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CRITERIA FOR QUASI-SYMMETRICITY AND THE HOLOMORPHIC SECTIONAL CURVATURE OF A HOMOGENEOUS BOUNDED DOMAIN

Dedicated to Professor Ichiro Satake on his sixtieth birthday

KAZUO AZUKAWA

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Introduction. It is well-known that to every biholomorphic equivalence class of homogeneous bounded domains in $\boldsymbol{C}^n$, there naturally corresponds bijectively an isomorphism class of normal j-algebras of dimension $2n$. On the base of normal j-algebras, curvature properties of the Bergman metric on a homogeneous bounded domain were discussed in [4], [5], [6], [7], [8], [9].

In this paper we first define the notion of a "strong j-ideal" of a normal j-algebra such that the decomposition of a normal j-algebra into simple strong j-ideals is precisely related to the decomposition of the corresponding homogeneous bounded domain into irreducible ones as Riemannian manifolds with respect to the Bergman metrics (Lemma 1.4).

Let $(\mathfrak{g}, j)$ be a normal j-algebra with $\mathfrak{n}=[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{n}$. Let

$$\mathfrak{g}=\sum \mathfrak{n}_{ab}+\sum j\mathfrak{n}_{ab}+\sum \mathfrak{n}_{a\infty}$$

be the root space decomposition of $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{a}$ on $\mathfrak{n}$. Let $\mathscr{L}=\sum \mathfrak{n}_{ab}$, $\mathscr{U}=\sum \mathfrak{n}_{a\infty}$. For $x\in\mathscr{L}$, we define two endomorphisms of $\mathscr{L}$ by $A(x)=2^{-1}((\mathrm{ad}_{\mathscr{L}}jx)+(\mathrm{ad}_{\mathscr{L}}jx)^t)$, $D(x)=2^{-1}((\mathrm{ad}_{\mathscr{L}}jx)-(\mathrm{ad}_{\mathscr{L}}jx)^t)$, and an endomorphism of $\mathscr{U}$ by $\varphi(x)=(\mathrm{ad}_{\mathscr{U}}jx)+(\mathrm{ad}_{\mathscr{U}}jx)^t$. For $x, y\in\mathscr{L}$, let $x\cdot y=A(x)y$. Then, $(\mathscr{L}, \cdot)$ is a commutative distributive algebra over $\boldsymbol{R}$. If $(\mathscr{L}, \cdot)$ is a Jordan algebra, the corresponding homogeneous bounded domain is said to be quasi-symmetric in the sense of Satake [16] (cf. [11]). We consider the following conditions on $(\mathfrak{g}, j)$:

(J) $(\mathscr{L}, \cdot)$ is a Jordan algebra.

(D) For every $x\in\mathscr{L}$, $D(x)$ is a derivation of $(\mathscr{L}, \cdot)$.

(M) For each simple strong j-ideal $\tilde{\mathfrak{g}}$ of $(\mathfrak{g}, j)$, all root spaces $\mathfrak{n}_{ab}\subset\tilde{\mathfrak{g}}$ $(a<b)$ are of the same dimension and so are all $\mathfrak{n}_{a\infty}\subset\tilde{\mathfrak{g}}$.

(A) $2\varphi(x\cdot y)=\varphi(x)\circ\varphi(y)+\varphi(y)\circ\varphi(x)$ on $\mathscr{U}$ for all $x, y\in\mathscr{L}$.

The main purpose of the present paper is to show that (J), (D), and (M) are mutually equivalent, that (M) implies (A), and that if $(\mathfrak{g}, j)$ is simple with $\mathscr{U}\neq\{0\}$, then (A) implies

(M) (Theorems 2.7 and 3.7). Parts of these assertions are well-known; the equivalence of (J) and (M) was first proved by D'Atri and Miatello [9], and the equivalence of (A) and (J) was proved by Dorfmeister [11]. Their original proofs are based on other concepts representing homogeneous bounded domains, than normal j-algebras. Our proofs are based only on the theory of normal j-algebras.

In the final section, we establish a formula giving the holomorphic sectional curvature of the Bergman metric on any homogeneous bounded domain in terms of the corresponding normal j-algebra (Theorem 4.9). As a corollary, employing results in preceding sections, we obtain Zelow's formula [21], [22] of the holomorphic sectional curvature on any quasi-symmetric bounded domain.

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1. Preliminary. Let D be a homogeneous bounded domain in $\boldsymbol{C}^n$. Let G be a maximal triangular analytic Lie subgroup of the group of all biholomorphic transformations of D, and let $\mathfrak{g}$ be its Lie algebra. Fix any point p in D. Then the mapping $\Phi: G \ni f \mapsto f(p) \in D$ becomes a diffeomorphism, and we get two $\boldsymbol{R}$-linear isomorphisms $\rho: \mathfrak{g} \ni x \mapsto x_e \in T_e G$ and $\Phi_*: T_e G \to T_p^{\boldsymbol{R}} D$, where $T_e G$ and $T_p^{\boldsymbol{R}} D$ are the tangent space at the identity mapping $e \in G$ and the real tangent space at $p \in D$, respectively. Let $j \in \mathrm{End}(\mathfrak{g})$ be the endomorphism induced from the complex structure of $T_p^{\boldsymbol{R}} D$ via $\Phi_* \circ \rho$, and $\langle\, ,\rangle$ be the j-invariant inner product on $\mathfrak{g}$ induced from the Bergman metric at $T_p^{\boldsymbol{R}} D$ via $\Phi_* \circ \rho$. Let $\omega \in \mathfrak{g}^*$ be the Koszul form on $\mathfrak{g}$, i.e., $\omega(x) = 2^{-1}\,\mathrm{trace}((\mathrm{ad}\, jx) - j \circ (\mathrm{ad}\, x))$, $x \in \mathfrak{g}$. Then it is known ([14]) that $\langle x, y\rangle = \omega[jx, y]$. The Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ with complex structure j obtained in the above manner becomes a normal j-algebra (see [15]), that is, $(\mathfrak{g}, j)$ satisfies the following three conditions:

(j1) The algebra $\mathfrak{g}$ is triangular, that is, it is solvable and every eigenvalue of $\mathrm{ad}\, x$ is real for any $x \in \mathfrak{g}$.

(j2) Nijenhuis' condition holds, that is, $[jx, jy] = j[jx, y] + j[x, jy] + [x, y]$ for all $x, y \in \mathfrak{g}$.

(j3) The bilinear form $\langle x, y\rangle = \omega[jx, y]$ defined from the Koszul form ω is a j-invariant inner product on $\mathfrak{g}$.

Set $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{a} = \mathfrak{n}^{\perp}$. The dimension of $\mathfrak{a}$ is called the rank of D. For any form $\alpha \in \mathfrak{a}^*$ on $\mathfrak{a}$, set $\mathfrak{n}(\alpha) = \{x \in \mathfrak{n}; [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{a}\}$. Every element of the set $\Delta = \{\alpha \in \mathfrak{a}^*; \mathfrak{n}(\alpha) \neq \{0\}\}$ is called a root (under the adjoint representation of $\mathfrak{a}$ on $\mathfrak{n}$). The structure theorem of Pyatetskii-Shapiro [15] says the following:

(n1) The subspace $\mathfrak{a}$ is a non-zero abelian subalgebra of $\mathfrak{g}$, and $\mathfrak{n}$ has an orthogonal decomposition $\sum_{\alpha \in \Delta} \mathfrak{n}(\alpha)$.

(n2) There are R roots $\varepsilon_1, \cdots, \varepsilon_R$ such that $j\mathfrak{a}$ is the direct sum of the 1-dimensional root spaces $\mathfrak{n}(\varepsilon_a)$, $1 \leqq a \leqq R$, and that every other root is of one of the forms $(\varepsilon_a \pm \varepsilon_b)/2$, $\varepsilon_c/2$, where $a, b, c \in \{1, \cdots, R\}$ with $a<b$.

(n3) $j\mathfrak{n}((\varepsilon_a+\varepsilon_b)/2)=\mathfrak{n}((\varepsilon_a-\varepsilon_b)/2)$ for $a<b$, and $j\mathfrak{n}(\varepsilon_a/2)=\mathfrak{n}(\varepsilon_a/2)$ for all a.

We fix some notations. Let $r_a \in \mathfrak{n}(\varepsilon_a)$ be unique elements such that $\varepsilon_a(jr_b)=\delta_{ab}$ for $a, b \in \{1, \cdots, R\}$, and let $r=\sum_{a=1}^R r_a$. We say that any root in $\Delta_0=\{\varepsilon_1, \cdots, \varepsilon_R\}$ is of type 0, one in $\Delta_1:=\{(\varepsilon_a+\varepsilon_b)/2 \in \Delta;\ 1 \leqq a<b \leqq R\}$ of type 1, and one in $\Delta_2:=\{\varepsilon_a/2 \in \Delta;\ 1 \leqq a \leqq R\}$ of type 2. Put $\mathscr{L}_i=\sum_{\alpha\in\Delta_i}\mathfrak{n}(\alpha)$ $(i=0, 1)$, $\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1$, and $\mathscr{U}=\sum_{\alpha\in\Delta_2}\mathfrak{n}(\alpha)$. Thus, $j\mathscr{L}_0=\mathfrak{a}$, $\mathfrak{g}=\mathscr{L}+j\mathscr{L}+\mathscr{U}$. If $a, b, c \in \{1, \cdots, R\}$ and $a \leqq b$, then we set $\mathfrak{n}_{ab}=\mathfrak{n}((\varepsilon_a+\varepsilon_b)/2)$, $\mathfrak{n}_{c\infty}=\mathfrak{n}(\varepsilon_c/2)$. We sometimes denote a set $\{a_1, \cdots, a_k\}$ by $\{a_1 \cdots a_k\}$ (without commas), if it causes no confusion. If $a, b \in \{1, \cdots, R, \infty\}$ and $(a, b)\neq(\infty, \infty)$, then the symbol $\mathfrak{n}_{\{ab\}}$ stands for $\mathfrak{n}_{ab}$ when $a \leqq b$, and $\mathfrak{n}_{ba}$ when $a>b$. Set

$$n_{ab}=\dim \mathfrak{n}_{\{ab\}}\,. \tag{1.1}$$

We denote the cardinality of a set S by $\# S$, For two sets $\{ab\}$ and $\{cd\}$ with $\#(\{ab\}\cap\{cd\})=1$, letting $\{ab\}\cap\{cd\}=\{g\}$, $\{ab\}=\{ge\}$, and $\{cd\}=\{gf\}$, we denote the set $\{ef\}$ by the symbol $\{ab\}\sim\{cd\}$. For example, if $a\neq b$ then $\{aa\}\sim\{ab\}=\{ab\}$.

