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THE ORDER OF APPROXIMATION BY POSITIVE LINEAR OPERATORS

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1. Introduction. Let X be a compact Hausdorff space and let $B(X)$ denote the Banach lattice of all real-valued bounded functions on X with the supremum norm $\|\cdot\|$. $C(X)$ denotes the closed sublattice of $B(X)$ consisting of all real-valued continuous functions on X. Let A be a linear subspace of $C(X)$ which contains the unit function 1_X defined by $1_X(y) = 1$ for all $y \in X$. Let N denote the set of all non-negative integers. Let p be any fixed positive real number and let G be a subset of A separating the points of X. Suppose that A contains the set $\{|g - g(y)1_X|^p; g \in G, y \in X\}$. For a bounded linear operator T of A into $B(X)$ and a function $g \in G$, we define

$$\mu^{(p)}(T, g)(y) = T(|g - g(y)1_X|^p)(y) \quad (y \in X) .$$

Let $\{L_\alpha; \alpha \in D\}$ be a net of positive linear operators of A into $B(X)$ and put

$$\mu_\alpha^{(p)}(g) = \mu^{(p)}(L_\alpha, g) \quad (\alpha \in D, g \in G) ,$$

whose norm is called the p-th absolute moment for L_α with respect to g.

In [18] we proved the following convergence theorems, which may play an important role in the study of saturation property for $\{L_\alpha\}$:

THEOREM A. *Let U be a multiplication operator given by $U(f) = hf$ $(f \in A)$, where h is an arbitrary fixed non-negative function in $B(X)$. If $\lim_\alpha \|\mu_\alpha^{(p)}(g)\| = 0$ for all $g \in G$ and there exists a strictly positive function $u \in A$ such that $\lim_\alpha \|L_\alpha(u) - U(u)\| = 0$, then $\lim_\alpha \|L_\alpha(f) - U(f)\| = 0$ for every $f \in A$.*

THEOREM B. *Let T be a positive projection operator with $T \neq I$ (identity operator), $T(1_X) = 1_X$ and $L_\alpha T = T$ for every $\alpha \in D$. If $\mu^{(p)}(T, g) \in A$ and $\lim_\alpha \|L_\alpha(\mu^{(p)}(T, g))\| = 0$ for all $g \in G$, then $\lim_\alpha \|L_\alpha(f) - T(f)\| = 0$ for every $f \in A$.*

The purpose of this paper is to give a quantitative version of the above theorems in which we estimate the rate of convergence of $\{L_\alpha(f)\}$ by using a modulus of continuity of f. Furthermore, a particular attention is paid to the degree of approximation by iterations of positive linear operators of A into itself (cf. [17]).

Finally the results are applied to various summation processes and some ergodic theorems for positive linear operators from a quantitative point of view. Concrete examples of approximating operators can be provided by the multidimensional Bernstein operators and the semigroup of Markov operators induced by them (cf. [12]). For the basic theory of semigroups of operators on Banach spaces, one may consult the books of Butzer and Berens [2] and Hille and Phillips [4]. Actually, the results of the author [11], [13] can be improved by means of the higher order moments.

2. Degree of convergence. Here we assume that A contains the set $\{|g - g(y)1_X|^p; g \in G, y \in X, p \geqq 1\}$. Let $f \in B(X)$. If $\{g_1, g_2, \cdots, g_r\}$ is a finite subset of G and $\delta \geqq 0$, then we define

$$\omega(f; g_1, \cdots, g_r, \delta) = \sup\{|f(x) - f(y)|; x, y \in X, d(x, y) \leqq \delta\} ,$$

where

$$d(x, y) = \max\{|g_i(x) - g_i(y)|; i = 1, 2, \cdots, r\} .$$

This quantity is called the modulus of continuity of f with respect to $g_1, g_2, \cdots, g_r$ ([17]).

In order to achieve our purpose it is always supposed that the following condition is satisfied:

(1) There exist constants $C \geqq 1$ and $K > 0$ such that

$$\omega(f; g_1, \cdots, g_r, \xi\delta) \leqq (C + K\xi)\omega(f; g_1, \cdots, g_r, \delta)$$

for all $f \in B(X)$, ξ, $\delta \geqq 0$ and for all finite subsets $\{g_1, g_2, \cdots, g_r\}$ of G.

Now we have the following key estimate for positive linear functionals on A.

LEMMA. *Let L be a positive linear functional on A and $y \in X$. Let $\{g_1, g_2, \cdots, g_r\}$ be a finite subset of G, $p \geqq 1$ and $\delta > 0$. Then for all $g \in A$, we have*

$$|L(g) - g(y)L(1_X)| \leqq (CL(1_X) + a(y))\omega(f; g_1, \cdots, g_r, \delta) ,$$

where

$$a(y) = \min\{\delta^{-p}KL(\Phi(\cdot, y)), \delta^{-1}K(L(\Phi(\cdot, y)))^{1/p}(L(1_X))^{1-1/p}\}$$

with

$$\Phi(x, y) = \sum_{i=1}^{r} |g_i(x) - g_i(y)|^p \quad (x, y \in X) .$$

PROOF. Let $x \in X$. If $d(x, y) > \delta$, then it follows from (1) that

$$(2) \qquad \begin{aligned} |g(x) - g(y)| &\leqq (C + K(d(x, y)/\delta))\omega(g; g_1, \cdots, g_r, \delta) \\ &\leqq (C + K(d(x, y)/\delta)^p)\omega(g; g_1, \cdots, g_r, \delta) \\ &\leqq (C + \delta^{-p}K\Phi(x, y))\omega(g; g_1, \cdots, g_r, \delta) . \end{aligned}$$

If $d(x, y) \leqq \delta$, then (2) also holds since $C \geqq 1$. Consequently, we have

$$|g - g(y)1_X| \leqq \omega(g; g_1, \cdots, g_r, \delta)(C1_X + \delta^{-p}K\Phi(\cdot, y)) ,$$

and applying L to both sides of this inequality we get

$$(3) \qquad |L(g) - g(y)L(1_X)| \leqq \omega(g; g_1, \cdots, g_r, \delta)(CL(1_X) + \delta^{-p}KL(\Phi(\cdot, y))) .$$

On the other hand, there holds

$$(4) \qquad |g - g(y)1_X| \leqq \omega(g; g_1, \cdots, g_r, \delta)(C1_X + \delta^{-1}K(\Phi(\cdot, y))^{1/p}) .$$

Now we extend L to a positive linear functional on the whole space $C(X)$ and denote this functional by the same L. Then applying L to both sides of (4) and using Hölder's inequality, we obtain

$$|L(g) - g(y)L(1_X)| \leqq \omega(g; g_1, \cdots, g_r, \delta)(CL(1_X) + \delta^{-1}K(L(\Phi(\cdot, y)))^{1/p}(L(1_X))^{1-1/p}) ,$$

which together with (3) implies the claim of the lemma for $p > 1$. If $p = 1$, then (3) is obviously identical with the desired estimate. q.e.d.

