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LOGARITHMIC DEL PEZZO SURFACES WITH RATIONAL DOUBLE AND TRIPLE SINGULAR POINTS

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CONTENTS

Introduction. Throughout the present article, we work on an algebraically closed field k of characteristic zero. Whenever we consider problems of topological nature, we assume k to be the complex field $\mathbf{C}$.

DEFINITION 1. A logarithmic del Pezzo surface (henceforth called log del Pezzo surface, for short) $\bar{V}$ with contractible boundary is a projective normal algebraic surface satisfying:

(i) $\bar{V}$ is singular but has at most quotient singularities.

(ii) The anti-canonical divisor $-K_{\bar{V}}$ is ample.

$\bar{V}$ is said to have rank one if the Picard number $\rho(\bar{V})$ of $\bar{V}$ is equal to one.

Let $g: V \to \bar{V}$ be a minimal resolution of singularities of $\bar{V}$, $D := g^{-1}(\mathrm{Sing}\ \bar{V})$ and $V^0 := \bar{V} - \mathrm{Sing}(\bar{V}) = V - D$. We often denote (V, D) and $\bar{V}$ interchangeably (cf. [7]). A general theory on the structure of such singular surfaces is developed in Zhang [11]. When $\bar{V}$ with $\rho(\bar{V}) = 1$ admits only rational double points, we studied topological properties of $\bar{V} - \mathrm{Sing}(\bar{V})$ in Miyanishi-Zhang [9]. In the present article, we consider a special class of such surfaces admitting singularities of higher multiplicity. Namely, we consider a class specified in the following:

DEFINITION 2. Let $\bar{V}$ be a log del Pezzo surface of rank one with contractible boundary. $\bar{V}$ or (V, D) is called a dP3-surface if $\bar{V}$ has no singular points other than rational double points and a unique rational triple point.

In §2∼§5, we apply the results in [11] and classify all dP3-surfaces. In §6, we compute $H_1(V^0; \boldsymbol{Z})$ and $\pi_1(V^0)$. Let U^0 be the universal covering of V^0, which is an algebraic surface because it turns out that $\pi_1(V^0)$ is finite. We let $\bar{U}$ be the normalization of $\bar{V}$ in $k(U^0)$ and call $\bar{U}$ the quasi-universal covering of $\bar{V}$. We give an explicit method to construct $\bar{U}$. Some examples show that $\pi_1(V^0)$ is not necessarily abelian, contrary to the case admitting only rational double points (cf. [9]). Our main result is the following:

MAIN THEOREM. *Let $\bar{V}$ be a* dP3-*surface. In the previous notation, we have*:

(I) *There are altogether* 97 *singularity types of* dP3-*surfaces, each of which is realizable and given in terms of the dual graph of D in a table* (see Appendix). *We call this table just the Table, and a singularity type given with classifying number n in the Table just "the singularity of No. n.".*

(II) *Suppose (V, D) is not isomorphic to (Σ_3, M_3). Then we can find a (-1)-curve C and a $\boldsymbol{P}^1$-fibration $\Psi: V \to \boldsymbol{P}^1$ in such a way that $0 < -(C, D^{\#}+K_V) \leq -(E, D^{\#}+K_V)$ for every irreducible curve E on V which is not a component of D, and that the configuration of $C+D$ as well as all singular fibers of Ψ can be explicitly described. The configuration is given in Appendix, as the configuration (n) if $n \neq 15$, 18 and as the configuration (na) or (nb) if $n=15$, 18.*

(III) *$\pi_1(V^0)$ is a finite group which is not necessarily abelian, and the quasi-universal covering $\bar{U}$ of $\bar{V}$ is a rational log del Pezzo surface. The fundamental group $\pi_1(V^0)$ and the singularities of $\bar{U}$ are given in the Table together with other data.*

(IV) *Suppose $\pi_1(V^0)=(0)$. Then V^0 contains $\boldsymbol{C} \times \boldsymbol{C}^*$ as a Zariski open set, where $\boldsymbol{C}^* := \boldsymbol{C} - \{0\}$.*