It is well-known ([18], [4]) that $\omega(x)=0$ for $x \in \mathfrak{a}+j\mathscr{L}_1+\mathscr{L}_1+\mathscr{U}$, and

$$\omega_a:=\omega(r_a)=1+\frac{1}{2}\sum_{b\neq a, b\leqq R} n_{ab}+\frac{1}{4}n_{a\infty} \tag{1.2}$$

for $a \in \{1 \cdots R\}$. For an endomorphism $A \in \mathrm{End}(\mathfrak{g})$ we denote by A^t the adjoint operator of A with respect to $\langle\, ,\rangle$. The following two facts are useful in our argument.

Lemma 1.1 ([15; Theorem 2, p. 61]). *Let $1 \leqq a<b<c \leqq \infty$. If $x \in \mathfrak{n}_{ab}$ and $y \in \mathfrak{n}_{bc}$, then*

$$\langle [jx, y], [jx, y]\rangle=\langle x, x\rangle\langle y, y\rangle/2\omega_b\,, \quad \text{or}$$

$$(\mathrm{ad}\, jx)^t(\mathrm{ad}\, jx)y=(\langle x, x\rangle/2\omega_b)y\,.$$

Lemma 1.2. *Let $a<b<c \leqq \infty$, $b<d \leqq \infty$, and $c \neq d$. If $x \in \mathfrak{n}_{ab}$, $y \in \mathfrak{n}_{bc}$, $z \in \mathfrak{n}_{\{cd\}}$, and $w \in \mathfrak{n}_{ad}$, then*

$$\langle (\mathrm{ad}\, jw)^t(\mathrm{ad}\, jy)^t x, z\rangle=\langle (\mathrm{ad}\, jy)^t(\mathrm{ad}\, jw)^t x, z\rangle\,, \quad \text{or}$$

$$\langle (\mathrm{ad}\, j[jx, y])^t w, z\rangle=\langle (\mathrm{ad}\, jy)^t(\mathrm{ad}\, jx)^t w, z\rangle\,.$$

Proof. By Jacobi's identity we have $(\mathrm{ad}\, jw)(\mathrm{ad}\, jy)z=(\mathrm{ad}\, jy)(\mathrm{ad}\, jw)z$, which implies the first assertion. Since $(\mathrm{ad}\, j[jx, y])^t w=(\mathrm{ad}\, jw)^t[jx, y]$, $[jx, y]=(\mathrm{ad}\, jy)^t x$, and $(\mathrm{ad}\, jw)^t x=(\mathrm{ad}\, jx)^t w$, the second assertion follows.

The following definition and lemma are useful in our argument.

Definition 1.3. A subspace $\tilde{\mathfrak{g}}$ of a normal j-algebra $(\mathfrak{g}, j)$ is called a j-ideal of

$(\mathfrak{g}, j)$ if $\tilde{\mathfrak{g}}$ is a j-invariant ideal of $\mathfrak{g}$, and it is called a strong j-ideal of $(\mathfrak{g}, j)$ if $\tilde{\mathfrak{g}}$ is a non-zero j-ideal of $(\mathfrak{g}, j)$ and satisfies $(\mathrm{ad}\, x)^t \tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}}$ for all $x \in \tilde{\mathfrak{g}}$. If $\mathfrak{g}$ possesses no strong j-ideals except itself, $(\mathfrak{g}, j)$ is said to be simple.

LEMMA 1.4. *A normal j-algebra of rank R is simple if and only if the set* $\{1 \cdots R\}$ *of integers is connected in the sense (cf. Asano [2]) that for every pair a, b in* $\{1 \cdots R\}$ *there exists a sequence* $a = a_0, \cdots, a_k = b$ *in* $\{1 \cdots R\}$ *with* $n_{a_{i-1}a_i} \neq 0$, $i = 1, \cdots, k$ *(see (1.1)). Every normal j-algebra* $\mathfrak{g}$ *is a direct sum of simple strong j-ideals* $\mathfrak{g}_1, \cdots, \mathfrak{g}_p$, *and every simple strong j-ideal is one of the* $\mathfrak{g}_s$.

PROOF. Let $\tilde{\mathfrak{g}}$ be a strong j-ideal of a normal j-algebra $(\mathfrak{g}, j)$. Take an element $z \in \tilde{\mathfrak{g}} - \{0\}$, and express it in the form $z = x + jy + u$, where $x, y \in \mathscr{L}$ and $u \in \mathscr{U}$. Then, $\tilde{\mathfrak{g}} \ni [jr, z] = x + 2^{-1}u$, $\tilde{\mathfrak{g}} \ni -j[jr, jz] = jy + 2^{-1}u$, and $\tilde{\mathfrak{g}} \ni [jr, jy + 2^{-1}u] = 4^{-1}u$, so that x, jy, and u belong to $\tilde{\mathfrak{g}}$. Assume $x \neq 0$ or $y \neq 0$, say $x \neq 0$. Write $x = \sum_{a \leqq b} x_{ab}$, where $x_{ab} \in \mathfrak{n}_{ab}$. Then, for each pair a, b in $\{1 \cdots R\}$ with $a < b$, we have $\tilde{\mathfrak{g}} \ni [jr_b, [jr_a, x]] = 4^{-1} x_{ab}$, so that all x_{ab}, $a < b$, and $x' = \sum x_{aa}$ belong to $\tilde{\mathfrak{g}}$. Furthermore, $x_{aa} = [jr_a, x'] \in \tilde{\mathfrak{g}}$ for any a. If $x_{ab} \neq 0$ for some $a < b$ then $\tilde{\mathfrak{g}} \ni [jx_{ab}, x_{ab}] = \langle x_{ab}, x_{ab} \rangle r_a / \omega_a$; while if $x_{aa} \neq 0$ then $r_a \in \tilde{\mathfrak{g}}$. Thus, we obtain the following:

(1.3) There exists an a such that $r_a \in \tilde{\mathfrak{g}}$.

A similar argument shows that the assumption $u \neq 0$ also implies the claim (1.3). We shall show another claim:

(1.4) If $r_a \in \tilde{\mathfrak{g}}$, $b \neq a$, and $n_{ab} \neq 0$, then $r_b \in \tilde{\mathfrak{g}}$.

Indeed, let $x \in \mathfrak{n}_{\{ab\}} - \{0\}$. Then, $x = 2[jr_a, x] \in \tilde{\mathfrak{g}}$. If $a > b$ then $\tilde{\mathfrak{g}} \ni [jx, x] = \langle x, x \rangle r_b / \omega_b$, while if $a < b$ then by the strongness of $\tilde{\mathfrak{g}}$ we have $\tilde{\mathfrak{g}} \ni (\mathrm{ad}\, jx)^t x = \langle x, x \rangle r_b / \omega_b$. Thus, (1.4) is proved. Now, if we assume $\{1 \cdots R\}$ is connected in the prescribed sense, then by (1.3) and (1.4) we see that all r_a belong to $\tilde{\mathfrak{g}}$. It follows that $\tilde{\mathfrak{g}} = \mathfrak{g}$. Therefore, $\mathfrak{g}$ is simple. When $(\mathfrak{g}, j)$ is arbitrary, we decompose the set $\{1 \cdots R\}$ into connected components $I_1, \cdots, I_p$ in the prescribed sense. Since $\mathfrak{n}_{\{ab\}} = \{0\}$ for all $a \in I_s$, $b \in I_t$ with $s \neq t$, if for every s we set $\Delta_0^{(s)} = \{\varepsilon_a; a \in I_s\}$, $\Delta_1^{(s)} = \{(\varepsilon_a + \varepsilon_b)/2 \in \Delta; a, b \in I_s, a \neq b\}$, $\Delta_2^{(s)} = \{\varepsilon_a/2 \in \Delta; a \in I_s\}$, $\mathscr{L}^{(s)} = \sum\{\mathfrak{n}(\alpha); \alpha \in \Delta_0^{(s)} \cup \Delta_1^{(s)}\}$, and $\mathscr{U}^{(s)} = \sum\{\mathfrak{n}(\alpha); \alpha \in \Delta_2^{(s)}\}$, then $\mathfrak{g}_s := \mathscr{L}^{(s)} + j\mathscr{L}^{(s)} + \mathscr{U}^{(s)}$ are simple strong j-ideals of $(\mathfrak{g}, j)$ and $\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_p$ (direct sum of subspaces). To prove the last assertion of Lemma 1.4, let $\tilde{\mathfrak{g}}$ be a simple strong j-ideal of $(\mathfrak{g}, j)$. Set $I = \{a \in \{1 \cdots R\}; r_a \in \tilde{\mathfrak{g}}\}$, $\tilde{\Delta}_0 = \{\varepsilon_a; a \in I\}$, $\tilde{\Delta}_1 = \{(\varepsilon_a + \varepsilon_b)/2 \in \Delta;\ a, b \in I, a \neq b\}$, $\tilde{\Delta}_2 = \{\varepsilon_a/2 \in \Delta; a \in I\}$, $\tilde{\mathscr{L}} = \sum\{\mathfrak{n}(\alpha); \alpha \in \tilde{\Delta}_0 \cup \tilde{\Delta}_1\}$, and $\tilde{\mathscr{U}} = \sum\{\mathfrak{n}(\alpha); \alpha \in \tilde{\Delta}_2\}$. Then, we see that $\tilde{\mathfrak{g}} = \tilde{\mathscr{L}} + j\tilde{\mathscr{L}} + \tilde{\mathscr{U}}$. Since $\tilde{\mathfrak{g}}$ is simple, I is connected so that $I \subset I_s$ for some s. By (1.4) we see $I = I_s$, so that $\tilde{\mathfrak{g}} = \mathfrak{g}_s$. The proof is completed.