We are now in a position to recast Theorem A in a quantitative form with the rate of convergence.

THEOREM 1. *Let U be as in Theorem A and let u be a strictly positive function in A. Then for all $f \in A$ and for all $\alpha \in D$,*

$$\begin{aligned} \|L_\alpha(f) - U(f)\| &\leqq \|f/u\| \, \|L_\alpha(u) - U(u)\| \\ &+ \inf\Big\{K_\alpha^{(p,\varepsilon)}\Big(\|f/u\|\omega\Big(u; g_1, \cdots, g_r, \varepsilon\Big\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\Big\|^{1/p}\Big) \\ &+ \omega\Big(f; g_1, \cdots, g_r, \varepsilon\Big\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\Big\|^{1/p}\Big)\Big) ; \\ &p \geqq 1, \varepsilon > 0, g_1, \cdots, g_r \in G, \Big\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\Big\| > 0, r = 1, 2, \cdots\Big\} , \end{aligned}$$

where

$$K_\alpha^{(p,\varepsilon)} = \|CL_\alpha(1_X) + \min\{\varepsilon^{-p}K1_X, \varepsilon^{-1}K(L_\alpha(1_X))^{1-1/p}\}\| .$$

PROOF. Let y be an arbitrary point of X. Then for all $f \in A$ and all $\alpha \in D$, we have

$$\begin{aligned} |L_\alpha(f)(y) - U(f)(y)| &\leqq |f(y)/u(y)| \, |L_\alpha(u)(y) - U(u)(y)| \\ &+ \{|f(y)/u(y)| \, |L_\alpha(u)(y) - u(y)L_\alpha(1_X)(y)| + |L_\alpha(f)(y) - f(y)L_\alpha(1_X)(y)|\} . \end{aligned}$$

Now making use of Lemma with $L(\cdot) = L_\alpha(\cdot)(y)$, the second term on the right hand side is majorized by

$$(CL_\alpha(1_X)(y) + a(y))(|f(y)/u(y)|\omega(u; g_1, \cdots, g_r, \delta) + \omega(f; g_1, \cdots, g_r, \delta)),$$

and

$$a(y) \leqq \min\left\{\delta^{-p}K\left\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\right\|, \delta^{-1}K\left\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\right\|^{1/p}(L_\alpha(1_X)(y))^{1-1/p}\right\}.$$

Therefore, putting $\delta = \varepsilon\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\|^{1/p} > 0$ and taking the supremum over all $y \in X$ we arrive at

$$\begin{aligned}\|L_\alpha(f) - U(f)\| &\leqq \|f/u\|\,\|L_\alpha(u) - U(u)\| \\ &+ K_\alpha^{(p,\varepsilon)}\left(\|f/u\|\omega\left(u; g_1, \cdots, g_r, \varepsilon\left\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\right\|^{1/p}\right)\right. \\ &\left.+\omega\left(f; g_1, \cdots, g_r, \varepsilon\left\|\sum_{i=1}^r \mu_\alpha^{(p)}(g_i)\right\|^{1/p}\right)\right),\end{aligned}$$

which implies the desired result. q.e.d.

REMARK 1. If A contains the set

$$F_q(G) = \{g^i; g \in G, i = 0, 1, 2, \cdots, q\}$$

for an even positive integer q, then we have

$$\left\|\sum_{i=1}^r \mu_\alpha^{(q)}(g_i)\right\| \leqq \sum_{i=1}^r \sum_{j=0}^q \binom{q}{j}\|g_i\|^{q-j}\|L_\alpha(g_i^j) - U(g_i^j)\|,$$

and so Theorem 1 yields the estimate for $\|L_\alpha(f) - U(f)\|$ in terms of the corresponding quantities for the test system $F_q(G)$.

Concerning the degree of convergence in Theorem B we have the following:

THEOREM 2. *Let T be as in Theorem* B. *Then for all $f \in A$ and all $\alpha \in D$,*

$$\begin{aligned}\|L_\alpha(f) - T(f)\| \leqq \inf\Bigg\{&C^{(p,\varepsilon)}\omega\left(f; g_1, \cdots, g_r, \varepsilon\left\|\sum_{i=1}^r L_\alpha(\mu^{(p)}(T, g_i))\right\|^{1/p}\right); \\ &p \geqq 1, \varepsilon > 0, g_1, \cdots, g_r \in G, \left\|\sum_{i=1}^r L_\alpha(\mu^{(p)}(T, g_i))\right\| > 0, r = 1, 2, \cdots\Bigg\},\end{aligned}$$

where

$$C^{(p,\varepsilon)} = C + \min\{K/\varepsilon^p, K/\varepsilon\}. \tag{5}$$

PROOF. Applying Lemma to $L(\cdot) = T(\cdot)(y)$ with any fixed point y of X, we get

$$|T(f) - f| \leqq \omega(f; g_1, \cdots, g_r, \delta)(C1_X + a), \tag{6}$$

where

$$a = \min\left\{\delta^{-p}K\sum_{i=1}^{r}\mu^{(p)}(T, g_i),\ \delta^{-1}K\Big(\sum_{i=1}^{r}\mu^{(p)}(T, g_i)\Big)^{1/p}\right\}.$$

