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DUALITY OF CLOSED S-DECOMPOSABLE OPERATORS

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1. Introduction. There are many efforts to extend the decomposable operator theory on a complex Banach space, which was introduced by Colojoară and Foiaş [3]. Bacalu [2] discovered S-decomposability for a restriction of a decomposable operator. Subsequently, many authors studied the S-decomposable operator theory and obtained many characterizations of bounded S-decomposable operators, which are similar to those of decomposable operators. Hence, now, it seems a natural problem to seek appropriate characterizations of a closed S-decomposable operators. (cf. Nagy [8], [10], Wang [16] and Wang and Erdelyi [17].)

In this paper we obtain several characterizations of a closed S-decomposable operator (Theorem 1), which generalize the results due to the author [12], [13] and Radjabalipour [11], etc.

After this, we consider the duality theorem. Vasilescu [15] proved that if T is a densely defined closed S-decomposable operator, then T^* is also an S-decomposable operator. Conversely, Wang and Liu [18] proved that if the dual operator T^* of $T \in B(X)$ is an S-decomposable operator, then T is also an S-decomposable operator. Using Theorem 1, we prove the duality theorem of a closed S-decomposable operator (Theorem 2) by a way similar to that of [18]. We remark that Theorem 2 is partly proved by Erdelyi and Wang [4], who proved the case $S=\{\infty\}$. (cf. Erdelyi and Wang [5] and Lange [6]).

2. Preliminaries. Let X be a complex Banach space. Let $C(X)$ (resp. $B(X)$) be the family of all closed (resp. bounded) linear operators on X. $\boldsymbol{C}$ is the complex plane and $\bar{\boldsymbol{C}}=\boldsymbol{C}\cup\{\infty\}$ is its one-point compactification.

$D(T)$ is the domain of $T\in C(X)$ and $\sigma_e(T)$ is its extended spectrum, i.e., $\sigma_e(T)=\sigma(T)$ if $T\in B(X)$ and $\sigma_e(T)=\sigma(T)\cup\{\infty\}$, otherwise. A closed subspace Y of X is an invariant subspace of T if $T(Y\cap D(T))\subset Y$. $I(T)$ is the family of all invariant subspaces of T. $T\,|\,Y$ is the restriction of T to $Y\in I(T)$ with domain $D(T\,|\,Y)=Y\cap D(T)$. T/Y is the quotient operator induced by T on X/Y with domain $D(T/Y)=\{\hat{x}\in X/Y\,|\,\hat{x}\cap D(T)\neq\emptyset\}$, i.e., $(T/Y)\hat{x}=(Tx)^\wedge$ for $x\in\hat{x}\cap D(T)$, where $\hat{x}=x+Y\in X/Y$ is the coset of $x\in X$. Ω_T is the maximal open set with the property that if $\omega\subset\Omega_T$ is open and if $f:\omega\to D(T)$ is an analytic function such that $(z-T)f(z)=0$ for $z\in\omega\cap\boldsymbol{C}$, then $f(z)=0$ for $z\in\omega$. Let $S_T=\bar{\boldsymbol{C}}\setminus\Omega_T$. For a closed set $F\subset\bar{\boldsymbol{C}}$, we denote by $X_T(F)$ the set of points $x\in X$ such that there exists an analytic function $f:\bar{\boldsymbol{C}}\setminus F\to D(T)$ with $(z-T)f(z)=x$ for $z\in\boldsymbol{C}\setminus F$.

For any set $E \in \bar{C}$, we denote by $X_T(E)$ the union of $X_T(F)$ for all closed sets $F \subset E$. We call $X_T(F)$, $X_T(E)$ spectral manifolds of X. We remark that $X_T(E) = X_T(E \cap \sigma_e(T))$ for all subsets $E \in \bar{C}$. These definitions of spectral manifolds are due to Radjabalipour [11] and different from those of [4], [7] and [14], etc. But if $S_T \subset E$, then they are equivalent.