(V) *Suppose $\pi_1(V^0) \neq (0)$. Then $\bar{V}$ is a quotient of $\boldsymbol{P}^2$ by a finite subgroup H of $PGL(2, \boldsymbol{C})$ if and only if the Picard number $\rho(\bar{U})=1$. If this is the case, then there exists a cyclic normal subgroup H_1 of H such that $H/H_1 \cong \pi_1(V^0)$ and $\boldsymbol{P}^2/H_1 \cong \bar{U}$.*

It remains to consider the following problem:

(*) *For a given singularity type, say of No. n, how many isomorphism classes of* dP3-*surfaces are there with the given singularity type*?

A singularity type does not necessarily determine uniquely the isomorphism class of a dP3-surface. Indeed, if we consider the singularity of No. n ($n=15$ or 18), there are two dP3-surfaces $\bar{V}(na)$ and $\bar{V}(nb)$ corresponding to the configurations (na) and (nb), respectively, such that Sing $\bar{V}(na)$ and Sing $\bar{V}(nb)$ are given as the same singularity type of No. n, while $\bar{V}(na)$ is not isomorphic to $\bar{V}(nb)$. For the proof, see the argument in Lemma 2.5. However, there is a result which suggests that a singularity type may determine uniquely the isomorphism class of a dP3-surface in the cases considered in §5. Namely, suppose that a dP3-surface (V, D) has a (-1)-curve E which meets a (-2)-curve D_1 and the unique (-3)-curve D_2 in D with $(E, D)=(E, D_1+D_2)=2$. Let $\eta: V \to W$ be the blowing-down of E. Then $\eta(D-D_1)$ is contractible to rational double singular points on a Gorenstein log del Pezzo surface $\bar{W}$ of rank one by Lemma 4.2. In [9; Lemma 7], it is proved that unless Sing $\bar{W}$ consists of two singular points of Dynkin type (D_4), $\bar{W}$ is uniquely determined by Sing $\bar{W}$ up to isomorphisms. But we do not know yet whether or not such a (-1)-curve E as above is unique.

TERMINOLOGY. A $(-n)$-curve is a nonsingular rational curve with self-intersection number $-n$. A (-2)-rod (resp. (-2)-fork) is a rod (resp. fork) whose irreducible components are (-2)-curves. For the definitions of rods, twigs, forks, admissible rods, rational forks, Bk(D), $D^{\#}$, we refer to Miyanishi-Tsunoda [7; pp. 203–204, pp. 211–213]. A (-2)-rod (resp. (-2)-fork) corresponds to the exceptional locus of a minimal resolution of a rational double point of Dynkin type A_n (resp. D_n, E_6, E_7 or E_8). A reduced effective divisor D is called an NC (resp. SNC) divisor if D has only normal (resp. simple normal) crossings. An irreducible component of D is called a $(-n)$-component if it is a $(-n)$-curve. A surface V^0 is said to be affine-uniruled (resp. affine-ruled) if there exists a dominant morphism (resp. an open immersion) $\varphi: \boldsymbol{A}^1 \times R \to V^0$, where R is an affine curve. We often denote $\boldsymbol{A}^1$ by $\boldsymbol{C}$ when the ground field $k=\boldsymbol{C}$. Given a $\boldsymbol{P}^1$-fibrataion Ψ: $V \to \boldsymbol{P}^1$, an irreducible curve B is called an n-section of Ψ if $n \geq 2$ and $(B, L)=n$ for a general fiber L of Ψ.

NOTATION.