Now let ψ be a positive linear functional on A with $\psi(1_X) = 1$ and denote an extension of ψ to the whole space $C(X)$ by the same ψ. Applying ψ to both sides of (6) and using Hölder's inequality, we obtain

$$|\psi(T(f)) - \psi(f)| \leqq (C + \psi(a))\omega(f; g_1, \cdots, g_r, \delta)$$

and

$$\psi(a) \leqq \min\left\{\delta^{-p}K\sum_{i=1}^{r}\psi(\mu^{(p)}(T, g_i)),\ \delta^{-1}K\Big(\sum_{i=1}^{r}\psi(\mu^{(p)}(T, g_i))\Big)^{1/p}\right\}.$$

Take $\psi(\cdot) = L_\alpha(\cdot)(y)$, where y is an arbitrary fixed point of X. Then, since $L_\alpha T = T$, we have

$$|T(f)(y) - L_\alpha(f)(y)| \leqq (C + M)\omega(f; g_1, \cdots, g_r, \delta),$$

where

$$M = \min\left\{\delta^{-p}K\left\|\sum_{i=1}^{r}L_\alpha(\mu^{(p)}(T, g_i))\right\|,\ \delta^{-1}K\left\|\sum_{i=1}^{r}L_\alpha(\mu^{(p)}(T, g_i))\right\|^{1/p}\right\}.$$

Thus putting $\delta = \varepsilon\|\sum_{i=1}^{r}L_\alpha(\mu^{(p)}(T, g_i))\|^{1/p} > 0$ and taking the supremum over all $y \in X$, we obtain

$$\|L_\alpha(f) - T(f)\| \leqq C^{(p,\varepsilon)}\omega\Big(f; g_1, \cdots, g_r, \varepsilon\left\|\sum_{i=1}^{r}L_\alpha(\mu^{(p)}(T, g_i))\right\|^{1/p}\Big),$$

which establishes the desired result. q.e.d.

REMARK 2. If A contains $F_q(G)$ for an even positive integer q and

(7) $$T(g^i) = g^i \quad (g \in G,\ i = 0, 1, 2, \cdots, q-1),$$

then we have

$$\left\|\sum_{i=1}^{r}L_\alpha(\mu^{(q)}(T, g_i))\right\| \leqq \sum_{i=1}^{r}\|L_\alpha(g_i^q) - T(g_i^q)\|,$$

and so Theorem 2 gives an estimate for $\|L_\alpha(f) - T(f)\|$ in terms of the corresponding quantities for the test system $G^q = \{g^q; g \in G\}$.

In the rest of this section A is assumed to contain $F_q(G)$ for an even positive integer q. Let T be a positive projection operator on A with $T \neq I$, which satisfies (7) and $L_\alpha T = T$ for every $\alpha \in D$. Suppose that each L_α maps A into itself and $L_\alpha(g^q) = g^q + \xi_\alpha(T(g^q) - g^q)$ for all $\alpha \in D$, $g \in G$, where $\{\xi_\alpha\}$ is a net of real numbers with $0 < \xi_\alpha < 1$.

For $f \in B(X)$ and $\delta > 0$, we define

$$\Psi(f, \delta) = \inf\left\{C^{(q,\varepsilon)}\omega\left(f; g_1 \cdots, g_r, \delta\varepsilon\left\|\sum_{i=1}^{r}(T(g_i^q) - g_i^q)\right\|^{1/q}\right);\right.$$
$$\left.\varepsilon > 0, g_1, \cdots, g_r \in G, \left\|\sum_{i=1}^{r}(T(g_i^q) - g_i^q)\right\| > 0, r = 1, 2, \cdots\right\},$$

where $C^{(q,\varepsilon)}$ is given by (5) with $p = q$.

As a consequence of Theorems 1 and 2, we have the following corollary which is more convenient for later applications.

COROLLARY 1. *Let $\{k_\alpha; \alpha \in D\}$ be a net of positive integers and let $L_\alpha^{k_\alpha}$ denote the k_α-iteration of L_α for each $\alpha \in D$. Then for all $f \in A$ and all $\alpha \in D$,*

$$\|L_\alpha^{k_\alpha}(f) - f\| \leqq \Psi(f, (1 - (1 - \xi_\alpha)^{k_\alpha})^{1/q}) \leqq \Psi(f, (k_\alpha \xi_\alpha)^{1/q})$$

and

$$\|L_\alpha^{k_\alpha}(f) - T(f)\| \leqq \Psi(f, (1 - \xi_\alpha)^{k_\alpha/q}) .$$

In [18; Theorem 3], we showed that if $\lim_\alpha k_\alpha \xi_\alpha = 0$, then $\{L_\alpha^{k_\alpha}; \alpha \in D\}$ is saturated in A with order $1 - (1 - \xi_\alpha)^{k_\alpha}$, or equivalently, with order $k_\alpha \xi_\alpha$, and its trivial class coincides with the range of T. Thus the above corollary may give the optimal estimate for the order of approximation by $L_\alpha^{k_\alpha}$.

3. Applications. Let A be a closed linear subspace of $C(X)$. A mapping L of A into itself is called a Markov operator on A if it is a positive linear operator with $L(1_X) = 1_X$. Let $\{a_{\alpha,m}; \alpha \in D, m \in N\}$ be a family of non-negative real numbers with $\sum_{m=0}^{\infty} a_{\alpha,m} = 1$ for each $\alpha \in D$. For examples of such families, see, for instance, [14] and [16]. Let $\{i_m; m \in N\}$ be a sequence of non-negative integers and $\{j_m; m \in N\}$ a sequence of positive integers. Let $\{S_\gamma; \gamma \in \Gamma\}$ be a net of Markov operators on A and $\{T_m; m \geqq 1\}$ a sequence of Markov operators on A. For any $f \in A$, we define

$$S_{\alpha,\gamma}(f) = \sum_{m=0}^{\infty} a_{\alpha,m} S_\gamma^{i_m}(f) \quad (\alpha \in D, \gamma \in \Gamma) \tag{8}$$

and

$$T_{\alpha,k}(f) = \sum_{m=0}^{\infty} a_{\alpha,m} T_{j_m k}^{i_m}(f) \quad (\alpha \in D, k \geqq 1) , \tag{9}$$

which converge in A. Let $\{W(t); t \geqq 0\}$ be a family of Markov operators on A such that for each $f \in A$, the map $t \to W(t)(f)$ is strongly continuous on $[0, \infty)$. For any $f \in A$, we define

$$C_{\xi,\lambda}(f) = (1/\xi)\int_0^\xi W(t + \lambda)(f)dt \quad (\xi > 0, \lambda \geqq 0) \tag{10}$$

and

$$R_{\xi,\lambda}(f) = \xi\int_0^\infty \exp(-\xi t)\, W(t+\lambda)(f)dt \quad (\xi, \lambda \geqq 0)\,, \tag{11}$$

which exist in A.