For a closed set $F \subset \bar{C}$, let $X(T, F)$ be an invariant subspace of T such that (1) $\sigma_e(T|X(T, F)) \subset F$ and that (2) if $Y \in I(T)$ satisfies $\sigma_e(T|Y) \subset F$, then $Y \subset X(T, F)$. Naturally, $X(T, F)$ may or may not exists, but if such an invariant subspace exists, then it is obviously unique. We call $X(T, F)$ a spectral maximal space of T. $SM(T)$ is the family of all spectral maximal spaces of T.

Let $S \subset \bar{C}$ be a closed set. A family of open sets $\{G_1, \cdots, G_n; G_0\}$ is called an S-covering of $\sigma_e(T)$ if $\sigma_e(T) \cup S \subset G_1 \cup \cdots \cup G_n \cup G_0$ and $\bar{G}_i \cap S = \varnothing$ for $i = 1, \cdots, n$. $T \in C(X)$ is called an S-decomposable operator if for every S-covering $\{G_1, \cdots, G_n; G_0\}$ of $\sigma_e(T)$, there exists a family $\{X_1, \cdots, X_n; X_0\}$ of spectral maximal spaces of T such that (1) $X = X_1 + \cdots + X_n + X_0$ with $X_1, \cdots, X_n \subset D(T)$ and that (2) $\sigma_e(T|X_i) \subset G_i$ for $i = 1, \cdots, n, 0$. If $T \in C(X)$ is S-decomposable, then T is $S \cap \sigma_e(T)$-decomposable. Hence we may assume $S \subset \sigma_e(T)$. If $T \in C(X)$ is S-decomposable and if $\infty \notin S$, then $T \in B(X)$.

3. Main results. We always assume $\sigma_e(T) \neq \bar{C}$ in this paper. We need some lemmas to prove Theorem 1. Lemma 1 is due to Nagy [8, Lemma 2].

LEMMA 1. *Let $T \in C(X)$. If $F \subset \bar{C}$ is a closed set with $S_T \subset F$ and if $X_T(F)$ is closed, then $X_T(F) = X(T, F)$ and $\sigma_e(T | X_T(F)) \subset F \cap \sigma_e(T)$.*

Lemma 2 is a modification of Nagy [10, Lemma, 3.1], but it plays an essential role in this paper. Although [10, Lemma 3.1] used the assumption $Y \subset D(T)$, we can prove Lemma 2 without this assumption.

LEMMA 2. *Let $T \in C(X)$ and $Y \in I(T)$. Let $\sigma_e(T) \cup \sigma_e(T|Y) \neq \bar{C}$. Let $D = (D(T), \| \ \|_T)$ denote the linear manifold $D(T)$ endowed with the graph norm $\|x\|_T = \|x\| + \|Tx\|$. Then D is a Banach space and $Y \cap D(T)$ is closed in D. Moreover, T/Y is closed and $(D(T/Y), \| \ \|_{T/Y}) \simeq D/(Y \cap D(T))$.*

The proof of Lemma 3 is similar to that of [12, Lemma 2] by Lemma 2.

LEMMA 3. *Let $T \in C(X)$, $T \in I(T)$ and $x \in X$. If $\hat{x} \in (X/Y)_{T/Y}(F)$ for a closed set $F \subset \bar{C}$, then $x \in X_T(F \cup \sigma_e(T|Y) \cup S_T)$.*

LEMMA 4. *Let $T \in C(X)$ and $x \in X_T(F)$ for some closed set $F \subset \bar{C}$, i.e., there exists an analytic function $f: \bar{C} \setminus F \to D(T)$ such that $(z - T)f(z) = x$ for $z \in C \setminus F$. Then $f(z) \in X_T(F)$ for $z \in \bar{C} \setminus F$.*

PROOF. If $z = \infty$, then $f(z) = 0$ by the same argument as in the proof of Lemma

3. The rest of the proof is similar to that of [14, Proposition 2.2]. q.e.d.

The proof of Lemma 5 is similar to that of [12, Lemma 3] by Lemma 4.