$\#(D)$:	the number of irreducible components in D
$\rho(V)$:	Picard number of V
K_V:	canonical divisor of V
$q(V)$:	irregularity of V
$p_a(A)$:	arithmetic genus of an irreducible curve A
$h^i(D)$:	$\dim H^i(V, D)$
$D \sim D'$:	D and D' are linearly equivalent divisors
$D \equiv D'$:	D and D' are numerically equivalent divisors
(D, D'):	the intersection number of two divisors D and D'
$\lvert D \rvert$:	complete linear system defined by D
$\Phi_{\lvert D \rvert}$:	the rational map $V \cdots \to \boldsymbol{P}^{\dim \lvert D \rvert}$ defined by $\lvert D \rvert$
f^*D:	the total transform of a divisor D by a morphism f
$f'D$:	the proper transform of a divisor D by a morphism f
(Σ_n, M_n):	Σ_n is a Hirzebruch surface of degree n and M_n is a minimal section
$\otimes$, o, $*$:	stand for a (-1)-curve, a (-2)-curve and a (-3)-curve, respectively (see Figures (10)$\sim$(12) and the Table)
$*$—o^n—$*$:	a rod (i.e., a linear chain) consisting of two (-3)-curves (as tips) and n (-2)-curve (see the Table)
$(-n)$:	a $(-n)$-curve in the dual graph of the exceptional divisor coming from a resolution of Sing($\bar{U}$) (see the Table)
$A_l+D_m+(-n)$:	disjoint union of Dynkin types A_l and D_m and a $(-n)$-curve (see Lemma 5.1 and the Table).

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1. Preliminary results. We employ the notation in the Introduction. Let (V, D) be a log del Pezzo surface with contractible boundary. There is a natural number N such that for any Weil divisor $\bar{F}$ on $\bar{V}$, $N\bar{F}$ is linearly equivalent to a Cartier divisor. Thus, for Weil divisors $\bar{F}_1$ and $\bar{F}_2$ on $\bar{V}$ one can define the intersection number by $(\bar{F}_1, \bar{F}_2) := (1/N^2)(g^*N\bar{F}_1, g^*N\bar{F}_2)$. One can also define the direct image g_*F of a divisor F on V as usual, and if $F_1 \sim F_2$ on V then $g_*F_1 \sim g_*F_2$. For relevant results, we refer to Artin [1; Cor. 2.6] and [7; Lemma 2.4].

Let (V, D) be a dP3-surface and let Δ be a connected component of D. Since any singular point of $\bar{V}$ is a quotient singularity, the dual graph of Δ is as described in Brieskorn [3; Satz 2.10]. On the other hand, the dual graph of the exceptional divisor of a minimal resolution of a rational double (or triple) singular point is given in Artin [2; p. 135]. Combining these results, we know all possibilities of the dual graph of Δ. In particular, $g(\Delta)$ is a rational double point if and only if Δ is either a (-2)-rod or a (-2)-fork, and $g(\Delta)$ is a rational triple point if and only if Δ consists of only one (-3)-curve and several (-2)-curves and the dual graph of Δ is either a rod or a fork.

LEMMA 1.1 *Let (V, D) be a log del Pezzo surface with contractible boundary. Then we have*:

(1) $g^*K_{\bar{V}} \equiv D^\# + K_V$ *and* $-(D^\# + K_V, F) \geq 0$ *for any curve F where the inequality becomes an equality if and only if F is a component of D. Moreover,* $\rho(V) = \#(D) + \rho(\bar{V})$.

(2) *Any $(-n)$-curve with $n \geq 2$ is a component of D. Hence if (V, D) is a* dP3-*surface, there are a unique (-3)-curve in D and no other $(-n)$-curves on V with $n \geq 3$.*

(3) *If* $\rho(\bar{V}) = 1$ *then* $\mathrm{Pic}(\bar{V}) \cong H^2(\bar{V}; \boldsymbol{Z}) \cong \boldsymbol{Z}$ *and V is rational.*

PROOF. (1) For the first assertion, see [7; Lemma 2.5]. The second follows from the first and the hypothesis that $-K_{\bar{V}}$ is ample. The last assertion is obvious.