All the operators given above are Markov operators on A and our general results obtained in the preceding section are applicable to them. As illustrations of these general results we restrict ourselves to the following setting:

Let X be a compact convex subset of a real locally convex Hausdorff vector space E with its dual space E^*, and $G = \{v|_X; v \in E^*\}$, where $v|_X$ denotes the restriction of v to X. Note that Condition (1) holds for $C = K = 1$ (see, [11; Lemma 1]). Let T be a positive projection operator of $C(X)$ onto a closed linear subspace of $C(X)$ containing 1_X and G (which is the case where $A = C(X)$ and $q = 2$).

For applications to Corollary 1 it is convenient to make the following definition: Let $\{P_\lambda; \lambda \in \Lambda\}$ be a family of Markov operators on $C(X)$ and $\{x_\lambda; \lambda \in \Lambda\}$ a family of non-negative real numbers. We say that $\{P_\lambda\}$ is of type $[T; x_\lambda]$ if $P_\lambda T = T$ and $P_\lambda(g^2) = g^2 + x_\lambda(T(g^2) - g^2)$ for all $\lambda \in \Lambda$ and all $g \in G$.

Now we first consider the case where $E = \boldsymbol{R}^r$, the r-dimensional Euclidean space equipped with the metric

$$\rho(x, y) = \max\{|x_i - y_i|; i = 1, 2, \cdots, r\}$$

for $x = (x_1, x_2, \cdots, x_r)$ and $y = (y_1, y_2, \cdots, y_r)$. Let e_i denote the i-th coordinate function on X. Then $\omega(f; e_1, \cdots, e_r, \delta)$ reduces to the usual modulus of continuity of f, given by

$$\omega(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in X, \rho(x, y) \leqq \delta\}\,.$$

In view of Remarks 1 and 2, we have a quantitative version of the Korovkin type convergence theorem due to Karlin and Ziegler [5; Theorem 1 and Remark 2] for multidimensional case.

Take $X = \boldsymbol{I}_r$, the unit r-cube, i.e.,

$$\boldsymbol{I}_r = \{(x_1, \cdots, x_r) \in \boldsymbol{R}^r; 0 \leqq x_i \leqq 1, i = 1, 2, \cdots, r\}$$

and let F be the closed linear subspace of $C(X)$ spanned by the set

$$\{e_1^{k_1} e_2^{k_2} \cdots e_r^{k_r}; k_i \in \{0, 1\}, i = 1, 2, \cdots, r\}\,.$$

Let $\{B_n; n \geqq 1\}$ be the sequence of Bernstein operators on $C(X)$ given by

$$B_n(f)(x) = \sum_{k_1=0}^{n} \cdots \sum_{k_r=0}^{n} f(k_1/n, \cdots, k_r/n) \prod_{i=1}^{r} \binom{n}{k_i} x_i^{k_i}(1 - x_i)^{n-k_i} \tag{12}$$

for $f \in C(X)$ and $x = (x_1, x_2, \cdots, x_r) \in X$ (see, e.g., [8]). It can be verified that B_1 is a positive projection operator of $C(X)$ onto F and $\{B_n\}$ is of type $[B_1; 1/n]$. Consequently, if $\Gamma = N\setminus\{0\}$, $S_\gamma = B_\gamma$ and $T_m = B_m$, then $\{S_{\alpha,\gamma}\}$ and $\{T_{\alpha,k}\}$ are of types $[B_1; 1 - x_{\alpha,\gamma}]$ and $[B_1; 1 - y_{\alpha,k}]$, respectively, where

$$x_{\alpha,\gamma} = \sum_{m=0}^{\infty} a_{\alpha,m}(1 - 1/\gamma)^{i_m} \quad \text{and} \quad y_{\alpha,k} = \sum_{m=0}^{\infty} a_{\alpha,m}(1 - 1/(j_m k))^{i_m} ,$$

and so Corollary 1 can be applied to these operators. In particular, concerning the order of approximation by iterations of the Bernstein operators we have the following estimates: For all $f \in C(X)$ and all $n \geqq 1$,

$$\begin{aligned} \|B_{j_n}^{i_n}(f) - f\| &\leqq (1 + \min\{r/4, r^{1/2}/2\})\omega(f, (1 - (1 - 1/j_n)^{i_n})^{1/2}) \\ &\leqq (1 + \min\{r/4, r^{1/2}/2\})\omega(f, (i_n/j_n)^{1/2}) \end{aligned} \tag{13}$$

and

$$\|B_{j_n}^{i_n}(f) - B_1(f)\| \leqq (1 + \min\{r/4, r^{1/2}/2\})\omega(f, (1 - 1/j_n)^{i_n/2}) . \tag{14}$$

Therefore, on account of (13) and (14), we have $\lim_{n\to\infty} i_n/j_n = 0$ if and only if $\lim_{n\to\infty} \|B_{j_n}^{i_n}(f) - f\| = 0$ for every $f \in C(I_r)$, and $\lim_{n\to\infty} i_n/j_n = +\infty$ if and only if $\lim_{n\to\infty} \|B_{j_n}^{i_n}(f) - B_1(f)\| = 0$ for every $f \in C(I_r)$. When $r = 1$ and $j_n = n$ for all $n \geqq 1$, this result reduces to that of Kelisky and Rivlin [6] (cf. [5], [9], [10]). For extensive approximation properties by iterations of positive linear operators, we refer to [18]. If $\{i_n\} = \{1\}$, then we can sharpen (13) further as

$$\|B_{j_n}(f) - f\| \leqq (1 + \min\{3r/16, (3r/16)^{1/4}\})\omega(f, (1/j_n)^{1/2}) ,$$

by taking the fourth absolute moment and making use of Theorem 1 (cf. [14], [15], [16], where quantitative Korovkin type estimates can be treated in the setting of an arbitrary compact metric space).