We denote by $\mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto \nabla_x y \in \mathfrak{g}$ the bilinear mapping induced from the Levi-Civita connection of the Bergman metric on D via $\Phi_* \circ \rho$. It is given by

$$\nabla_x y = 2^{-1}([x, y] - (\mathrm{ad}\, x)^t y - (\mathrm{ad}\, y)^t x)\,.$$

The mapping $R(x, y)=[\nabla_x, \nabla_y]-\nabla_{[x,y]}$ then corresponds to the Riemannian curvature tensor of the Bergman metric on D via $\Phi_* \circ \rho$. Since the Bergman metric is Kählerian, we have $\nabla_x \circ j = j \circ \nabla_x$, $R(x, y)\circ j = j \circ R(x, y)$, and $R(jx, jy)=R(x, y)$. If $x, y \in \mathscr{L}$ and $u, v \in \mathscr{U}$, it is well-known ([4]) that $\nabla_{jx}\mathscr{L} \subset \mathscr{L}$, $\nabla_{jx}\mathscr{U} \subset \mathscr{U}$, $\nabla_x \mathscr{L} \subset j\mathscr{L}$, $\nabla_x \mathscr{U} \subset \mathscr{U}$, and

(1.5) $$\nabla_{jx}=((\mathrm{ad}\, jx)-(\mathrm{ad}\, jx)^t)/2 \quad \text{on} \quad \mathscr{L}+\mathscr{U}\,,$$

(1.6) $$\nabla_x = j\circ((\mathrm{ad}\, jx)+\mathrm{ad}\, jx)^t)/2 \quad \text{on} \quad \mathscr{L}+\mathscr{U}\,,$$

(1.7) $$\nabla_u v=[u, v]/2+j[ju, v]/2 \in \mathscr{L}+j\mathscr{L}\,.$$

2. Quasi-symmetricity (1).

DEFINITION 2.1. For $x \in \mathscr{L}$, we define two endomorphisms $A(x)$ and $D(x)$ of $\mathscr{L}$ by

$$A(x)=-j\circ\nabla_x|_{\mathscr{L}}=2^{-1}((ad_{\mathscr{L}}jx)+(\mathrm{ad}_{\mathscr{L}}jx)^t)\,,$$
$$D(x)=\nabla_{jx}|_{\mathscr{L}}=2^{-1}((\mathrm{ad}_{\mathscr{L}}jx)-(\mathrm{ad}_{\mathscr{L}}jx)^t)\,,$$

where $\mathrm{ad}_{\mathscr{L}}jx=(\mathrm{ad}\, jx)|_{\mathscr{L}}$ (see (1.5) and (1.6)).

The following is easily verified.

LEMMA 2.2 *Let $x \in \mathfrak{n}_{\{ab\}} \subset \mathscr{L}$ and $y \in \mathfrak{n}_{\{cd\}} \subset \mathscr{L}$.*

(i) *If $\{ab\} \cap \{cd\} = \varnothing$, then $A(x)y=0$ and $D(x)y=0$.*

(ii) *$A(r_a)y=2^{-1}(\delta_{ac}+\delta_{ad})y$, and $D(r_a)y=0$.*

(iii)$_1$ *If $a<b$, $a \in \{cd\}$, and $b \notin \{cd\}$, then $A(x)y=2^{-1}(\mathrm{ad}\, jx)^t y=-D(x)y \in \mathfrak{n}_{\{ab\}\sim\{cd\}} \subset \mathscr{L}_1$.*

(iii)$_2$ *If $a<b$, $a \notin \{cd\}$, and $b \in \{cd\}$, then $A(x)y=2^{-1}(\mathrm{ad}\, jx)y=D(x)y \in \mathfrak{n}_{\{ab\}\sim\{cd\}} \subset \mathscr{L}_1$.*

(iv) *If $a<b$ and $(a, b)=(c, d)$, then*

$$A(x)y=2^{-1}\langle x, y\rangle(r_a/\omega_a+r_b/\omega_b) \in \mathscr{L}_0\,,$$
$$D(x)y=2^{-1}\langle x, y\rangle(r_a/\omega_a-r_b/\omega_b) \in \mathscr{L}_0\,.$$

DEFINITION 2.3. For $x, y \in \mathscr{L}$, let $xy=x\cdot y=A(x)y$. Then $(\mathscr{L}, \cdot)$ is a commutative distributive algebra over $\boldsymbol{R}$. We call this the algebra induced from the connection of $(\mathfrak{g}, j)$, or simply, the connection algebra of $(\mathfrak{g}, j)$ (cf. Vinberg [20]).

DEFINITION 2.4. A commutative distributive algebra C is called a Jordan algebra if $(xy)x^2=x(yx^2)$ holds for all $x, y \in C$.

It is well-known (Albert [1], Satake [17]) that if C is a Jordan algebra, then $[A(x), A(y)]$ is a derivation of C for all $x, y \in C$, where $A(x)$ is the left (or right) multiplication operator on C by x. It is also known (Vinberg [19]) that if the bilinear form $\langle x, y\rangle_t := \mathrm{trace}\, A(xy)$ $(x, y \in C)$ is non-degenerate, then the converse is true. As was proved by Zelow [23], if $(\mathscr{L}, \cdot)$ is the connection algebra of a normal j-algebra $(\mathfrak{g}, j)$, then $\langle\ ,\ \rangle_t$ is positive definite.

DEFINITION 2.5 (Dorfmeister [11]). A normal j-algebra $(\mathfrak{g}, j)$, or the corresponding homogeneous bounded domain D, is said to be quasi-symmetric (in the sense of Satake [16]) if the connection algebra $(\mathscr{L}, \cdot)$ of $(\mathfrak{g}, j)$ is a Jordan algebra.

DEFINITION 2.6. A normal j-algebra $(\mathfrak{g}, j)$, or the corresponding homogeneous bounded domain D, is said to satisfy the multiplicity condition if for every simple strong j-ideal $\tilde{\mathfrak{g}}$ of $(\mathfrak{g}, j)$, all roots of type 1 and type 2, respectively, corresponding to the subalgebra $(\tilde{\mathfrak{g}}, j)$ have the same multiplicities.

We can now formulate our main result of this section.

THEOREM 2.7. *For a normal j-algebra* $(\mathfrak{g}, j)$, *the following three conditions are mutually equivalent*:

(J) $(\mathfrak{g}, j)$ *is quasi-symmetric.*

(D) *For every* $x \in \mathscr{L}$, $D(x)$ *is a derivation of* $(\mathscr{L}, \cdot)$.

(M) $(\mathfrak{g}, j)$ *satisfies the multiplicity condition.*

REMARK 2.8. D'Atri and Miatello [9] proved the equivalence (J)$\Leftrightarrow$(M) on the base of the theory of T-algebras by Vinberg [20].

For the proof of Theorem 2.7 we need several lemmas. First, the argument mentioned above yields the following.

LEMMA 2.9 ([1], [17], [19], [23]). *For a normal j-algebra* $(\mathfrak{g}, j)$, *the condition* (J) *in Theorem* 2.7 *is equivalent to the following*:

(JD) $[A(x), A(y)]$ *is a derivation on* $(\mathscr{L}, \cdot)$ *for any* $x, y \in \mathscr{L}$.

LEMMA 2.10. *For a normal j-algebra* $(\mathfrak{g}, j)$, *the condition* (D) *in Theorem* 2.7 *is equivalent to the condition* (JD) *in Lemma* 2.9.

PROOF. By definition we see that $R(x, y)|_{\mathscr{L}} = -[A(x), A(Y)]$ for $x, y \in \mathscr{L}$. As equalities between elements in $\mathrm{End}(\mathscr{L})$ we have

$$\begin{aligned}[A(x), A(y)] &= -R(x, y) = -R(jx, jy) = -[\nabla_{jx}, \nabla_{jy}] + \nabla_{j[jx,y]+j[x,jy]} \\ &= -[D(x), D(y)] + D([jx, y]) + D([x, jy]) .\end{aligned}$$

From this we have the implication (D)$\Rightarrow$(JD) and the following formula (since $D(r_a) = 0$ by Lemma 2.2): For $x \in \mathfrak{n}_{ab}$ with $a < b \leqq R$, $[A(r_a), A(x)] = 2^{-1}D(x)$, which yields the implication (JD)$\Rightarrow$(D). The proof is completed.

LEMMA 2.11. *For a normal j-algebra* $(\mathfrak{g}, j)$, *the condition* (D) *in Theorem* 2.7 *is equivalent to the totality of the following three conditions*:

(d1) *If* $x \in \mathfrak{n}_{ab} \subset \mathscr{L}_1$, $y \in \mathfrak{n}_{cb} \subset \mathscr{L}_1$, *and* $a \neq c$, *then* $4A(x)^2 y = (\langle x, x\rangle/2\omega_b)y$, *or* $\langle [jx,y], [jx, y]\rangle = \langle x, x\rangle\langle y, y\rangle/2\omega_b$.

(d2) *If* $x \in \mathfrak{n}_{ab} \subset \mathscr{L}_1$, $y \in \mathfrak{n}_{ac} \subset \mathscr{L}_1$, *and* $b \neq c$, *then* $4A(x)^2 y = (\langle x, x\rangle/2\omega_a)y$, *or* $\langle(\mathrm{ad}\, jx)^t y, (\mathrm{ad}\, jx)^t y\rangle = \langle x, x\rangle\langle y, y\rangle/2\omega_a$.

(d3) *If* $n_{ab} \neq 0$, *then* $\omega_a = \omega_b$.

To prove Lemma 2.11 it is convenient to employ the following definition and result.

DEFINITION 2.12. For $x_a \in \mathscr{L}$, set

$$d(x_1, x_2, x_3) = (D(x_1)x_2)x_3 + (D(x_1)x_3)x_2 - D(x_1)(x_2x_3),$$

$$\delta(x_1, x_2, x_3, x_4) = \langle d(x_1, x_2, x_3), x_4\rangle = \langle D(x_1)x_2, x_3x_4\rangle + \langle D(x_1)x_3, x_4x_2\rangle + \langle D(x_1)x_4, x_2x_3\rangle .$$

We note that for an $x \in \mathscr{L}$, the operator $D(x)$ is a derivation on $(\mathscr{L}, \cdot)$ if and only if $d(x, \cdot, \cdot) = 0$, or $\delta(x, \cdot, \cdot, \cdot) = 0$.