LEMMA 5. *Let $T \in C(X)$. If $X_T(H) = X(T, H)$ for all closed sets H with $S \subset H$, then $X_T(F) = X(T, F)$ for all closed sets F with $F \cap S = \emptyset$.*

The proof of Lemma 6 is routine, but the formulation seems most general. (cf. [1].)

LEMMA 6. *Let $T \in C(X)$. If F_1 and F_2 are disjoint closed sets, then $X_T(F_1 \cup F_2) = X_T(F_1) + X_T(F_2)$.*

THEOREM 1. *Let $T \in C(X)$ and $S \subset \bar{C}$ be a closed set with $\infty \in S$. Then the following assertions are equivalent.*

(1) *T is S-decomposable.*

(2) *$X_T(F) = X(T, F)$ for all closed sets F with $S \subset F$ and $X_T(G_1 \cup G_0) = X_T(G_1) + X_T(G_0)$ for all open sets G_1, G_0 with $\bar{G}_1 \cap S = \emptyset$ and $S \subset G_0$.*

(3) *$X_T(F) = X(T, F)$, $T/X_T(F)$ is closed and $\sigma_e(T/X_T(F)) \subset (\bar{C} \setminus F^i) \cup S$ for all closed sets F with $S \subset F$. (F^i is the interior of F.)*

(4) *For all open sets G with $S \subset G$, there exists $Y \in I(T)$ such that $\sigma_e(T \mid Y) \subset \bar{G}$, T/Y is closed and $\sigma_e(T/Y) \subset (\bar{C} \setminus G) \cup S$.*

PROOF. The proof of Theorem 1 is similar to that of [12]. We prove the implications (1)→(3)→(4)→(2)→(1).

(1)→(3). Let T be S-decomposable and F be a closed set with $S \subset F$. Nagy [8] proved that $S_T \subset S$ and $X_T(F) = X(T, F)$. Since $X_T(F) = X_T(F \cap \sigma_e(T))$, we obtain $\sigma_e(T) \cup \sigma_e(T \mid X_T(F)) \subset \sigma_e(T) \neq \bar{C}$. Hence $T/X_T(F)$ is closed by [9, Lemma 3].

The rest of the proof is similar to that of [12] by Lemma 4.

(3)→(4). Let $Y = X_T(\bar{G})$.

(4)→(2). We can prove that $S_T \subset S$ by a similar argument as in [12].

Next we prove that $X_T(F) = X(T, F)$ for all closed sets F with $S \subset F$. By Lemma 1, we have only to prove that $X_T(F)$ is closed. Let G be any open set with $F \subset G$. Then there exists $Y \in I(T)$ such that $\sigma_e(T \mid Y) \subset \bar{G}$, T/Y is closed and $\sigma_e(T/Y) \subset (\bar{C} \setminus G) \cup S$. Since $S \subset G$, we can write $X/Y = Z_1 \oplus Z_0$ where $Z_1, Z_0 \in I(T/Y)$, $\sigma_e((T/Y) \mid Z_1) = \sigma_e(T/Y) \cap (\bar{C} \setminus G) \subset (\bar{C} \setminus G)$ and $\sigma_e((T/Y) \mid Z_0) = \sigma_e(T/Y) \cap S \subset S$. Let P_i be the projection of X/Y onto Z_i along Z_j for $i \neq j$. Let $x \in X_T(F)$. Then there exists an analytic function f: $\bar{C} \setminus F \to D(T)$ such that $(z - T)f(z) = x$ for $z \in C \setminus F$. Since P_i commutes with T/Y, we obtain $P_i\widehat{f(z)} \in D(T/Y)$ for $i = 1, 0$. Then we can write