Statements analogous to the above-mentioned results may be derived for the case where B_n are the Bernstein operators on $C(\Delta_r)$ with the standard r-simplex

$$\Delta_r = \{(x_1, \cdots, x_r) \in \boldsymbol{R}^r; x_i \geqq 0, i = 1, 2, \cdots, r, x_1 + \cdots + x_r \leqq 1\} ,$$

given by

$$\begin{aligned} B_n(f)(x) = \sum_{k_i \geqq 0, k_1 + \cdots + k_r \leqq n} & f(k_1/n, \cdots, k_r/n) \\ & \times n!/((k_1!k_2! \cdots k_r!)(n - k_1 - k_2 - \cdots - k_r)!) \\ & \times x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}(1 - x_1 - \cdots - x_r)^{n-k_1-\cdots-k_r} \end{aligned} \tag{15}$$

for $f \in C(\Delta_r)$ and $x = (x_1, x_2, \cdots, x_r) \in \Delta_r$ (see, e.g., [8]). These can be obtained in the following very general setting.

Again let X be a compact convex subset of a real locally convex Hausdorff vector space E and $G=\{v|_X;\, v\in E^*\}$. Let $A(X)$ denote the space of all real-valued continuous affine functions on X. If L is a Markov operator on $C(X)$, then for a point $x\in X$, a Radon probability measure ν_x on X is called an $L(A(X))$-representing measure for x if

$$L(f)(x)=\int_X f d\nu_x$$

for every $f\in A(X)$. Let $M=\{M_n;\, n\geqq 1\}$ be a sequence of Markov operators on $C(X)$, $\nu^{(M)}=\{\nu_{x,n};\, x\in X,\, n\geqq 1\}$ a family of Radon probability measures on X such that $\nu_{x,n}$ is an $M_n(A(X))$-representing measure for x, $P=(p_{nj})_{n,j\geqq 1}$ an infinite lower triangular stochastic matrix, $Y=\{y_x;\, x\in X\}$ a family of points of X, and $\rho=\{\rho_n;\, n\geqq 1\}$ a sequence of functions mapping X into [0, 1]. Then we define

$$\nu^{(M,Y)}_{x,n,\rho}=\rho_n(x)\nu_{x,n}+(1-\rho_n(x))\varepsilon_{y_x}\circ M_n\ ,$$

where ε_t denotes the Dirac measure at t, and also define the mapping

$$\pi_{n,P}\colon X^n\to X \quad \text{by} \quad (x_1, x_2, \cdots, x_n)\to\sum_{j\geqq 1} p_{nj}x_j\ .$$

For a function $f\in C(X)$, the n-th Bernstein-Lototsky-Schnabl function of f on X with respect to $\nu^{(M)}$, P, Y and ρ is defined by

$$B_n(f)(x)=B^{(\nu^{(M)},Y)}_{n,P,\rho}(f)(x)=\int_{X^n} f\circ\pi_{n,P}\, d\bigotimes_{1\leqq j\leqq n}\nu^{(M,Y)}_{x,j,\rho}$$

([13], cf. [3], [19]).

Now take

$$i_m=1\quad (m=0, 1, 2, \cdots)\ ,\qquad T_m=B_m\quad (m=1, 2, \cdots)$$

and let $\{T_{\alpha,k};\, \alpha\in D,\, k\geqq 1\}$ be the family of operators given by (9). Then we have the following:

THEOREM 3. *Suppose that $M_n(g)=g$ for all $n\geqq 1$ and all $g\in A(X)$. Then the following statements hold:*

(i) *If $y_x=x$ for every $x\in X$, then for all $f\in C(X)$, $\alpha\in D$ and all $k\geqq 1$,*

$$\|T_{\alpha,k}(f)-f\|\leqq\zeta_{\alpha,k}(f)\ , \tag{16}$$

where

$$\zeta_{\alpha,k}(f)=\inf\{(1+\min\{\varepsilon^{-1}, \varepsilon^{-2}\})\omega(f;\, g_1, \cdots, g_r, \varepsilon\|h_{\alpha,k}(g_1, \cdots, g_r)\|^{1/2})\ ;\\ \varepsilon>0,\, g_1, \cdots, g_r\in G,\, \|h_{\alpha,k}(g_1, \cdots, g_r)\|>0,\, r=1, 2, \cdots\}\ ,$$

with

$$h_{\alpha,k}(g_1, \cdots, g_r)(x) = \sum_{m=0}^{\infty} a_{\alpha,m} \sum_{i\geqq 1} p_{j_m k i}^2 \rho_i(x) \sum_{n=1}^{r} (\nu_{x,i}(g_n^2) - g_n^2(x)) \quad (x \in X) .$$

(ii) *If* $\rho_n = 1_X$ *for all* $n \geqq 1$, *then* (16) *holds for*

$$h_{\alpha,k}(g_1, \cdots, g_r)(x) = \sum_{m=0}^{\infty} a_{\alpha,m} \sum_{i\geqq 1} p_{j_m k i}^2 \sum_{n=1}^{r} (\nu_{x,i}(g_n^2) - g_n^2(x)) \quad (x \in X) .$$

PROOF. Assume that $y_x = x$ for every $x \in X$. Then, by [13; Lemma 4], it can be seen that for all $\alpha \in D$, $k \geqq 1$ and all $g \in G$,

$$T_{\alpha,k}(1_X) = 1_X , \qquad T_{\alpha,k}(g) = g$$

and

$$\begin{aligned}\mu^{(2)}(T_{\alpha,k}, g)(x) &= T_{\alpha,k}(g^2)(x) - g^2(x) \\ &= \sum_{m=0}^{\infty} a_{\alpha,m} \sum_{i\geqq 1} p_{j_m k i}^2 \rho_i(x)(\nu_{x,i}(g^2) - g^2(x)) \quad (x \in X) .\end{aligned}$$

Therefore, the desired estimate (16) follows from Theorem 1 with $h = u = 1_X$. The proof of Part (ii) is similar. q.e.d.

