_{SF}(T)$ containing $\lambda = 0$. Put* $Y = \mathrm{clm}\{\ker T^k\}_{k=1}^\infty$, $Z = \mathrm{clm}\{\ker(T - \lambda)\}_{\lambda \in G}$. *Then $Y = Z$ and $0 \in \rho_r(T_Y)$.*

PROOF. The proof proceeds as follows:

(i) $Y \subset Z$. Clearly $\ker T \subset Z$. Suppose $\ker T^k \subset Z$ for $k \leqq n-1$ and $x \in \ker T^n$. Then $Tx \in \ker T^{n-1}$ and hence $Tx \in Z$. By Lemmas 1.5 and 1.4 we have $Z \in \mathrm{Lat}(T)$ and $0 \in \rho_r(T_Z)$. So there is $y \in Z$ such that $Tx = Ty$. Set $z = x - y$. Then $z \in \ker T$ and hence $x = y + z \in Z$.

(ii) $0 \in \rho_r(T_Y)$. Put $Y_\circ = \mathrm{lm}\{\ker(T - \lambda)^k\}_{k=1}^\infty$. It is trivial that $TY_\circ \subset Y_\circ$. Conversely, let $x \in \ker T^n$. Since $x \in Z$, we have that $x = Ty$ for some $y \in Z$ and hence $y \in \ker T^{n+1}$. Thus $Y_\circ = TY_\circ$ is proved. Then we follow the proof of Lemma 1.5 to obtain $TY = Y$.

(iii) $Z \subset Y$. Let S be the pseudo-inverse of T. By Proposition 4.3 we have

$$\ker(T - \mu) \subset Y \quad \text{for all} \quad |\mu| < \|S\|^{-1} .$$

Therefore if $x \perp Y$, then $x \perp \ker(T - \mu)$ for $|\mu| < \|S\|^{-1}$ and hence for $\mu \in G$ by Proposition 4.1. Thus $Y^\perp \subset Z^\perp$, i.e., $Z \subset Y$.

REMARK 4.5. [1, Lemma 1.7] is a corollary of Proposition 4.4.

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