LEMMA 2.13. *The trilinear operator* $d\colon \mathscr{L}^3 \to \mathscr{L}$ *and the quartic form* δ *on* $\mathscr{L}$ *are both symmetric with respect to the prescribed variables.*

We will prove Lemma 2.13 in § 4.

PROOF OF LEMMA 2.11. Since $D(r_a) = 0$, Lemma 2.13 implies that the condition (D) is equivalent to

$$\delta(x_1, x_2, x_3, x_4) = 0 \quad \text{for all} \quad x_1, \cdots, x_4 \in \mathscr{L}_1 . \tag{2.1}$$

Furthermore, (2.1) is equivalent to

$$\delta(x_1, x_2, x_3, x_4) = 0 \quad \text{for all} \quad x_1, \cdots, x_4 \in \mathscr{L}_1 \quad \text{with} \quad \langle D(x_1)x_2, x_3x_4\rangle \neq 0 . \tag{2.2}$$

We now assume that

$$x_1 \in \mathfrak{n}_{ab}, \quad x_2 \in \mathfrak{n}_{cd}, \quad x_3 \in \mathfrak{n}_{ef}, \quad x_4 \in \mathfrak{n}_{gh} \quad \text{with} \quad a<b, \quad c<d, \quad e<f, \quad g<h . \tag{2.3}$$

By Lemma 2.2, $D(x_1)x_2 \neq 0$ (resp. $x_3x_4 \neq 0$) implies $\{ab\} \cap \{cd\} \neq \varnothing$ (resp. $\{ef\} \cap \{gh\} \neq \varnothing$). Furthermore, $\#(\{ab\} \cap \{cd\}) = 2$ (resp. $= 1$) implies that x_1x_2 and $D(x_1)x_2$ belong to $\mathscr{L}_0$ (resp. to $\mathscr{L}_1$). Therefore, if $\langle D(x_1)x_2, x_3x_4\rangle \neq 0$, then one of the following two cases (a), (b) occurs: (a) $\#(\{ab\} \cap \{cd\}) = 2$, $\#(\{ef\} \cap \{gh\}) = 2$, and $\{ab\} \cap \{ef\} \neq \varnothing$; (b) $\#(\{ab\} \cap \{cd\}) = 1$, $\#(\{ef\} \cap \{gh\}) = 1$, and $\{ab\} \sim \{cd\} = \{ef\} \sim \{gh\}$. It follows from the symmetry of δ that (2.2) is equivalent to the following condition:

(2.4) If $x_1, \cdots, x_4$ are as in (2.3), then $\delta(x_1, x_2, x_3, x_4) = 0$ for each of the following cases:

- (i) $(a, b) = (c, d) = (e, f) = (g, h)$.
- (ii)$_1$ $(a, b) = (c, d)$, $(e, f) = (g, h)$, and $b = e$.
- (ii)$_2$ $(a, b) = (c, d)$, $(e, f) = (g, h)$, $b = f$ and $a < e$.
- (ii)$_3$ $(a, b) = (c, d)$, $(e, f) = (g, h)$, $a = e$ and $b > f$.
- (iii)$_1$ $g = a$, $b = c$, $d = e$, $f = h$.

(iii)$_2$ $g=a$, $b=c$, $d=f$, $e=h$, and $b<e$.
(iii)$_3$ $a=e$, $b=d$, $c=g$, $f=h$, $a<c$, and $b<f$.

We shall examine the condition (2.4) separately in each case.

Case (i): By Lemma 2.2 we have

$$4\delta(x_1, x_2, x_3, x_4) \\ =(\langle x_1, x_2\rangle\langle x_3, x_4\rangle+\langle x_1, x_3\rangle\langle x_4, x_2\rangle+\langle x_1, x_4\rangle\langle x_2, x_3\rangle)(\omega_a^{-1}-\omega_b^{-1}).$$

Thus, (2.4) in case (i) is equivalent to (d3).

Case (ii)$_1$ or (ii)$_2$: By Lemma 2.2 we have

$$4\delta(x_1, x_2, x_3, x_4) \\ =-\langle x_1, x_2\rangle\langle x_3, x_4\rangle/\omega_b+\langle[jx_1, x_3], [jx_2, x_4]\rangle+\langle[jx_1, x_4], [jx_2, x_3]\rangle.$$

Thus, (2.4) in case (ii)$_2$ is equivalent to (d1). On the other hand, (2.4) in case (ii)$_1$ a priori holds by Lemma 1.1.

Case (ii)$_3$: By Lemma 2.2 we have

$$4\delta(x_1, x_2, x_3, x_4) \\ =\langle x_1, x_2\rangle\langle x_3, x_4\rangle/\omega_a-\langle(\mathrm{ad}\, jx_1)^t x_3, (\mathrm{ad}\, jx_2)^t x_4\rangle-\langle(\mathrm{ad}\, jx_1)^t x_4, (\mathrm{ad}\, jx_2)^t x_3\rangle.$$

Thus, (2.4) in case (ii)$_3$ is equivalent to (d2).

Case (iii)$_1$ or (iii)$_2$: By Lemma 2.2 we have

$$4\delta(x_1, x_2, x_3, x_4)=\langle[jx_1, x_2], [jx_4, x_3]\rangle-\langle(\mathrm{ad}\, jx_1)^t x_4, [jx_2, x_3]\rangle \\ =\langle(\mathrm{ad}\, jx_2)^t x_1, [jx_4, x_3]\rangle-\langle(\mathrm{ad}\, jx_4)^t x_1, [jx_2, x_3]\rangle.$$

Thus, (2.4) in case (iii)$_1$ or (iii)$_2$ a priori holds by Lemma 1.2.

Case (iii)$_3$: By Lemma 2.2 we see

$$4\delta(x_1, x_2, x_3, x_4)=\langle[jx_1, x_2], [jx_3, x_4]\rangle-\langle(\mathrm{ad}\, jx_1)^t x_3, (\mathrm{ad}\, jx_2)^t x_4\rangle \\ =\langle[jx_2, x_1], [jx_3, x_4]\rangle-\langle(\mathrm{ad}\, jx_3)^t x_1, (\mathrm{ad}\, jx_2)^t x_4\rangle.$$

Thus, (2.4) in case (iii)$_3$ is equivalent to $A(x_3)A(x_2)x_1=A(x_2)A(x_3)x_1$. On the other hand, Lemma 1.1 asserts that $A(x_2)A(x_3)y=A(x_3)A(x_2)y$ for all $y\in\mathfrak{n}_{ac}$, so that $A(x_2)^2A(x_3)A(x_2)x_1=A(x_2)A(x_3)A(x_2)^2x_1$. Observing (d1) and (d2) to the effect that $4A(x_2)^2x_1=(\langle x_2, x_2\rangle/2\omega_b)x_1$, $4A(x_2)^2y=(\langle x_2, x_2\rangle/2\omega_c)y$ for all $y\in\mathfrak{n}_{cf}$, we see that (2.4) in case (iii)$_3$ is a consequence of (d1), (d2), and (d3). The proof of Lemma 2.11 is completed.

LEMMA 2.14. *Assume that a normal j-algebra* $(\mathfrak{g}, j)$ *is simple and satisfies the conditions* (d1) *and* (d2) *in Lemma* 2.11. *Then* $n_{ab}=b_{12}$ *for all a, b with* $a<b\leqq R$.

PROOF. Let $x\in\mathfrak{n}_{ab}-\{0\}$ with $a<b\leqq R$. By (d1) as well as Lemma 1.1 we see that for every $c\in\{1\cdots R\}-\{ab\}$, $A(x)$ is an injection from $\mathfrak{n}_{\{bc\}}$ to $\mathfrak{n}_{\{ac\}}$, so that $n_{bc}\leqq n_{ac}$.

Similarly, by (d2) as well as Lemma 1.1 we see that for every $c \in \{1 \cdots R\} - \{ab\}$, $A(x)$ is also an injection from $\mathfrak{n}_{\{ac\}}$ to $\mathfrak{n}_{\{bc\}}$, so that $n_{ac} \leqq n_{bc}$. Thus, for every a, b in $\{1 \cdots R\}$ with $a \neq b$, if $n_{ab} \neq 0$ then $n_{ac} = n_{bc}$ for all $c \in \{1 \cdots R\} - \{ab\}$. From this we see that for distinct $a, b, c \in \{1 \cdots R\}$, $n_{ab} \neq 0$ and $n_{bc} \neq 0$ imply $n_{ac} = n_{ab} = n_{bc}$. It follows from the connectedness of $\{1 \cdots R\}$ (Lemma 1.4) that $n_{ab} = n_{12}$ for all a, b with $a < b$. Lemma 2.14 is proved.

PROOF OF THEORM 2.7. If the rank R of $(\mathfrak{g}, j)$ is one, then all three conditions trivially hold. Thus, we assume $R \geqq 2$. The equivalence (J)$\Leftrightarrow$(D) follows from Lemmas 2.9 and 2.10. To show the equivalence (D)$\Leftrightarrow$(M), by Lemma 2.11 we may prove the equivalence (d1) $\wedge$ (d2) $\wedge$ (d3)$\Leftrightarrow$(M). Assume (d1), (d2), and (d3) hold, and take any simple strong j-ideal $\tilde{\mathfrak{g}}$ of $(\mathfrak{g}, j)$ with $I = \{a \in \{1 \cdots R\}; r_a \in \tilde{\mathfrak{g}}\}$. Then by Lemma 2.14 we see that n_{ab} $(a, b \in I, a \neq b)$ are constant. In view of (1.2), (d3) implies (M). To prove the converse we assume that (M) is satisfied. Decompose $\mathfrak{g}$ into simple strong j-ideals $\mathfrak{g}_1, \cdots, \mathfrak{g}_p$ with $I_s = \{a \in \{1 \cdots R\}; r_a \in \mathfrak{g}_s\}$ (Lemma 1.4). If $a \in I_s$, $b \in I_t$, and $s \neq t$, then $n_{ab} = 0$. It follows from (1.2) that (d3) holds. To show (d2), let $a < b < c \leqq R$, $x \in \mathfrak{n}_{ab} - \{0\}$, $y \in \mathfrak{n}_{ac} - \{0\}$. By Lemma 1.1 we see that

$$\langle [jx, z], [jx, z] \rangle = \langle x, x \rangle \langle z, z \rangle / 2\omega_b \qquad \text{for} \quad z \in \mathfrak{n}_{bc}. \tag{2.5}$$