COROLLARY 2. *Let M be as in Theorem 3. Then the following statements hold:*

(i) *If* $y_x = x$ *for every* $x \in X$, *then for all* $f \in C(X)$ *and all* $n \geqq 1$,

$$\|B_n(f) - f\| \leqq \omega_n(f) , \tag{17}$$

where

$$\omega_n(f) = \inf\{(1 + \min\{\varepsilon^{-1}, \varepsilon^{-2}\})\omega(f; g_1, \cdots, g_r, \varepsilon\delta_n(g_1, \cdots, g_r)) ; \\ \varepsilon > 0, g_1, \cdots, g_r \in G, \delta_n(g_1, \cdots, g_r) > 0, r = 1, 2, \cdots\} ,$$

with

$$\delta_n(g_1, \cdots, g_r) = \left(\sup\left\{\sum_{j\geqq 1} p_{nj}^2 \rho_j(x) \sum_{i=1}^{r} (\nu_{x,j}(g_i^2) - g_i^2(x)); x \in X\right\}\right)^{1/2} .$$

(ii) *If* $\rho_n = 1_X$ *for all* $n \geqq 1$, *then* (17) *holds for*

$$\delta_n(g_1, \cdots, g_r) = \left(\sup\left\{\sum_{j\geqq 1} p_{nj}^2 \sum_{i=1}^{r} (\nu_{x,j}(g_i^2) - g_i^2(x)); x \in X\right\}\right)^{1/2} .$$

This corollary gives a quantitative version of the result ([cf. 19; Satz 1]) of Grossman [3] and it estimates the degree of strong convergence of $\{B_n\}$ to I on $C(X)$.

From now on we suppose that

$$M_n = I \quad (n \geqq 1) , \qquad y_x = x \quad (x \in X)$$
$$\rho_n = 1_X \quad (n \geqq 1) , \quad \text{and} \quad \nu_{x,n} = \nu_x \quad (x \in X, n \geqq 1) ,$$

where ν_x is a representing measure for x (i.e., an $I(A(X))$-representing measure for x) such that the map

$$x \to \nu_x(f) = \int_X f d\nu_x$$

belongs to $A(X)$ for every $f \in C(X)$. Thus each B_n maps $C(X)$ into itself and B_1 is a positive projection operator of $C(X)$ onto $A(X)$ (cf. [3; Proposition], [13; Remark 7]).

For any $f \in B(X)$ and $\delta > 0$, we define

$$\Omega(f, \delta) = \inf\{(1 + \min\{\varepsilon^{-1}, \varepsilon^{-2}\})\omega(f; g_1, \cdots, g_r, \delta\varepsilon\|\tau(g_1, \cdots, g_r)\|^{1/2}) ;\\ \varepsilon > 0, g_1, \cdots, g_r \in G, \|\tau(g_1, \cdots, g_r)\| > 0, r = 1, 2, \cdots\} ,$$

where

$$\tau(g_1, \cdots, g_r)(x) = \sum_{i=1}^{r} (\nu_x(g_i^2) - g_i^2(x)) \quad (x \in X) .$$

Now take $T_m = B_m$ $(m = 1, 2, \cdots)$, and let $\{T_{\alpha,k}; \alpha \in D, k \geqq 1\}$ be the family of operators given by (9). Then we have the following:

Theorem 4. *Let $\{m_\alpha; \alpha \in D\}$ be a net of positive integers. Then for all $f \in C(X)$, $\alpha \in D$ and all $k \geqq 1$,*

$$\|T_{\alpha,k}^{m_\alpha}(f) - f\| \leqq \Omega(f, (1 - x_{\alpha,k}^{m_\alpha})^{1/2})$$

and

$$\|T_{\alpha,k}^{m_\alpha}(f) - B_1(f)\| \leqq \Omega(f, x_{\alpha,k}^{m_\alpha/2}) ,$$

where

$$x_{\alpha,k} = \sum_{m=0}^{\infty} a_{\alpha,m}(1 - \sum_{i\geqq 1} p_{j_m k i}^2)^{i_m} .$$

Proof. By induction on the degree m of iteration of B_n, it can be verified that $\{B_n^m\}$ is of type $[B_1; 1 - (1 - \sum_{j\geqq 1} p_{nj}^2)^m]$ $(n, m = 1, 2, \cdots)$. Therefore, $\{T_{\alpha,k}\}$ is of type $[B_1; 1 - x_{\alpha,k}]$ and so the desired result follows from Corollary 1. q.e.d.

Corollary 3. *For all $f \in C(X)$ and all $n \in N$,*

$$\|B_{j_n}^{i_n}(f) - f\| \leqq \Omega(f, (1 - (1 - \sum_{m\geqq 1} p_{j_n m}^2)^{i_n})^{1/2}) \leqq \Omega(f, (i_n \sum_{m\geqq 1} p_{j_n m}^2)^{1/2}) ,$$

and

$$\|B_{j_n}^{i_n}(f) - B_1(f)\| \leqq \Omega(f, (1 - \sum_{m\geqq 1} p_{j_n m}^2)^{i_n/2}) .$$

From this result we conclude that if $\lim_{n\to\infty} \sum_{m\geqq 1} p_{j_n m}^2 = 0$, then

$$\lim_{n\to\infty} i_n \sum_{m\geq 1} p_{j_n m}^2 = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \|B_{j_n}^{i_n}(f) - f\| = 0$$

for all $f \in C(X)$, and

$$\lim_{n\to\infty} i_n \sum_{m\geq 1} p_{j_n m}^2 = +\infty \quad \text{if and only if} \quad \lim_{n\to\infty} \|B_{j_n}^{i_n}(f) - B_1(f)\| = 0$$

for all $f \in C(X)$. Also, by [18; Theorem 3(b)] we see that if $\lim_{n\to\infty} \sum_{m\geq 1} p_{j_n m}^2 = 0$, then $\{B_{j_n}^{i_n}\}$ is saturated in $C(X)$ with order $1 - (1 - \sum_{m\geq 1} p_{j_n m}^2)^{i_n}$, or equivalently, with order $i_n \sum_{m\geq 1} p_{j_n m}^2$, and its trivial class coincides with $A(X)$ (cf. [18; Theorem 4]). Therefore the first part of Corollary 3 seems to be useful for the characterization of the saturation class of $\{B_{j_n}^{i_n}\}$ by structural properties on the functions f.