$$(z - T/Y)\widehat{f(z)} = (z - U_1)g_1(z) \oplus (z - U_0)g_0(z) = \hat{x}_1 \oplus \hat{x}_0 = \hat{x}$$

for $z \in C \setminus F$ where $U_i = (T/Y) \mid Z_i$, $g_i(z) = P_i\widehat{f(z)}$ and $\hat{x}_i = P_i\hat{x}$ for $i = 1, 0$. Then $\hat{x}_1 \in Z_{1U_1}(F) = Z_{1U_1}(F \cap \sigma_e(U_1)) = Z_{1U_1}(\emptyset) = \{\hat{0}\}$. Hence $\hat{x} = \hat{x}_0 \in Z_0$. Hence $X_T(F) \subset \Pi^{-1}(Z_0)$ where $\Pi : X \to X/Y$ is the canonical mapping. Since $\sigma_e((T/Y) \mid Z_0) \subset S$, we obtain $Z_0 \subset X/Y_{T/Y}(S)$, and hence $\Pi^{-1}Z_0 \subset X_T(S \cup \sigma_e(T \mid Y) \cup S_T) \subset X_T(\bar{G})$ by Lemma 3. Since G

is any open set with $F\subset G$, we obtain $X_T(F)\subset\bigcap\Pi^{-1}(Z_0)\subset\bigcap\{X_T(G)\,|\,G$ is open and $F\subset G\}=X_T(F)$ because $S_T\subset S\subset F$. Hence $X_T(F)=\bigcap\Pi^{-1}Z_0$, and hence $X_T(F)$ is closed.

The rest of the proof is similar to that of [12] by Lemmas 3 and 6.

(2)→(1). The proof is similar to that of [12] by Lemma 5. q.e.d.

REMARK. Theorem 1 is proved partly by many authors. (See introduction.) We remark that Wang [16] proved the implication (1)→(2) implicitly. Also [16] proved that if $T\in C(X)$ is S-decomposable and if G is an open set with $\bar{G}\cap S=\varnothing$, then $T/X_T(\bar{G})$ is closed and $\sigma_e(T/X_T(\bar{G}))\subset\bar{C}\setminus G$. (cf. [13].)

Next we prove the duality theorem of closed S-decomposability (Theorem 2). For the proof of Theorem 2, we need some lemmas. We assume some density conditions in the following, i.e.,

(*) $D(T)$ is dense,
(**) $D(T)$ and $D(T^*)$ are dense,
(***) $D(T)$, $D(T^*)$ and $D(T^{**})$ are dense,
(****) $D(T)$, $D(T^*)$, $D(T^{**})$ and $D(T^{***})$ are dense,
(*****) $D(T)$, $D(T^*)$, $D(T^{**})$, $D(T^{***})$ and $D(T^{****})$ are dense.

LEMMA 7. *Let $T\in C(X)$ be an S-decomposable operator with* (*). *Then $X_T(G)^\perp=X^*_{T^*}(C\setminus G)$ for all open sets G with $S\subset G$ or $\bar{G}\cap S=\varnothing$.*

PROOF. Vasilescu [15, Proposition 2.9] proved the case $\bar{G}\cap S=\varnothing$ implicitly. We prove the case G is an open set with $S\subset G$. We may assume $S\subset\sigma_e(T)$. If $\infty\notin S$, then $T\in B(X)$, and hence $X_T(G)^\perp=X^*_{T^*}(\bar{C}\setminus G)$ by [13]. Hence we may assume $\infty\in S$.