(2) Suppose that a $(-n)$-curve E, with $n \geq 2$, is not a component of D. Then $(E, K_V) \leq (E, D^\# + K_V) < 0$ by (1) and hence $(E^2) = -2 - (E, K_V) \geq -1$. This is a contradiction.

(3) For the first assertion we refer to [9; Lemma 1]. Let P be a natural number such that $PD^\#$ is an integral divisor. By (1), we have $0 = h^0(nP(D^\# + K_V)) \geq h^0(nPK_V)$ for any $n > 0$. Hence $\kappa(V) = -\infty$ and there is a $\boldsymbol{P}^1$-fibration Φ: $V \to B$ onto a nonsingular curve B with genus $q(V)$. If $q(V) > 0$, then every component D_i of D which is rational is contained in a fiber of Φ. Let H be a section of Φ. Then H, a fiber f of Φ, and D_i's are numerically independent. Hence $\rho(V) \geq \#(D) + 2$. This is absurd by (1) and by the hypothesis that $\rho(\bar{V}) = 1$. Thus, $\kappa(V) = -\infty$ and $q(V) = 0$. Hence V is rational. q.e.d.

By Lemma 1.1, (1), if C is an irreducible curve not contained in D, $-(C, D^\# + K_V)$

takes value in $(1/P)\boldsymbol{N}:=\{n/P\mid n\in\boldsymbol{N}\}$, where P is a natural number such that $PD^{\#}$ is an integral divisor. So, we can find an irreducible curve C such that $-(C, D^{\#}+K_V)$ attains the smallest positive value.

DEFINITION 1.2. Let (V, D) be a log del Pezzo surface with contractible boundary. (V, D) is said to be of the first kind if there exists an irreducible curve C such that $|C+D+K_V|\neq\emptyset$ and that $-(C, D^{\#}+K_V)$ attains the smallest positive value. (V, D) is said to be of the second kind if (V, D) is not of the first kind, i.e., if $|C+D+K_V|=\emptyset$ for each irreducible curve C for which $-(C, D^{\#}+K_V)$ attains the smallest positive value.

LEMMA 1.3. *Let $\bar{V}$ be a projective normal algebraic surface with only quotient singularities. Let $g: V\to\bar{V}$ be a minimal resolution of singularities of $\bar{V}$ and let $D=g^{-1}(\mathrm{Sing}\,\bar{V})$. Then we have*:

(1) *Assume that $\rho(\bar{V})=1$. Then for any (-1)-curve E on V and any birational morphism $\sigma: V\to W$ with W nonsingular, not every connected component of $\sigma_*(E+D)$ is an admissible rational rod or fork.*

(2) *Assume that V is rational and that any $(-n)$-curve with $n\geq 2$ is a component of D. (These assumptions are satisfied by* dP3*-surfaces by Lemma* 1.1*). Let $\Phi: V\to\boldsymbol{P}^1$ be a $\boldsymbol{P}^1$-fibration. Then $\rho(V)-\#(D)-1+\#\{$irreducible components of D not contained in any fiber of $\Phi\}=1+\sum_f\{\#((-1)$-curves in $f)-1\}$, where f moves over all singular fibers of Φ. If a singular fiber f contains only (-1)-curves and (-2)-curves then the dual graph of f is of type* (I) *or* (II) *in Figure* (1).

(I)

(II)

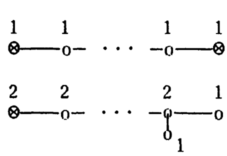


FIGURE (1)

In Figure (1), $\otimes$ (*resp.* o) *stands for a* (-1)*-curve* (*resp.* (-2)*-curve*) *and each number is the multiplicity of the corresponding curve in f.*

(3) *With the notation and assumptions in* (2), *assume further that $\bar{V}$ is a* dP3*-surface. If there is a singular fiber f of type* (II) *whose (-1)-curve E satisfies that $-(E, D^{\#}+K_V)$ attains the smallest positive value, then the unique (-3)-curve is not contained in any fiber and every (-1)-curve E' contained in a singular fiber f' satisfies $-(E', D^{\#}+K_V)=-(E, D^{\#}+K_V)$.*

PROOF. (1) If $\sigma_*(E+D)$ consists of admissible rational rods or forks, i.e., if $\sigma_*(E+D)$ is contractible to quotient singularities, then $\#(D)+1=\rho(V)\geq\#\{$irreducible components of $E+D$ contracted by $\sigma\}+\rho(W)\geqq\#\{$irreducible components of $E+D$ contracted by $\sigma\}+\#(\sigma_*(E+D))+1=\#(D)+2$. This is absurd.