If L is a Markov operator on $C(X)$, then for any $f \in C(X)$, we define

$$\sigma_{n,i}(L; f) = (1/(n+1)) \sum_{m=0}^{n} L^{m+i}(f) \quad (n, i \in \boldsymbol{N})$$

and

$$A_{t,i}(L; f) = (1-t) \sum_{m=0}^{\infty} t^m L^{m+i}(f) \quad (0 < t < 1, i \in \boldsymbol{N}),$$

which is a particular case of (8). Note that if $\{S_\gamma\}$ is of type $[B_1; x_\gamma]$, then $\{S_{\alpha,\gamma}\}$ is of type $[B_1; 1 - \sum_{m=0}^{\infty} a_{\alpha,m}(1 - x_\gamma)^{im}]$. Thus, in view of this fact, making use of Corollary 1 we have the following quantitative ergodic type theorem for iterations of the discrete Cesàro and Abel means of the Bernstein-Lototsky-Schnabl operators.

THEOREM 5. *Let $m, j \geq 1$ be fixed, and set $\beta = \beta(m, j) = (1 - \sum_{i\geq 1} p_{mi}^2)^j$. Then the following statements hold:*

(i) *Let $\{k_n; n \in \boldsymbol{N}\}$ be a sequence of positive integers. Then for all $f \in C(X)$, $n \in \boldsymbol{N}$ and all $i \in \boldsymbol{N}$,*

$$\|\sigma_{n,i}^{k_n}(B_m^j; f) - B_1(f)\| \leq \Omega(f, x_{n,i}),$$

where

$$x_{n,i} = (\beta^i(1 - \beta^{n+1})/((1-\beta)(n+1)))^{k_n/2}. \tag{18}$$

(ii) *Let $\{n_t; 0 < t < 1\}$ be a family of positive integers. Then for all $f \in C(X)$, $t \in (0, 1)$ and all $i \in \boldsymbol{N}$,*

$$\|A_{t,i}^{n_t}(B_m^j; f) - B_1(f)\| \leq \Omega(f, y_{t,i}),$$

where

$$y_{t,i} = (\beta^i(1-t)/(1-t\beta))^{n_t/2}. \tag{19}$$

In particular, for the sequence $\{B_n; n \geq 1\}$ of the Bernstein operators on $C(\Delta_r)$ given by (15) we have:

COROLLARY 4. *Let m, $j \geqq 1$ be fixed. Let $x_{n,i}$ and $y_{t,i}$ be given by* (18) *and* (19) *with $\beta = \beta(m, j) = (1 - 1/m)^j$, respectively. Then for all $f \in C(\Delta_r)$, n, $i \in N$ and all $t \in (0, 1)$,*

$$\|\sigma_{n,i}^{k_n}(B_m^j; f) - B_1(f)\| \leqq \left(1 + \min\left\{\left\|\sum_{i=1}^r (e_i - e_i^2)\right\|, \left\|\sum_{i=1}^r (e_i - e_i^2)\right\|^{1/2}\right\}\right)\omega(f, x_{n,i})$$
$$\leqq (1 + \min\{r/4, r^{1/2}/2\})\omega(f, x_{n,i}) ,$$

and

$$\|A_{t,i}^{n_t}(B_m^j; f) - B_1(f)\| \leqq \left(1 + \min\left\{\left\|\sum_{i=1}^r (e_i - e_i^2)\right\|, \left\|\sum_{i=1}^r (e_i - e_i^2)\right\|^{1/2}\right\}\right)\omega(f, y_{t,i})$$
$$\leqq (1 + \min\{r/4, r^{1/2}/2\})\omega(f, y_{t,i}) .$$

We also note that the corresponding result of Corollary 4 holds for the Bernstein operators on $C(I_r)$ given by (12).

Finally, we restrict ourselves to the case where $P = (p_{nj})_{n,j\geqq 1}$ is the arithmetic Toeplitz matrix, i.e., $p_{nj} = 1/n$ for $n \geqq 1$, $1 \leqq j \leqq n$, and $p_{nj} = 0$ otherwise. In [12] we showed that there exists a unique strongly continuous semigroup $\{S(t); t \geqq 0\}$ of Markov operators on $C(X)$ such that for every $f \in C(X)$ and every sequence $\{k_n\}$ of positive integers with $\lim_{n\to\infty} k_n/n = t$,

$$\lim_{n\to\infty} \|B_n^{k_n}(f) - S(t)(f)\| = 0$$

and

$$\lim_{n\to\infty} \left\|(1/(k_n + 1)) \sum_{i=0}^{k_n} B_n^i(f) - \int_0^1 S(tu)(f)du\right\| = 0$$

whenever $t \geqq 0$.

Now take $W(t) = S(t)$ $(t \geqq 0)$ and let $\{C_{\xi,\lambda}; \xi > 0, \lambda \geqq 0\}$ and $\{R_{\xi,\lambda}; \xi, \lambda \geqq 0\}$ be the families of operators defined by (10) and (11), respectively. Then we have the following quantitative ergodic type theorem for iterations of continuous Cesàro and Abel means of the semigroup $\{S(t)\}$.

THEOREM 6. *Let $\{m_\xi; \xi > 0\}$ be a family of positive integers. Then for all $f \in C(X)$, $\xi > 0$ and all $\lambda \geqq 0$,*

$$\|C_{\xi,\lambda}^{m_\xi}(f) - B_1(f)\| \leqq \Omega(f, \exp(-\lambda m_\xi/2)((1 - \exp(-\xi))/\xi)^{m_\xi/2})$$

and

$$\|R_{\xi,\lambda}^{m_\xi}(f) - B_1(f)\| \leqq \Omega(f, \exp(-\lambda m_\xi/2)(\xi/(\xi + 1))^{m_\xi/2}) .$$

PROOF. From the proof of [12; Theorem 4], $\{S(t)\}$ is of type $[B_1; 1 - \exp(-t)]$. Therefore, $\{C_{\xi,\lambda}\}$ and $\{R_{\xi,\lambda}\}$ are of types $[B_1; 1 - (1/\xi)(1 - \exp(-\xi))\exp(-\lambda)]$ and $[B_1; 1 - (\xi/(\xi + 1))\exp(-\lambda)]$, respectively. Thus the desired result follows from Corollary 1. q.e.d.