We prove $X_T(G)^\perp\subset X^*_{T^*}(\bar{C}\setminus G)$. Let $x^*\in X_T(G)^\perp$. We can write $G=\bigcup H_\alpha$ where H_α are the components of G. Let $S_\alpha=S\cap H_\alpha$. Then S_α is closed and $\{\alpha\,|\,S\cap H_\alpha\neq\varnothing\}$ is finite. We write

$$S=S_1\cup\cdots\cup S_n\cup S_0\quad\text{where}\quad S_k=\varnothing\quad\text{and}\quad\infty\in S_0\,.$$

Then there exist connected open sets H_k', H_k'' such that $S_k\subset H_k'$, $\overline{H_k'}\subset H_k''$. $\overline{H_k''}\subset H_k$ for $k=1,\cdots,n,0$. Let $G_0=\bigcup H_k''$ and $G_1=\bigcap(\bar{C}\setminus\overline{H_k'})$. Then $\{G_1;G_0\}$ is an S-covering of $\sigma_e(T)$, and hence there exist $X_1, X_0\in SM(T)$ such that $X=X_1+X_0$, $X_1\subset D(T)$ and $\sigma_e(T\,|\,X_i)\subset G_i$ for $i=1,0$. Let $x\in X$. Then we can write $x=x_1+x_0$ for some $x_i\in X_i$ for $i=1,0$. Since $x_1\in X_1\subset X_T(\bar{G})$, there exists an analytic function $f\colon\bigcup H_k'\to D(T)$ such that $(z-T)f(z)=x_1$ for $z\in(\bigcup H_k')\cap C$ because $S_T\subset S$ and H_k' is connected.

We define $(g(z))(x)=x^*(f(z))$ for $z\in\bigcup H_k'$. Then we can prove that $g(z)\in X^*$ for $z\in\bigcup H_k'$, and also $g(z)((z-T)x)=x^*(x)$ for $z\in(\bigcup H_k')\cap C$ and for $x\in D(T)$ by an argument similar to that of [15, Proposition 2.9] by Lemma 4.

Also $g(\infty)=0\in D(T^*)$. This implies that $x^*\in X^*_{T^*}(\bar{C}\setminus(\bigcup H_k'))$, hence $x^*\in X^*_{T^*}(\bar{C}\setminus G)$. Thus $X_T(G)^\perp\subset X^*_{T^*}(\bar{C}\setminus G)$. The converse inclusion is easy. q.e.d.

Let $T\in C(X)$, $Y\in I(T)$ and $\Omega\subset\bar{C}$ be an open set. We say Y is Ω-analytically invariant under T if $f(z)\in Y$ for $z\in\omega$ for all open sets $\omega\subset\Omega$ and for all analytic functions $f\colon\omega\to D(T)$ which satisfy $(z-T)f(z)\in Y$ for $z\in\omega\cap\boldsymbol{C}$. We say T has Ω-svep if $\Omega\subset\Omega_T$. The proof of Lemma 8 is similar to that of [18, Proposition 2.1] by Lemma 2.

LEMMA 8. *Let $T\in C(X)$, $Y\in I(T)$ and $\sigma_e(T)\cup\sigma_e(T\,|\,Y)\neq\bar{\boldsymbol{C}}$. Then Y is Ω-analytically invariant under T if and only if T/Y has Ω-svep.*

Let $J\colon X\to X^{**}$ and $K\colon X^*\to X^{***}$ be the canonical embeddings. Then $X^{***}=KX^*\oplus(JX)^{\perp}$ (cf. [18].) Let P be the projection of X^{***} onto KX^* along $(JX)^{\perp}$.

LEMMA 9. *Let $T\in C(X)$ with $(****)$ and let $\Omega\subset\bar{\boldsymbol{C}}$ be an open set. If T^{****} has Ω-svep, then JX is Ω-analytically invariant under T^{**}.*

PROOF. [18, theorem 2.2] proved that $(X^{**}/JX)^*\simeq(JX)^{\perp}=N(P)$, $(X^{***}/KX^*)^*\simeq(KX^*)^{\perp}=N(P^*)$ and $X^{***}/KX^*\simeq(JX)^{\perp}$ where $N(P)$ (resp. $N(P^*)$) denotes the null space of P (resp. P^*). Since P commutes with T^{***} by [4, Lemma 2.6], $T^{***}\,|\,N(P)$ is densely defined. Hence $(T^{**}/JX)^*$ is similar to $T^{***}\,|\,N(P)$ by [4, Lemma 2.2]. (We remark that the assumption $Y\subset D(T)$ in [4, Lemma 2.2] is unnecessary.) The rest of the proof is similar to that of [18, Corollary 2.3] by Lemma 8. q.e.d.