This means that $(\mathrm{ad}\, jx)|_{\mathfrak{n}_{bc}} : \mathfrak{n}_{bc} \to \mathfrak{n}_{ac}$ is injective, so that it is an isomorphism by (M) because a, b, c all belong to an I_s. Take $z_0 \in \mathfrak{n}_{bc}$ such that $(\mathrm{ad}\, jx) z_0 = y$. Then, by (2.5) we see that $\langle (\mathrm{ad}\, jx)^t y, z \rangle = \langle (\langle x, x \rangle / 2\omega_c) z_0, z \rangle$ for all $z \in \mathfrak{n}_{bc}$. Thus

$$(\mathrm{ad}\, jx)^t |_{\mathfrak{n}_{ac}} = (\langle x, x \rangle / 2\omega_b)((\mathrm{ad}\, jx)|_{\mathfrak{n}_{bc}})^{-1},$$

so that once more (2.5) yields $\langle (\mathrm{ad}\, jx)^t y, (\mathrm{ad}\, jx)^t y \rangle = \langle x, x \rangle \langle y, y \rangle / 2\omega_b$; therefore by (d3) we have (d2). Next, to show (d1), let $a < c < b \leqq R$, and $x \in \mathfrak{n}_{ab} - \{0\}$. Similarly to the above argument, instead of Lemma 1.1, (d2) implies that $(\mathrm{ad}\, jx)^t |_{\mathfrak{n}_{ac}} : \mathfrak{n}_{ac} \to \mathfrak{n}_{cb}$ is an isomorphism and that

$$(\mathrm{ad}\, jx)|_{\mathfrak{n}_{cb}} = (\langle x, x \rangle / 2\omega_a)((\mathrm{ad}\, jx)^t |_{\mathfrak{n}_{ac}})^{-1},$$

which yields (d1) because $\omega_a = \omega_b$. The proof of Theorem 2.7 is completed.

3. Quasi-symmetricity (2). In this section we study a condition for a normal j-algebra $(\mathfrak{g}, j)$ to be quasi-symmetric, which is related to the subspace $\mathscr{U}$, where $\mathfrak{g} = \mathscr{L} + j\mathscr{L} + \mathscr{U}$. We denote by $\mathrm{End}(\mathscr{U}, j)$ the totality of all j-invariant endomorphisms of $\mathscr{U}$. It is well-known (e.g., [17]) that the space $\mathrm{End}(\mathscr{U}, j)$ endowed with the product

$$XY = (X \circ Y + Y \circ X)/2 \tag{3.1}$$

is a Jordan algebra with the identity transformation as the unit.

DEFINITION 3.1. For $x \in \mathscr{L}$, we define an endomorphism $\varphi(x)$ of $\mathscr{U}$ by

(φ0) $$\varphi(x) = -2j\circ\nabla_x|_{\mathscr{U}} = (\mathrm{ad}_{\mathscr{U}}\, jx) + (\mathrm{ad}_{\mathscr{U}}\, jx)^t$$

(see (1.6)). In fact, $\varphi(x)\in \mathrm{End}(\mathscr{U}, j)$.

The following is easily verified.

LEMMA 3.2. *Let* $u\in \mathfrak{n}_{c\infty}$.

(i) $\varphi(r_a)u = \delta_{ac}u$ *for all* $a\in\{1 \cdots R\}$.

(ii) *For every* $x\in\mathfrak{n}_{ab}$ *with* $a<b\leqq R$, $\varphi(x)u = \delta_{bc}(\mathrm{ad}\, jx)u + \delta_{ac}(\mathrm{ad}\, jx)^t u \in \delta_{bc}\mathfrak{n}_{a\infty} + \delta_{ac}\mathfrak{n}_{b\infty}$.

We denote by $V^c = V + iV$ the complexification of a real vector space V.

DEFINITION 3.3 ([15]). For $u, v\in\mathscr{U}$, we define $F(u, v)\in\mathscr{L}^c$ by $4F(u, v) = [ju, v] + i[u, v]$.

We extend the inner product $\langle\, ,\rangle$ on $\mathfrak{g}$ to a unique complex symmetric bilinear form on $\mathfrak{g}^c$ (cf. [13]).

LEMMA 3.4. *For every* $x\in\mathscr{L}$, $\varphi(x)\in\mathrm{End}(\mathscr{U})$ *is characterized by the following mutually equivalent conditions*:

(φ1) $\langle\varphi(x)u, v\rangle = \langle x, [ju, v]\rangle$ *for all* $u, v\in\mathscr{U}$.

(φ2) $\langle r, [\varphi(x)u, v]\rangle = \langle x, [u, v]\rangle$ *for all* $u, v\in\mathscr{U}$.

(φ3) $\langle r, F(\varphi(x)u, v)\rangle = \langle x, F(u, v)\rangle$ *for all* $u, v\in\mathscr{U}$.

PROOF. The equivalence (φ0)$\Leftrightarrow$(φ1) follows from

$$\begin{aligned}\langle((\mathrm{ad}\, jx) + (\mathrm{ad}\, jx)^t)u, v\rangle &= \langle[jx, u], v\rangle + \langle u, [jx, v]\rangle = \langle[jx, u], v\rangle + \langle ju, [jx, jv]\rangle \\ &= \langle x, [ju, v]\rangle\,.\end{aligned}$$

We note that each condition (φi) implies that $\varphi(x)\circ j = j\circ\varphi(x)$, so that the equivalence (φ1)$\Leftrightarrow$(φ2) follows from the fact $\langle r, [ju, v]\rangle = \langle u, v\rangle$, and the equivalence ($\varphi$2)$\Leftrightarrow$($\varphi$3) follows from the definition of F and the extension of $\langle\, ,\rangle$.

DEFINITION 3.5 ([16]). Let $x\in\mathscr{L}$. We say that $\varphi(x)\in\mathrm{End}(\mathscr{U}, j)$ is associated to $A(x)\in\mathrm{End}(\mathscr{L})$ if they satisfy

(α0) $2A(x)F(u, v) = F(\varphi(x)u, v) + F(u, \varphi(x)v)$ for all $u, v\in\mathscr{U}$.

Here we regard $A(x)$ as a $\boldsymbol{C}$-linear endomorphism of $\mathscr{L}^c$.

LEMMA 3.6. *For* $x\in\mathscr{L}$, *the condition* (α0) *is equivalent to each of the following seven conditions*:

(α0)′ $2A(x)[u, v] = [\varphi(x)u, v] + [u, \varphi(x)v]$ *for all* $u, v\in\mathscr{U}$.

(α0)″ $(\mathrm{ad}\, jx)^t[u, v] = [(\mathrm{ad}\, jx)^t u, v] + [u, (\mathrm{ad}\, jx)^t v]$ *for all* $u, v\in\mathscr{U}$.

(α0)‴ $j\nabla_x[u, v] = [j\nabla_x u, v] + [u, j\nabla_x v]$ *for all* $u, v\in\mathscr{U}$.

(α1) $2\varphi(xy) = \varphi(x)\circ\varphi(y) + \varphi(y)\circ\varphi(x)$ *on* $\mathscr{U}$ *for all* $y\in\mathscr{L}$.

(α1)′ $\varphi((\mathrm{ad}\, jx)y) = (\mathrm{ad}_{\mathscr{U}}\, jx) \circ \varphi(y) + \varphi(y) \circ (\mathrm{ad}_{\mathscr{U}}\, jx)^t$ *on* $\mathscr{U}$ *for all* $y \in \mathscr{L}$.

(α2) $2\langle xy, F(u, v)\rangle = \langle r, F(\varphi(x)u, \varphi(y)v) + F(\varphi(y)u, \varphi(x)v)\rangle$ *for all* $y \in \mathscr{L}$ *and* $u, v \in \mathscr{U}$.

(α2)′ $2\langle xy, [u, v]\rangle = \langle r, [\varphi(x)u, \varphi(y)v] + [\varphi(y)u, \varphi(x)v]\rangle$ *for all* $y \in \mathscr{L}$ *and* $u, v \in \mathscr{U}$.

We remark that (α1) means that φ is an algebra homomorphism from the connection algebra $(\mathscr{L}, \cdot)$ to the Jordan algebra $\mathrm{End}(\mathscr{U}, j)$ with the product (3.1).

PROOF OF LEMMA 3.6. The equivalence (α0)′$\Leftrightarrow$(α1) follows from the equality

$$\langle 2\varphi(xy)u - \varphi(x)\varphi(y)u - \varphi(y)\varphi(x)u, v\rangle = \langle y, 2A(x)[ju, v] - [ju, \varphi(x)v] - [\varphi(x)ju, v]\rangle ,$$

which follows from (φ1). The equivalence (α0)′$\Leftrightarrow$(α2)′ follows from the equality

$$\begin{aligned}&\langle 2A(x)[u, v] - [\varphi(x)u, v] - [u, \varphi(x)v], y\rangle \\ &\quad = 2\langle [u, v], xy\rangle - \langle r, [\varphi(x)u, \varphi(y)v] + [\varphi(y)u, \varphi(x)v]\rangle ,\end{aligned}$$

which follows from (φ2). The equivalences of the conditions (α0), (α0)′, (α0)″, (α0)‴ are easy to see, and so is the equivalence (α2)$\Leftrightarrow$(α2)′. To prove the equivalence (α1)$\Leftrightarrow$(α1)′, it is sufficient to prove the following: $\varphi((\mathrm{ad}\, jx)^t y) = (\mathrm{ad}_{\mathscr{U}}\, jx)^t \circ \varphi(y) + \varphi(y) \circ (\mathrm{ad}_{\mathscr{U}}\, jx)$ on $\mathscr{U}$. Using (φ1) we see this as follows:

$$\begin{aligned}&\langle \varphi((\mathrm{ad}\, jx)^t y)u - (\mathrm{ad}\, jx)^t \varphi(y)u - \varphi(y)(\mathrm{ad}\, jx)u, v\rangle \\ &\quad = \langle y, [jx, [ju, v]] - [ju, [jx, v]] - [j[jx, u], v]\rangle \\ &\quad = \langle y, [jx, [ju, v]] - [ju, [jx, v]] - [[jx, ju], v]\rangle = 0\end{aligned}$$

for all $u, v \in \mathscr{U}$. The proof is completed.