(2) By contracting components of singular fibers of Ψ, we can blow down V to

a Hirzebruch surface Σ_n. Since $\rho(\Sigma_n)=2$, one verifies the first assertion. For the second assertion, one just argues by induction on $\#(f)$.

(3) Suppose that the (-3)-curve of D is in a fiber of Φ, say f_1. Then the sum of coefficients of all (-1)-curves in f_1 is greater than 2 by [11; Lemma 1.6] and hence $-2(E, D^{\#}+K_V)=-(f, D^{\#}+K_V)=-(f_1, D^{\#}+K_V)\geq -3(E, D^{\#}+K_V)$ by the minimality of $-(E, D^{\#}+K_V)$. This is absurd. Thus, the (-3)-curve is transversal to Φ and every singular fiber f' has type (I) or (II). Hence, by $-2(E, D^{\#}+K_V)=-(f', D^{\#}+K_V)$ and by the minimality of $-(E, D^{\#}+K_V)$, one verifies the second assertion. q.e.d.

By the definition and the computation of $D^{\#}$ given in [7; p. 213], one verifies straightforwardly the following:

Lemma 1.4. *Let (V, D) be a* dP3-*surface and let Δ be the connected component of D containing the (-3)-curve. Then according to the dual graph of Δ, one has the following results.*

(1) $$D^{\#}=\frac{s-i+1}{(s+1)+i(s-i+1)}D_1+\cdots+\frac{(s-i+1)(i-1)}{(s+1)+i(s-i+1)}D_{i-1}+\frac{(s-i+1)i}{(s+1)+i(s-i+1)}D_i$$
$$+\frac{(s-i)i}{(s+1)+i(s-i+1)}D_{i+1}+\cdots+\frac{i}{(s+1)+i(s-i+1)}D_s,$$

if Δ has the dual graph as shown in Figure (2).

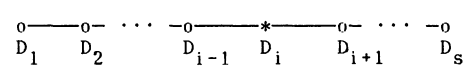


FIGURE (2)

(2) $$D^{\#}=\frac{1}{i+1}D_1+\cdots+\frac{i-1}{i+1}D_{i-1}+\frac{i}{i+1}D_i+\cdots+\frac{i}{i+1}D_{s-2}$$
$$+\frac{i}{2(i+1)}D_{s-1}+\frac{i}{2(i+1)}D_s,$$

if Δ has the dual graph as shown in Figure (3).

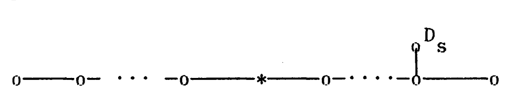


FIGURE (3)

(3) $$D^{\#}=\frac{2}{7}D_1+\frac{4}{7}D_2+\frac{6}{7}D_3+\frac{4}{7}D_4+\frac{2}{7}D_5+\frac{3}{7}D_6,$$

if Δ has the dual graph as shown in Figure (4).

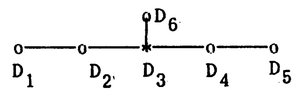


FIGURE (4)

We also use the following result.