Let $\{m_\xi; \xi > 0\}$ be a family of positive integers. Then, by [18; Theorem 3(b)], we have the following result: If $\lim_{\xi\to+0} m_\xi(\xi - 1 + \exp(-\xi))/\xi = 0$, then $\{C_{\xi,0}^{m_\xi}\}$ is saturated in $C(X)$ with order $1 - ((1 - \exp(-\xi))/\xi)^{m_\xi}$, or equivalently, with order $m_\xi(\xi - 1 + \exp(-\xi))/\xi$, and its trivial class coincides with $A(X)$. Also, if $\lim_{\xi\to\infty} m_\xi/(\xi + 1) = 0$, then $\{R_{\xi,0}^{m_\xi}\}$ is saturated in $C(X)$ with order $1 - (\xi/(\xi + 1))^{m_\xi}$, or equivalently, with order $m_\xi/(\xi + 1)$, and its trivial class coincides with $A(X)$. Concerning the direct estimates of the degree of approximation for these processes we have, by Corollary 1, the following:

THEOREM 7. *Let $\{m_\xi; \xi > 0\}$ be a family of positive integers. Then for all $f \in C(X)$ and all $\xi > 0$,*

$$\begin{aligned}\|C_{\xi,0}^{m_\xi}(f) - f\| &\leqq \Omega(f, (1 - ((1 - \exp(-\xi))/\xi)^{m_\xi})^{1/2}) \\ &\leqq \Omega(f, (m_\xi(\xi - 1 + \exp(-\xi))/\xi)^{1/2}) \leqq \Omega(f, (m_\xi \xi)^{1/2})\end{aligned}$$

and

$$\|R_{\xi,0}^{m_\xi}(f) - f\| \leqq \Omega(f, (1 - (\xi/(\xi + 1))^{m_\xi})^{1/2}) \leqq \Omega(f, (m_\xi/(\xi + 1))^{1/2}) .$$

REMARK 3. Let $u > 0$ be fixed. Then the following statements hold:

(i) Let $\{k_n; n \in \boldsymbol{N}\}$ be a sequence of positive integers. Then for all $f \in C(X)$, $n \in \boldsymbol{N}$ and all $i \in \boldsymbol{N}$,

$$\|\sigma_{n,i}^{k_n}(S(u); f) - B_1(f)\| \leqq \Omega(f, x_{n,i}) ,$$

where

$$x_{n,i} = \exp(-iuk_n/2)((1 - \exp(-u(n + 1)))/((1 - \exp(-u))(n + 1)))^{k_n/2} .$$

(ii) Let $\{n_t; 0 < t < 1\}$ be a family of positive integers. Then for all $f \in C(X)$, $t \in (0, 1)$ and all $i \in \boldsymbol{N}$,

$$\|A_{t,i}^{n_t}(S(u); f) - B_1(f)\| \leqq \Omega(f, y_{t,i}) ,$$

where

$$y_{t,i} = \exp(-iun_t/2)((1 - t)/(1 - t\exp(-u)))^{n_t/2} .$$

Consequently, for $\{k_n\} = \{n_t\} = \{m_\xi\} = \{1\}$, Theorems 5 and 6 and Remark 3 give quantitative versions of [18; Theorem 5] and they enable us to estimate the rate of convergence.

REMARK 4. Let X be a compact connected Hausdorff abelian group and let G be an independent subset of the character group of X. Then, under the setting of complex-valued functions, Condition (1) holds for $C = K = \pi$ (see, [1; Lemma 3]). Thus it should be possible to apply our general results (which are valid for the case of complex-valued functions) to this situation and we are able to derive a sharp improvement of the results of Bloom and Sussich [1]. Consequently, we have quantitative estimates for the degree of approximation by various positive convolution operators on $C(\boldsymbol{T}^r)$, where $\boldsymbol{T}^r$ is the r-dimensional torus. We omit the details.

We also note that results analogous to those of this paper is obtained for approximation processes in the sense of the author [13], whose results can be actually improved by means of the higher order moments. As illustrations of general results in this direction, for instance, concerning the degree of almost convergence (F-summability) (in the sense of Lorentz [7]) of $\{B_n^{k_n}; n \geqq 1\}$ with a sequence $\{k_n\}$ of positive integers, we have the following estimates for all $f \in C(X)$ and all $n \geqq 1$:

$$\sup\left\{\left\|(1/n)\sum_{i=m}^{n+m-1} B_i^{k_i}(f) - f\right\|, m \in N\right\} \leqq \Omega(f, x_n) ,$$

where

$$x_n = \left(\sup\left\{(1/n)\sum_{i=m}^{n+m-1}(1-(1-1/i)^{k_i}); m \in N\right\}\right)^{1/2}$$
$$\leqq \left(\sup\left\{(1/n)\sum_{i=m}^{n+m-1} k_i/i; m \in N\right\}\right)^{1/2} .$$

In particular, if $k_n = 1$ for all $n \geqq 1$, then

$$x_n = \left(\sup\left\{(1/n)\sum_{i=m}^{n+m-1}(1/i); m \in N\right\}\right)^{1/2}$$
$$\leqq ((\gamma + \log(n+1))/n)^{1/2} ,$$

where $\gamma = 0.57721566490153286\cdots$ is Euler's constant.

$$\sup\left\{\left\|(1/n)\sum_{i=m}^{n+m-1} B_i^{k_i}(f) - B_1(f)\right\|; m \in N\right\} \leqq \Omega(f, y_n) ,$$

where

$$y_n = \left(\sup\left\{(1/n)\sum_{i=m}^{n+m-1}(1-1/i)^{k_i}; m \in N\right\}\right)^{1/2} .$$

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