We shall give an alternative proof of the following result of Dorfmeister [11] by using the theory of normal j-algebras.

THEOREM 3.7 ([11]). *If a normal j-algebra* $(\mathfrak{g}, j)$ *is quasi-symmetric, then the following holds*:

(A) *For every* $x \in \mathscr{L}$, $\varphi(x)$ *is associated to* $A(x)$.

Furthermore, the condition (A) *is equivalent to the quasi-symmetricity of* $(\mathfrak{g}, j)$, *provided that every strong j-ideal* $\tilde{\mathfrak{g}}$ *of* $(\mathfrak{g}, j)$ *satisfies* $\tilde{\mathscr{U}} \neq \{0\}$, *where* $\tilde{\mathfrak{g}} = \tilde{\mathscr{L}} + j\tilde{\mathscr{L}} + \tilde{\mathscr{U}}$.

To prove Theorem 3.7 we need some lemmas.

LEMMA 3.8. *For a normal j-algebra* $(\mathfrak{g}, j)$, *the condition* (A) *in Theorem* 3.7 *is equivalent to the totality of the following two conditions*:

(a1) *If* $x \in \mathfrak{n}_{ab}$, $y \in \mathfrak{n}_{cb}$, $a < c < b \leqq R$, *and* $u \in \mathfrak{n}_{a\infty}$, *then* $\varphi(y)\varphi(x)u = 2\varphi(xy)u$, *or* $\langle (\mathrm{ad}\, jx)^t u, (\mathrm{ad}\, jy)^t v\rangle = \langle [jx, y], [ju, v]\rangle$ *for all* $v \in \mathfrak{n}_{c\infty}$.

(a2) *If* $x \in \mathfrak{n}_{ab}$, $a < b \leqq R$, *and* $u \in \mathfrak{n}_{a\infty}$, *then* $\varphi(x)^2 u = (\langle x, x\rangle / 2\omega_a)u$, *or* $\langle (\mathrm{ad}\, jx)^t u, (\mathrm{ad}\, jx)^t v\rangle = \langle u, v\rangle \langle x, x\rangle / 2\omega_a$ *for all* $v \in \mathfrak{n}_{a\infty}$.

PROOF. By Lemma 3.6, (A) is equivalent to the following condition:

(3.2) $$2\varphi(xy)u=\varphi(x)\varphi(y)u+\varphi(y)\varphi(x)u \text{ for all } x\in\mathfrak{n}_{ab},\ y\in\mathfrak{n}_{cd}, \text{ and } u\in\mathfrak{n}_{e\infty} \text{ with } a\leqq b\leqq R,\ c\leqq d\leqq R,\ e\leqq R.$$

We notice that (3.2) is symmetric in x and y. If $a=b$ and $x=r_a$, then $2\varphi(xy)u=(\delta_{ac}+\delta_{ad})\varphi(y)u$. Since $\varphi(x)u=\delta_{ae}u$ and $\varphi(y)u\in\delta_{ce}\mathfrak{n}_{d\infty}+\delta_{de}\mathfrak{n}_{c\infty}$, we see that (3.2) a priori holds. If $\{ab\}\cap\{cd\}=\varnothing$, then (3.2) holds trivially. We assume that $a<b$, $c<d$, and $\#(\{ab\}\cap\{cd\})=1$. First suppose $b=c$. Then, $2xy=(\mathrm{ad}\,jx)y\in\mathfrak{n}_{ad}$. If $e\notin\{ad\}$, then (3.2) trivially holds. If $e=a$, then (3.2) becomes $(\mathrm{ad}\,j[jx,y])^t u=(\mathrm{ad}\,jy)^t(\mathrm{ad}\,jx)^t u$, which a priori holds (Lemma 1.2). If $e=d$, then (3.2) becomes $[j[jx,y],u]=[jx,[jy,u]]$, which also a priori holds (Lemma 1.2). Next, suppose $a=c$. Then, $2xy=(\mathrm{ad}\,jx)^t y\in\mathfrak{n}_{\{bd\}}$. As in the first case, if $e\notin\{bd\}$, then (3.2) holds trivially. Assume $e\in\{bd\}$, say $e=b$, without loss of generality. Then, (3.2) becomes $\varphi((\mathrm{ad}\,jx)^t y)u=(\mathrm{ad}\,jy)^t(\mathrm{ad}\,jx)u$, or $\langle(\mathrm{ad}\,jx)^t y,[ju,v]\rangle=\langle\varphi((\mathrm{ad}\,jx)^t y)u,v\rangle=\langle[jx,u],[jy,v]\rangle$ for all $v\in\mathfrak{n}_{d\infty}$ (by (φ1) in Lemma 3.4), which a priori holds (Lemma 1.2). Therefore, we have proved that the condition (3.2) for $\#(\{ab\}\cap\{cd\})=1$ is equivalent to (a1). Finally, we assume that $(a,b)=(c,d)$ with $a<b$. Then,

$$2\varphi(xy)u=\langle x,y\rangle(\varphi(r_a)u/\omega_a+\varphi(r_b)u/\omega_b),$$

so that (3.2) with $e=b$ a priori holds (Lemma 1.1), and that (3.2) for $\#(\{ab\}\cap\{cd\})=2$ is equivalent to (a2). The proof is completed.

LEMMA 3.9 ([8; item (4)]). *If $n_{b\infty}\neq 0$, then $\mathfrak{n}_{ab}=[\mathfrak{n}_{a\infty},\mathfrak{n}_{b\infty}]$ for all $a<b$. More precisely, $(\langle u,u\rangle/2\omega_b)x=[(\mathrm{ad}\,jx)ju,u]$ for all $x\in\mathfrak{n}_{ab}$ and $u\in\mathfrak{n}_{b\infty}$.*

PROOF OF THEOREM 3.7. In view of Theorem 2.7, we may assume that the condition (M) holds. We may further assume that $(\mathfrak{g},j)$ is simple and that the rank R of $(\mathfrak{g},j)$ is greater than one. Then, $n_{ab}=n_{12}>0$ for all $a<b\leqq R$, and $n_{a\infty}=n_{1\infty}$ for all $a\leqq R$, so that

(3.3) $$\omega_a=\omega_1 \quad \text{for all} \quad a\leqq R.$$

We shall first derive (a2) in Lemma 3.8. To do so, let $x\in\mathfrak{n}_{ab}$ with $a<b\leqq R$. Observing (3.3), by Lemma 1.1 we see that

(3.4) $$\langle[jx,v],[jx,v']\rangle=\langle x,x\rangle\langle v,v'\rangle/2\omega_1 \quad \text{for all} \quad v,v'\in\mathfrak{n}_{b\infty}.$$

This means that $(\mathrm{ad}\,jx)|_{\mathfrak{n}_{b\infty}}:\mathfrak{n}_{b\infty}\to\mathfrak{n}_{a\infty}$ is injective, so that it is an isomorphism. For every $u_0\in\mathfrak{n}_{a\infty}$, take $v_0\in\mathfrak{n}_{b\infty}$ such that $(\mathrm{ad}\,jx)v_0=u_0$. Then, by (3.4) we see that $\langle(\mathrm{ad}\,jx)^t u_0,v\rangle=\langle(\langle x,x\rangle/2\omega_1)v_0,v\rangle$ for all $v\in\mathfrak{n}_{b\infty}$. Since $(\mathrm{ad}\,jx)^t\mathfrak{n}_{a\infty}\subset\mathfrak{n}_{b\infty}$, it follows that

$$(\mathrm{ad}\,jx)^t|_{\mathfrak{n}_{a\infty}}=(\langle x,x\rangle/2\omega_1)((\mathrm{ad}\,jx)|_{\mathfrak{n}_{b\infty}})^{-1},$$

so that (3.3) yields (a2). To derive (a1) in Lemma 3.8, let $x\in\mathfrak{n}_{ab}-\{0\}$, $y\in\mathfrak{n}_{cb}-\{0\}$, $a<c<b\leqq R$, and $u\in\mathfrak{n}_{a\infty}$, $v\in\mathfrak{n}_{c\infty}$. Set $u'=\varphi(x)u\in\mathfrak{n}_{b\infty}$, $v'=\varphi(y)v\in\mathfrak{n}_{b\infty}$. Since (a2) holds, we have $\varphi(x)u'=(\langle x,x\rangle/2\omega_1)u$, and $\varphi(y)v'=(\langle y,y\rangle/2\omega_1)v$. Thus,

$$[ju, v]=(4\omega_1^2/\langle x, x\rangle\langle y, y\rangle)[j\varphi(x)u', \varphi(y)v'].$$

On the other hand, we see

$$[j\varphi(x)u', \varphi(y)v']=[j\varphi(x)u', [jy, v']]=[jy, [j\varphi(x)u', v']] \quad \text{(Jacobi's identity)}$$
$$=[j[j\varphi(x)u', v'], y].$$

By (d1) in Lemma 2.11, utilizing Theorem 2.7, we see

$$\langle[j[j\varphi(x)u', v'], y], [jx, y]\rangle=\langle[j\varphi(x)u', v'], x\rangle\langle y, y\rangle/2\omega_1$$
$$=\langle[jv', \varphi(x)u'], x\rangle\langle y, y\rangle/2\omega_1=\langle\varphi(x)v', \varphi(x)u'\rangle\langle y, y\rangle/2\omega_1 \quad \text{(Lemma 3.4 }(\varphi 1))$$
$$=\langle x, x\rangle\langle u', v'\rangle\langle y, y\rangle/(2\omega_1)^2 \quad \text{(Lemma 1.1)}.$$

Therefore, we get $\langle[ju, v], [jx, y]\rangle=\langle u', v'\rangle$, which is (a1). Thus, by Lemma 3.8 we conclude that (A) holds.