LEMMA 1.5. *Let $\Phi\colon V\to \boldsymbol{P}^1$ be a $\boldsymbol{P}^1$-fibration on a nonsingular projective rational surface V. Then we have:*

(1) *Suppose that there are two cross-sections H_1 and H_2 of Φ, where H_1 is a (-2)-curve. Let $v\colon V\to\Sigma_2$ be a contraction of all (-1)-curves and consecutively contractible curves in singular fibers so that $(v_*H_1)^2=-2$. Then $(v_*H_2)^2=2+2(H_1, H_2)$.*

(2) *Let D be an* SNC *divisor on V. If the following three conditions are satisfied then $V-D$ is affine-ruled:*

(i) *There are two cross-sections H_1 and H_2 of Φ contained in D such that $D-H_1-H_2$ is contained in fibers, and $(H_1, H_2)=0$ (resp. 1);*

(ii) *For every fiber f, except at most two (resp. one), say f_k's where $k\le 2$ (resp. $k\le 1$), one of H_1 and H_2 meets a component of f not in D;*

(iii) *If f_1 and f_2 (resp. f_1) as above exist, then one of f_1 and f_2 (resp. f_1), say f_1, is a singular fiber, and H_1 and H_2 meet different connected components of the reduced effective divisor formed by all common components in f_1 and D.*

(3) *Suppose that there is an irreducible rational curve H such that H is a 2-section of Φ and that $p_a(H)\le 1$. Then there are at most two (resp. one) singular fibers of type* (II) *in* (2) *of Lemma* 1.3 *of which H meets only in the (-1)-curves if H is nonsingular or nodal (resp. cuspidal). Let $v\colon V\to\Sigma_n$ be a contraction of all (-1)-curves and consecutively contractible curves in singular fibers. Then $(v_*H)^2=4m$ for some $m\ge 1$, and $(v_*H, v_*F)\equiv 0$* (mod 2) *for any 2-section F of Φ. Moreover, if $n=2$ and v_*H does not meet the minimal section of Σ_2, then $(v_*H)^2=8$.*

PROOF. (1) Since H_1 does not meet any curve contracted by v, one has $(v_*H_1, v_*H_2)=(H_1, H_2)$ and $v_*H_2\sim v_*H_1+(2+(H_1, H_2))L$, where L is a general fiber of $\Phi\circ v^{-1}\colon \Sigma_2\to\boldsymbol{P}^1$. Thence follows (1).

(2) The affine-ruledness of $V-D$ can be proved in the same way as in [11; Lemma 3.3] where the case $(H_1, H_2)=0$ is treated. Indeed, for the case $(H_1, H_2)=1$, we only need to replace the claim there by the following:

CLAIM. Let A_1 and A_2 be two cross-sections of a $\boldsymbol{P}^1$-fibration $\pi\colon\Sigma_m\to\boldsymbol{P}^1$ such that $(A_1, A_2)=1$. Then we have $A_1+A_2+L+K_{\Sigma_m}\sim 0$ with a general fiber L of π.

Since this can be easily verified, we omit the proof.

(3) Note that if f is a fiber of type (II) and if H meets only the (-1)-curve in the fiber f then $f\cap H$ is a ramification point of $\Phi_{|H}$ and that $\#\{$ramification points of $\Phi_{|H}\}=2$ provided H is nonsingular. If H is singular, by extending $\Phi_{|H}\colon H\to\boldsymbol{P}^1$ to $\tilde{\Phi}\colon\tilde{H}\to\boldsymbol{P}^1$ where $\tilde{H}$ is the normalization of H, one verifies the first assertion. The second assertion

follows from the fact that v_*H and v_*F are 2-sections of $\Phi \circ v^{-1}: \Sigma_n \to \boldsymbol{P}^1$. The last assertion is obvious. q.e.d.