LEMMA 10. *Let $T\in C(X)$ with $(****)$ and let $S\subset\bar{\boldsymbol{C}}$ be a closed set. Let T^{****} have $(\bar{\boldsymbol{C}}\setminus S)$-svep and let F be a closed set with $S\subset F$ or $F\cap S=\varnothing$. Then $JX_T(F)=X^{**}_{T^{**}}(F)\cap JX$.*

PROOF. First we prove the case $S\subset F$. Let $x\in X$ and $Jx\in X^{**}_{T^{**}}(F)\cap JX$. Then there exists an analytic function $f\colon\bar{\boldsymbol{C}}\setminus F\to D(T^{**})$ such that $(z-T^{**})f(z)=Jx$ for $z\in\boldsymbol{C}\setminus F$. We can write $\bar{\boldsymbol{C}}\setminus F=\bigcup G_\alpha$ where G_α are the components of $\bar{\boldsymbol{C}}\setminus F$. Then $G_\alpha\cap(\bar{\boldsymbol{C}}\setminus S)\neq\varnothing$. Since $S_{T^{****}}\subset S$, JX is $(\bar{\boldsymbol{C}}\setminus S)$-analytically invariant under T^{**} by Lemma 9. Hence $f(z)\in JX\cap D(T^{**})=JD(T)$ for $z\in\boldsymbol{C}\setminus F$ by [4, Theorem 2.7]. The rest of the proof is similar to that of [18, Corollary 2.4].

Next we prove the case $F\cap S=\varnothing$. We can prove that $f(z)\in JD(T)=JX\cap D(T^{**})$ for $z\in G_\alpha\cap(\bar{\boldsymbol{C}}\setminus S)$ similarly as above, hence $f(z)\in JD(T)$ for $z\in G_\alpha$. The rest of the proof is similar to the case $S\subset F$. q.e.d.

LEMMA 11. *Let $T\in C(X)$ with $(*****)$ and let $S\subset\bar{\boldsymbol{C}}$ be a closed set. Let T^{*****} have $(\bar{\boldsymbol{C}}\setminus S)$-svep and let F be a closed set with $S\subset F$ or $F\subset S=\varnothing$. Then $KX^*_{T^*}(F)=PX^{***}_{T^{***}}(F)$.*

PROOF. Since P commutes with T^{***} by [4, Lemma 2.6], the proof of Lemma 11 is similar to that of [18, Corollary 2.5]. q.e.d.

The proof of following lemma is similar to that of [18, Proposition 2.6].

LEMMA 12. *Let $T\in C(X)$ with $(*****)$ and let $S\subset\bar{\boldsymbol{C}}$ be a closed set. Let T^* be S-decomposable and let F be a closed set with $S\subset F$ or $S\cap F=\varnothing$. Then $X^*_{T^*}(F)$ is closed in the w^*-topology.*

THEOREM 2. *Let $T \in C(x)$ with $(*****)$ and let $S \subset \bar{\boldsymbol{C}}$ be a closed set. If T^* is S-decomposable, then T is S-decomposable.*