To prove the second assertion of Theorem 3.7, we assume that every simple strong j-ideal $\tilde{\mathfrak{g}}$ of $(\mathfrak{g}, j)$ satisfies $\tilde{\mathscr{U}}\neq\{0\}$, where $\tilde{\mathfrak{g}}=\tilde{\mathscr{L}}+j\tilde{\mathscr{L}}+\tilde{\mathscr{U}}$, and assume that (A) holds, i.e., (a1) and (a2) hold. By Lemma 1.4 we may assume that $(\mathfrak{g}, j)$ is simple. Then, $\mathscr{U}$ itself is non-zero. By (a2) we see that if $n_{ab}\neq 0$ then $n_{a\infty}=n_{b\infty}$. It follows from the simplicity of $(\mathfrak{g}, j)$ and from the fact $\mathscr{U}\neq\{0\}$ that

$$(3.5) \qquad n_{a\infty}=n_{1\infty}>0 \qquad \text{for all} \quad a\leqq R.$$

We shall derive (d1) and (d2) in Lemma 2.11. Let $x\in\mathfrak{n}_{ab}$, $y\in\mathfrak{n}_{cb}$ with $a<c<b\leqq R$. Observing (3.5), by Lemma 3.9 we can take $u\in\mathfrak{n}_{c\infty}$ and $v\in\mathfrak{n}_{b\infty}$ such that $y=[u, jv]$. Then, $[jx, y]=[u, j[jx, v]]$. It follows from (a1) that

$$(3.6) \qquad \langle(\operatorname{ad} jx)^t[jx, v], (\operatorname{ad} jy)^t u\rangle=\langle[jx, y], [j[jx, v], u]\rangle.$$

The right hand side of (3.6) becomes $-\langle[jx, y], [jx, y]\rangle$. The left hand side of (3.6) becomes

$$\langle\varphi(x)^2 v, \varphi(y)u\rangle=\langle\varphi(x^2)v, \varphi(y)u\rangle \quad \text{(by (A))}$$
$$=(\langle x, x\rangle/2\omega_b)\langle v, \varphi(y)u\rangle=(\langle x, x\rangle/2\omega_b)\langle y, [ju, v]\rangle \quad \text{(Lemma 3.4 }(\varphi 1))$$
$$=-\langle x, x\rangle\langle y, y\rangle/2\omega_b.$$

Thus, (d1) is established. To derive (d2), let $x\in\mathfrak{n}_{ab}$, $y\in\mathfrak{n}_{ac}$ with $a<c<b\leqq R$. Take $u\in\mathfrak{n}_{a\infty}$ and $v\in\mathfrak{n}_{c\infty}$ such that $y=[ju, v]$ (Lemma 3.9). Put $z=(\operatorname{ad} jx)^t y\in\mathfrak{n}_{cb}$. It follows from (a1) that

$$(3.7) \qquad \langle(\operatorname{ad} jx)^t u, (\operatorname{ad} jz)^t v\rangle=\langle[jx, z], [ju, v]\rangle.$$

The right hand side of (3.7) coincides with $\langle z, z\rangle$. The left hand side of (3.7) becomes

$$\langle\varphi(x)u, \varphi(z)v\rangle=2\langle\varphi(x)u, \varphi(xy)v\rangle=\langle\varphi(x)u, \varphi(x)\varphi(y)v\rangle \quad \text{(by (A))}$$
$$=\langle\varphi(x)^2 u, \varphi(y)v\rangle=\langle\varphi(x^2)u, \varphi(y)v\rangle \quad \text{(by (A))}$$

$$=(\langle x, x\rangle/2\omega_a)\langle u, \varphi(y)v\rangle=(\langle x, x\rangle/2\omega_a)\langle y, [jv, u]\rangle \quad \text{(Lemma 3.4 }(\varphi 1))$$
$$=\langle x, x\rangle\langle y, y\rangle/2\omega_a .$$

Thus, (d2) is established. By Lemma 2.14 as well as (3.5) we conclude that (M) holds. The proof of Theorem 3.7 is completed.

4. The holomorphic sectional curvature. In this section we consider the holomorphic sectional curvature of the Bergman metric on any homogeneous bounded domain D with the corresponding normal j-algebra $(\mathfrak{g}, j)$.

DEFINITION 4.1 ([12]). We define a quartic form $R(\ ,\ ,\ ,\ )$ on $\mathfrak{g}$ by

$$\begin{aligned} R(x_1, x_2, x_3, x_4) &= \langle R(x_3, x_4)x_2, x_1\rangle \\ &= \langle \nabla_{x_3}x_2, \nabla_{x_4}x_1\rangle - \langle \nabla_{x_4}x_2, \nabla_{x_3}x_1\rangle - \langle \nabla_{[x_3, x_4]}x_2, x_1\rangle \end{aligned}$$

for $x_a \in \mathfrak{g}$ $(a=1, \cdots, 4)$ (see § 1).

In § 3 we have extended the inner product $\langle\ ,\ \rangle$ to a unique complex symmetric bilinear form on $\mathfrak{g}^c$. We also extend $[\ ,\ ]$ and ∇ (resp. $R(\ ,\ )$) to complex bilinear mappings from $\mathfrak{g}^c \times \mathfrak{g}^c$ to $\mathfrak{g}^c$ (resp. to $\mathrm{End}(\mathfrak{g}^c)$). Similarly, we extend $R(\ ,\ ,\ ,\ )$ to a complex quartic form on $\mathfrak{g}^c$. We notice that

$$\overline{R(z_1, z_2, z_3, z_4)} = R(\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}) \tag{4.1}$$

for $z_a \in \mathfrak{g}^c$ $(a=1, \cdots, 4)$.

DEFINITION 4.2. We consider the holomorphic part $\mathfrak{g}^h=\{x\in\mathfrak{g}^c; jx=ix\}$ of $\mathfrak{g}^c$, which corresponds to the holomorphic tangent space T_pD at the point p via $\Phi_* \circ \rho$ (see § 1), and the natural mapping χ from $\mathfrak{g}$ onto $\mathfrak{g}^h$ given by $\chi(x)=(x-ijx)/2$, $x\in\mathfrak{g}$.

NOTATION 4.3. Set $L=\chi(\mathscr{L})$, $U=\chi(\mathscr{U})$, $L_{ab}=\chi(\mathfrak{n}_{ab})$, and $U_a=\chi(\mathfrak{n}_{a\infty})$. Thus, we have direct sum decompositions

$$\mathfrak{g}^h = L + U, \qquad L=\textstyle\sum_{a\leq b} L_{ab}, \quad U=\sum_a U_a . \tag{4.2}$$

We notice that $\langle\ ,\ \bar{}\ \rangle$ is an Hermitian inner product on $\mathfrak{g}^h$ satisfying $\langle \chi(x), \overline{\chi(x)}\rangle = \langle x, x\rangle/2$ for $x\in\mathfrak{g}$, and that the decompositions (4.2) are orthogonal with respect to $\langle\ ,\ \bar{}\ \rangle$. We note that $R(\overline{\chi(x_1)}, \overline{\chi(x_2)}, z_3, z_4)=0$, $R(z_3, z_4, \overline{\chi(x_1)}, \overline{\chi(x_2)})=0$ for all $x_1, x_2\in\mathfrak{g}$ and $z_3, z_4\in\mathfrak{g}^c$.

NOTATION 4.4. For $x_a\in\mathfrak{g}$ $(a=1, \cdots, 4)$ set

$$R_{x_1\overline{x_2}x_3\overline{x_4}} = R(\chi(x_1), \overline{\chi(x_2)}, \chi(x_3), \overline{\chi(x_4)}) .$$

One can easily see that

$$4R_{x_1\overline{x_2}x_3\overline{x_4}} = S(x_1, x_2, x_3, x_4) + iS(x_1, x_2, x_3, jx_4) , \tag{4.3}$$

where $S(x_1, x_2, x_3, x_4)=R(x_1, x_2, x_3, x_4)-R(x_1, jx_2, x_3, jx_4)$. We note that

$$R_{x_1\bar{x}_2x_3\bar{x}_4}=R_{x_3\bar{x}_2x_1\bar{x}_4}=R_{x_1\bar{x}_4x_3\bar{x}_2}\,. \tag{4.4}$$

We need the following result.

Lemma 4.5 ([4; Lemma 4.1]). *For $x_a\in\mathcal{L}$ and $u_b\in\mathcal{U}$, the following hold:*

(LL | UU) $$R_{x_1\bar{u}_2x_3\bar{u}_4}=0\,;$$

(LL | LU) $$R_{x_1\bar{u}_2x_3\bar{x}_4}=0\,;$$

(LU | UU) $$R_{u_1\bar{u}_2x_3\bar{u}_4}=0\,.$$

We have left Lemma 2.13 unproved and used it in § 2. Here, we shall prove both Lemma 2.13 and the following lemma together.

Lemma 4.6 *For $x_a\in\mathcal{L}$, it holds that*

(LL | LL) $$4R_{x_1\bar{x}_2x_3\bar{x}_4}=\langle x_1x_2, x_3x_4\rangle+\langle x_1x_4, x_2x_3\rangle-\langle x_1x_3, x_2x_4\rangle+\delta(x_1, x_2, x_3, x_4)$$

(*see Definition* 2.12).

Proof of Lemma 2.13 and 4.6. It was proved in [4; Lemma 4.2 (i)] that

$$4R_{x_1\bar{x}_2x_3\bar{x}_4}=\langle x_3x_2, [jx_4, x_1]\rangle+\langle x_1x_2, [jx_4, x_3]\rangle-\langle x_3x_1, (\operatorname{ad} jx_4)^t x_2\rangle\,.$$

It follows from the formulas $(\operatorname{ad} jx_4)=A(x_4)+D(x_4)$ and $(\operatorname{ad} jx_4)^t=A(x_4)-D(x_4)$ that

$$4R_{x_1\bar{x}_2x_3\bar{x}_4}=\langle x_1x_2, x_3x_4\rangle+\langle x_1x_4, x_2x_3\rangle-\langle x_1x_3, x_2x_4\rangle+\delta(x_4, x_1, x_2, x_3)\,. \tag{4.5}$$

We shall show Lemma 2.13 on the base of formula (4.5). From the definition of d it is trivial that $d(x_1, x_2, x_3)=d(x_1, x_3, x_2)$. By (4.5) we have

$$4R_{x_2\bar{x}_3x_4\bar{x}_1}=\langle x_2x_3, x_4x_1\rangle+\langle x_2x_1, x_3x_4\rangle-\langle x_2x_4, x_3x_1\rangle+\langle d(x_1, x_2, x_3), x_4\rangle\,.$$

Since $R_{x_2\bar{x}_3x_4\bar{x}_1}=R_{x_2\bar{x}_1x_4\bar{x}_3}$ (by (4.4)), it follows that $\langle d(x_1, x_2, x_3), x_4\rangle=\langle d(x_3, x_2, x_1), x_4\rangle$ for all $x_4\in\mathcal{L}$. Thus, $d(x_1, x_2, x_3)=d(x_3, x_2, x_1)$. Therefore, the symmetricity of d follows. It is seen by the definition of δ that $\delta(x_1, x_2, x_3, x_4)=\delta(x_1, x_2, x_4, x_3)$. Combining this with the assertion just proved, we get the symmetricity of δ. Lemma 2.13 is completely proved. Formula (LL | LL) in Lemma 4.6 is now obtained by (4.5) and the symmetricity of δ.