We can now explain very roughly what we are going to do in the subsequent sections. Given any dP3-surface (V, D), we shall find below a $\boldsymbol{P}^1$-fibration $\Psi: V \to \boldsymbol{P}^1$, which must satisfy the following conditions on singular fibers. First of all, by (2) of Lemma 1.1, each singular fiber of Ψ consists of (-1)-curves, (-2)-curves or the unique (-3)-curve. Secondly, there are exactly five possible types of the singular fiber containing the (-3)-curve which are described in [11; Lemma 1.6]. Thirdly, there are exactly two possible types, type (I) and type (II) in (2) of Lemma 1.3, of singular fibers consisting only of (-1)-curves or (-2)-curves. The divisor D will consist of irreducible components of singular fibers, cross-sections and 2-sections of the fiberation Ψ. An explicit configuration of D is given in Appendix, where the $\boldsymbol{P}^1$-fibration is given vertically. We can compute $\pi_1(V^0)$ or construct the quasi-universal covering $\bar{U}$ of $\bar{V}$ only by making use of the $\boldsymbol{P}^1$-fibration Ψ.

Conversely, starting with a minimal ruled surface Σ_m $(m \leq 3)$ and blowing up points on fibers of the $\boldsymbol{P}^1$-fibration, we can produce a $\boldsymbol{P}^1$-fibration Ψ with singular fibers as specified as above. In this way, we can produce a dP3-surface with any singularity type.

The rest of the present section is a preparation for the study of dP3-surfaces of the first kind. First of all, we need the following:

DEFINITION 1.6 (cf. [10], [11]). A pair (V, D) of a nonsingular projective rational surface V and a reduced effective divisor D on V is called a quasi-Iitaka surface if D admits a decomposition into integral divisors $D = A + N$, such that $A > 0$, $N \geq 0$, $A + K_V \sim 0$ and N consists of (-2)-rods or (-2)-forks.

Furthermore, if A is an SNC divisor, we call the pair (V, D) an Iitaka surface.

Given a quasi-Iitaka surface (V, D), we can consider smooth contractions of the following two types:

(*A*) the contraction of an irreducible component of the part A,

(*B*) the contraction of a rod $E + R$, where E is a (-1)-curve, R (might be zero) is a connected component of the part N, and E does not meet connected components of N other than R.

It is easy to show that if $u: V \to W$ is a birational morphism which is a composite of smooth contractions of the above type (*A*) or (*B*), then $(W, u_*(D))$ is again a quasi-Iitaka surface. We call a quasi-Iitaka surface (V, D) *minimal* if no further contractions of type (*A*) or (*B*) are possible on (V, D).

LEMMA 1.7. *Let (V, D) be a* dP3-*surface of the first kind with a curve C as in the Definition* 1.2. *Then the following assertions hold true.*

(1) *There exists a unique decomposition of D into effective integral divisors $D = D' + D''$ such that*:

(i) $C+D''+K_V \sim 0$;

(ii) $(C, D_i)=(D'', D_i)=(K_V, D_i)=0$ *for any component* D_i *of* D'. *Hence* $(V, (C+D'')+D')$ *is a quasi-Iitaka surface.*

(2) *C is a nonsingular rational curve and* $(C, D'')=2$. *Moreover, either* $C+D''$ *is an* SNC *rational loop or* $\#(D'')\leq 2$ *and* $C+D''$ *has no intersection except at a single point common to all components.* D'' *is the connected component of* D *containing the unique* (-3)*-curve. Furthermore,* $\mathrm{Supp}(D^{\#})=\mathrm{Supp}(D'')$ *and* $(C, D^{\#})>0$.

(3) *We have* $(K_V^2)\geq 0$ *and* $\#(D)=\rho(V)-1=9-(K_V^2)\leq 9$.

PROOF. (1) is proved in [11; Lemma 2.1].

(2) If $D''=0$ then $D=D'$, where D' consists of (-2)-curves. This is not the case since (V, D) is a dP3-surface. In view of (1), we can list up all possible configurations of $C+D''$. (Use Miyanishi [6; Lemma 2.1.3]). In particular, D'' is connected and $(C, D'')=2$. Hence D'' is the connected component of D containing the unique (-3)-curve because $D'\cap D''=\varnothing$ by (1). By the definition of $D^{\#}$, the coefficient of a component D_i in $D^{\#}$ is zero if and only if D_i is contained in a connected component of D which is a (-2)-rod or (-2)-fork (cf. [7; § 1.5]). Hence $\mathrm{Supp}(D^{\#})=\mathrm{Supp}(D'')$. Thus $(C, D^{\#})>0$ because $(C, D'')=2$.