PROOF. We prove that T satisfies the condition (4) of Theorem 1. If $\infty \notin S$, then $T^* \in B(X^*)$ and T is S-decomposable by [18, Theorem 2.7]. Hence we may assume $\infty \in S$. Let G be an open set with $S \subset G$. Let $H = \bar{\boldsymbol{C}} \setminus \bar{G}$ and let $Y = {}^{\perp}X^*_{T^*}(\bar{H}) = \{x \in X \mid x \perp X^*_{T^*}(\bar{H})\}$. Then $\bar{H} \cap S = \varnothing$ and $X^*_{T^*}(\bar{H}) = X^*(T^*, \bar{H}) \subset D(T^*)$ by Lemmas 1 and 5. Hence $Y \in I(T)$. Also $Y^{\perp} = X^*_{T^*}(\bar{H})$ and $Y^* = X^*/Y^{\perp} = X^*/X^*_{T^*}(\bar{H})$ by Lemma 12. First we prove that $\sigma_e(T \mid Y) \subset \bar{\boldsymbol{C}} \setminus H = \bar{G}$. Let V be an open ball with $V \subset H$. Let $z \in V$. We prove that $z - T \mid Y$ is bijective. Since there exist open sets G_1, G_0 such that $V \subset G_1$, $\bar{G} \subset H$, $\bar{V} \cap \bar{G}_0 = \varnothing$ and $\{G_1; G_0\}$ is an S-covering of $\sigma_e(T^*)$, we obtain $X^* = X^*_{T^*}(\bar{G}_1) + X^*_{T^*}(\bar{G}_0)$.

We prove that $z - T \mid Y$ is injective. Let $y \in Y \cap D(T)$ and $(z - T \mid Y)y = 0$. Let $x^* \in X^*$. Then we can write $x^* = x_1^* + x_0^*$ for some $x_i^* \in X^*_{T^*}(\bar{G}_i)$ for $i = 1, 0$. Since $X^*_{T^*}(\bar{G}_1) \subset X^*_{T^*}(\bar{H})$, we obtain

$$\begin{aligned}\langle y, x^* \rangle &= \langle y, x_1^* + x_0^* \rangle = \langle y, x_0^* \rangle = \langle y, (z - T^*)(z - T^* | X^*_{T^*}(\bar{G}_0))^{-1} x_0^* \rangle \\ &= \langle (z - T)y, (z - T^* \mid X^*_{T^*}(\bar{G}_0))^{-1} x_0^* \rangle = 0\,,\end{aligned}$$

and hence $y = 0$.

We prove that $z - T \mid Y$ is surjective. Let $y \in Y$. Let $x^* \in X^*$. Then we can write $x^* = x_1^* + x_0^*$ for some $x_i^* \in X^*_{T^*}(\bar{G}_i)$ for $i = 1, 0$. We define

$$(\varphi(z))(x^*) = \langle y, (z - T^* | X^*_{T^*}(\bar{G}))^{-1} x_0^* \rangle\,.$$

We prove that $\varphi(z) \in X^{**}$. If $x^* = x_1^* + x_0^* = a_1^* + a_0^*$ for some $a_i^* \in X^*_{T^*}(\bar{G}_i)$, then

$$x_1^* - a_1^* = a_0^* - x_0^* \in X^*_{T^*}(\bar{G}_1) \cap X^*_{T^*}(\bar{G}_0) = X^*_{T^*}(\bar{G}_1 \cap \bar{G}_0)$$

because $S_{T^*} \subset S \subset G_0$. Hence there exists an analytic function f: $\bar{\boldsymbol{C}} \setminus (\bar{G}_1 \cap \bar{G}_0) \to D(T^*)$ such that

$$(\lambda - T^*) f(\lambda) = x_1^* - a_1^* = a_0^* - x_0^* \qquad \text{for} \quad \lambda \in \boldsymbol{C} \setminus (\bar{G}_0 \cap \bar{G}_1)\,.$$

Since

$$f(\lambda) \in X^*_{T^*}(\bar{G}_1 \cap \bar{G}_0) \subset X^*_{T^*}(\bar{H}) \qquad \text{for} \quad \lambda \in \bar{\boldsymbol{C}} \setminus (\bar{G}_1 \cap \bar{G}_0)$$

by Lemma 4 and

$$f(\lambda) = (\lambda - T^* \mid X^*_{T^*}(\bar{G}_0))^{-1}(a_0^* - x_0^*) \qquad \text{for} \quad \lambda \in \boldsymbol{C} \setminus \bar{G}_0\,,$$

we obtain

$$(\lambda - T^* \mid X^*_{T^*}(\bar{G}_0))^{-1}(a_0^* - x_0^*) \in X^*_{T^*}(\bar{H}) \qquad \text{for} \quad \lambda \in V\,.$$