Lemma 4.7. *For $x_a\in\mathcal{L}$ and $u_b\in\mathcal{U}$, it holds that*

(LU | LU) $$4R_{x_1\bar{x}_2u_3\bar{u}_4}=-2\langle r, F(\varphi(x_1)u_3, \varphi(x_2)u_4)\rangle+4\langle x_1x_2, F(u_3, u_4)\rangle\,.$$

Proof. By (4.4) as well as (4.3) we have

$$\begin{aligned}4R_{x_1\bar{x}_2u_3\bar{u}_4}&=4R_{u_3\bar{x}_2x_1\bar{u}_4}\\&=2(\langle\nabla_{x_1}x_2, \nabla_{u_4}u_3\rangle-\langle\nabla_{u_4}x_2, \nabla_{x_1}u_3\rangle)+2i(\langle j\nabla_{x_1}x_2, \nabla_{u_4}u_3\rangle\\&\quad-\langle j\nabla_{u_4}x_2, \nabla_{x_1}u_3\rangle)\,.\end{aligned}$$

We have

$$\begin{aligned}
&2\langle \nabla_{x_1}x_2, \nabla_{u_4}u_3\rangle + 2i\langle j\nabla_{x_1}x_2, \nabla_{u_4}u_3\rangle \\
&\quad = \langle \nabla_{x_1}x_2, j[ju_4, u_3]\rangle + i\langle j\nabla_{x_1}x_2, [u_4, u_3]\rangle && \text{(by (1.7))} \\
&\quad = \langle x_1x_2, [ju_4, u_3]\rangle - i\langle x_1x_2, [u_4, u_3]\rangle && \text{(Definition 2.1)} \\
&\quad = 4\langle x_1x_2, F(u_3, u_4)\rangle && \text{(Definition 3.3)} .
\end{aligned}$$

We also obtain

$$\begin{aligned}
&2\langle \nabla_{u_4}x_2, \nabla_{x_1}u_3\rangle + 2i\langle j\nabla_{u_4}x_2, \nabla_{x_1}u_3\rangle = 2\langle \nabla_{x_2}u_4, \nabla_{x_1}u_3\rangle + 2i\langle j\nabla_{x_2}u_4, \nabla_{x_1}u_3\rangle \\
&\quad = 2^{-1}\langle \varphi(x_2)u_4, \varphi(x_1)u_3\rangle + 2^{-1}i\langle \varphi(x_2)ju_4, \varphi(x_1)u_3\rangle \\
&\quad = 2^{-1}\langle x_1, [ju_3, \varphi(x_2)u_4]\rangle + 2^{-1}i\langle x_1, [ju_3, \varphi(x_2)ju_4]\rangle && \text{(Lemma 3.4 } (\varphi 1)) \\
&\quad = 2\langle x_1, F(u_3, \varphi(x_2)u_4)\rangle = 2\langle r, F(\varphi(x_1)u_3, \varphi(x_2)u_4)\rangle && \text{(Lemma 3.4 } (\varphi 3)) .
\end{aligned}$$

Thus, we have the desired formula.

The following is well-known.

LEMMA 4.8 ([4; item (5.9)]). *For $u_b \in \mathscr{U}$, it holds that*

(UU | UU) $\qquad 4R_{u_1\overline{u_2}u_3\overline{u_4}} = 8\langle F(u_1, u_2), F(u_3, u_4)\rangle + 8\langle F(u_1, u_4), F(u_3, u_2)\rangle .$

For $z \in \mathfrak{g} - \{0\}$, the holomorphic sectional curvature $\mathrm{HSC}(\chi(z))$ of the direction $\chi(z) \in \mathfrak{g}^h - \{0\}$, or the sectional curvature of $\mathrm{span}_{\boldsymbol{R}}\{z, jz\}$, is given by

$$\mathrm{HSC}(\chi(z)) = R(z, jz, z, jz)/\langle z, z\rangle\langle jz, jz\rangle = -R_{z\bar{z}z\bar{z}}/\langle \chi(z), \overline{\chi(z)}\rangle^2 = -4R_{z\bar{z}z\bar{z}}/\langle z, z\rangle^2$$

(see [13], [3]). Concerning this we shall show the following.

THEOREM 4.9. *For $z = x + jy + u \in \mathfrak{g}$ with $x, y \in \mathscr{L}$ and $u \in \mathscr{U}$, the following holds:*

$$\begin{aligned}
4R_{z\bar{z}z\bar{z}} &= 4\langle x^2, y^2\rangle + \langle (x+y)^2, (x-y)^2\rangle + \delta(x, y) + 8\langle F(\varphi(x-iy)u, \varphi(x-iy)u), r\rangle \\
&\quad + 16(\langle F(u, u), x^2\rangle - \langle F(\varphi(x)u, \varphi(x)u), r\rangle) + 16(\langle F(u, u), y^2\rangle \\
&\quad - \langle F(\varphi(y)u, \varphi(y)u), r\rangle) + 16\langle F(u, u), F(u, u)\rangle ,
\end{aligned}$$

where $\delta(x, y) = \delta(x, x, x, x) + 2\delta(x, x, y, y) + \delta(y, y, y, y)$, and $\varphi(x + iy)u = \varphi(x)u + j\varphi(y)u$.

PROOF. By Lemma 4.5 we see

$$4R_{z\bar{z}z\bar{z}} = 4R_{x+jy\,\overline{x+jy}\,x+jy\,\overline{x+jy}} + 16R_{x+jy\,\overline{x+jy}\,u\bar{u}} + 4R_{u\bar{u}u\bar{u}} .$$

Since $R_{x_1\bar{x}_2x_3\bar{x}_4} \in \boldsymbol{R}$ for all $x_a \in \mathscr{L}$, we have

$$4R_{x+jy\,\overline{x+jy}\,x+jy\,\overline{x+jy}} = 4R_{x\bar{x}x\bar{x}} + 4R_{y\bar{y}y\bar{y}} + 16R_{x\bar{x}y\bar{y}} - 8R_{y\bar{x}y\bar{x}} .$$

It follows from Lemma 4.6 that

$$\begin{aligned}
&4R_{x+jy\overline{x+jy}x+jy\overline{x+jy}}\\
&=(\langle x^2, x^2\rangle+\delta(x,x,x,x))+(\langle y^2, y^2\rangle+\delta(y,y,y,y))\\
&\quad+4(\langle x^2, y^2\rangle+\delta(x,x,y,y))-2(2\langle xy, xy\rangle-\langle x^2, y^2\rangle+\delta(x,x,y,y))\\
&=\langle x^2, x^2\rangle+6\langle x^2, y^2\rangle+\langle y^2, y^2\rangle-4\langle xy, xy\rangle+\delta(x,y)\\
&=4\langle x^2, y^2\rangle+\langle (x+y)^2, (x-y)^2\rangle+\delta(x,y)\,.
\end{aligned}$$

By Lemma 4.7 we see

$$\begin{aligned}
&4R_{x+jy\overline{x+jy}u\bar{u}}=4R_{x\bar{x}u\bar{u}}+4R_{y\bar{y}u\bar{u}}+8\,\mathrm{Im}R_{x\bar{y}u\bar{u}}\\
&=-2\langle F(\varphi(x)u, \varphi(x)u), r\rangle+4\langle F(u,u), x^2\rangle\\
&\quad-2\langle F(\varphi(y)u, \varphi(y)u), r\rangle+4\langle F(u,u), y^2\rangle-4\langle \mathrm{Im}\,F(\varphi(x)u, \varphi(y)u), r\rangle\\
&=2\langle F(\varphi(x-iy)u, \varphi(x-iy)u), r\rangle+4(\langle F(u,u), x^2\rangle-\langle F(\varphi(x)u, \varphi(x)u), r\rangle)\\
&\quad+4(\langle F(u,u), y^2\rangle-\langle F(\varphi(y)u, \varphi(y)u), r\rangle)\,.
\end{aligned}$$

The last equality follows from the fact $F(v-jw, v-jw)=F(v,v)+F(w,w)-2\,\mathrm{Im}\,F(v,w)$ for $v, w\in\mathscr{U}$. Finally, by Lemma 4.8 we have $4R_{u\bar{u}u\bar{u}}=16\langle F(u,u), F(u,u)\rangle$. This completes the proof.

COROLLARY 4.10 (Zelow [21], [22]). *If* $(\mathfrak{g}, j)$ *is quasi-symmetric, then for* $z=x+jy+u$ *with* $x, y\in\mathscr{L}$ *and* $u\in\mathscr{U}$, *it holds that*

$$\begin{aligned}
4R_{z\bar{z}z\bar{z}}=&4\langle x^2, y^2\rangle+\langle (x+y)^2, (x-y)^2\rangle+8\langle F(\varphi(x-iy)u, \varphi(x-iy)u), r\rangle\\
&+16\langle F(u,u), F(u,u)\rangle\,.
\end{aligned}$$

PROOF. Assume $(\mathfrak{g}, j)$ is quasi-symmetric. Then, Theorem 2.7 implies that $D(x)$ is a derivation of $(\mathscr{L}, \cdot)$, so that $\delta(x,y)=0$ in the formula in Theorem 4.9. Furthermore, Theorem 3.7 and Lemma 3.6 imply that $\langle F(u,u), x^2\rangle=\langle F(\varphi(x)u, \varphi(x)u), r\rangle$ etc. Therefore, the desired formula follows from Theorem 4.9.

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Department of Mathematics
Faculty of Science
Toyama University
Gofuku, Toyama 930
Japan