Hence,

$$\langle y, (z - T^* \mid X^*_{T^*}(\bar{G}_0))^{-1}(a_0^* - x_0^*) \rangle = 0\,,$$

and hence $\varphi(z)$ is well-defined. By Banach's theorem, there exists an $M>0$ such that $\|x_0^*\|\le M\|x^*\|$ for all $x^*\in X^*$. Hence,

$$|x(z)(x^*)|=|\langle y,(z-T^*|X_{T^*}^*(\bar{G}_0))^{-1}x_0^*\rangle|\le\|y\|\,\|(z-T^*|X_{T^*}^*(\bar{G}_0))^{-1}\|M\|x^*\|,$$

and hence $\varphi(z)\in X^{**}$. Let $x^*\in D(T^*)$. Then we can write $x^*=x_1^*+x_0^*$ for some $x_i^*\in X_{T^*}^*(\bar{G}_i)$ for $i=1,0$. Since $X_{T^*}^*(\bar{G}_1)\subset D(T^*)$, we obtain $x_0^*=x^*-x_1^*\in D(T^*)$. Then

$$\langle\varphi(z),(z-T^*)x^*\rangle=\langle y,x_0^*\rangle=\langle y,x_1^*+x_0^*\rangle=\langle y,x^*\rangle=\langle Jy,x^*\rangle.$$

Hence $\varphi(z)\in D(T^{**})$ and $(z-T^{**})\varphi(z)=Jy$. Since T^{**} is S-decomposable by [15, Theorem 2.10], T^{**} has $(\bar{C}\setminus S)$-svep. Hence JX is $(\bar{C}\setminus S)$-analytically invariant under T^{**} by Lemma 7, and hence $\varphi(z)\in JX\cap D(T^{**})=JD(T)$ by [4, Theorem 2.7]. Hence $J^{-1}\varphi(z)\colon V\to D(T)$ is an analytic function such that $(z-T)J^{-1}\varphi(z)=y$ for $z\in V$. We can prove that $J^{-1}\varphi(z)\in Y$ by an argument similar to the proof of $\varphi(z)\in X^{**}$. This implies that $z-T|Y$ is surjective.

Since it is easy to prove the inclusion $\sigma_e(T\,|\,Y)\subset\sigma_e(T)$ because $X_{T^*}^*(\bar{H})\subset D(T^*)$, T/Y is closed by [9, Lemma 3].

Finally we prove that $\sigma_e(T/Y)\subset\bar{C}\setminus G$. Since $X_{T^*}^*(\bar{H})\subset D(T^*)$ and T/Y is closed, we can prove that

$$\sigma_e(T/Y)=\sigma_e((T/Y)^*)=\sigma_e(T^*\,|\,Y^{\perp})=\sigma_e(T^*\,|\,X_{T^*}^*(\bar{H}))\subset\bar{H}\subset\bar{C}\setminus G$$

similarly to the proof of [4, Lemma 2.2]. q.e.d.

REMARK. If $T\,|\,Y$ is densely defined, i.e., $\overline{Y\cap D(T)}=Y$, then $\sigma_e(T|\,Y)=\sigma_e((T|\,Y)^*)=\sigma_e(T^*/X_{T^*}^*(\bar{H}))\subset\bar{C}\setminus H\subset\bar{G}$ by [16, Theorem 2.6]. But we do not know whether $\overline{Y\cap D(T)}=Y$. Also we doubt whether the assumption (*****) is necessary, but it seems a difficult problem.

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