(3) Let α be the coefficient of the (-3)-curve in $D^{\#}$. Then $0<\alpha<1$ by the definition of $D^{\#}$. Hence $0<(K_{\bar{V}}{}^2)=(D^{\#}+K_V)^2=(D^{\#}+K_V, D^{\#})+(K_V^2)+(D^{\#}, K_V)=(K_V^2)+\alpha<(K_V^2)+1$ and $(K_V^2)\geq 0$ (cf. (1) of Lemma 1.1). q.e.d.

The following proposition is proved in [11; Th 3.1].

PROPOSITION 1.8. *Let* (V, D) *be a* dP3*-surface of the first kind. Then there exist an irreducible curve* C *and a birational morphism* $u: V\to V_*$ *such that* $|C+D+K_V|\neq\varnothing$ *and* $-(C, D^{\#}+K_V)$ *attains the smallest positive value and that the following assertions hold true*:

(1) *D is decomposed into* $D=D'+D''$ *such that* $(V, (C+D'')+D')$ *is a quasi-Iitaka surface as in Lemma* 1.7, *and u is a composite of smooth contractions of type* (A) *or* (B) *such that if* $A_*:=u_*(C+D'')$ *and* $N_*:=u_*(D')$ *then* (V_*, A_*+N_*) *is a minimal quasi-Iitaka surface.*

(2) *Each smooth contraction of type* (B) *constituting u has the exceptional divisor, i.e.,* $E+R$ *in the above notation, disjoint from* (*the image of*) C.

(3) *One of the following three cases takes place*:

CASE (X). $V_*\cong P^2$ *or* Σ_m $(m\geq 0)$. A_* *is an* NC *divisor and* $N_*=0$.

CASE (Y). *There is a* $\boldsymbol{P}^1$*-fibration* $\Phi: V_*\to \boldsymbol{P}^1$ *such that* A_* *consists of one* 2*-section* H *and one nonsingular fiber* l *with* $H\cap l=$ *two points, and that the components of* N_* *are contained in fibers of* Φ.

CASE (Z). A_* *is a singular irreducible curve with* $p_a(A_*)=1$.

(4) *Let t be the number of contractions of type* (B) *involved in* u. *Then* $t=\#(A_*)+\#(N_*)-\rho(V_*)$. *In CASE* (Y) *and CASE* (Z) *one has* $t=0$.

2. dP3-surfaces of the first kind. We shall classify dP3-surfaces of the first kind. For this porpose, we divide them into three types by making use of Proposition 1.8.

DEFINITION 2.1. Let (V, D) be a dP3-surface of the first kind. (V, D) is said to be of type (Ic) if there exist a curve C and a birational morphism u so that CASE (Z) in Proposition 1.8 takes place. (V, D) is said to be of type (Ib) if there exist a curve C and a birational morphism u so that CASE (Y) in Proposition 1.8 takes place but (V, D) is not of type (Ic). (V, D) is said to be of type (Ia) if there exist a curve C and a birational morphism u so that CASE (X) in Proposition 1.8 takes place but (V, D) is neither of type (Ic) nor type (Ib).

The following is the main result of the present section.

THEOREM 2.2. *Let (V, D) be a* dP3*-surface of the first kind, which is not isomorphic to (Σ_3, M_3). Then the following assertions hold*:

(1) *The dual graph of D (i.e., the singularity type of $\bar{V}$), is one of those given in the cases No. n in the Table with $2 \leq n \leq 27$.*

(2) *We can take a (-1)-curve as the curve C considered in Proposition* 1.8, *and find a $\boldsymbol{P}^1$-fibration $\Psi: V \to \boldsymbol{P}^1$ such that the configuration of $C+D$ as well as all singular fibers of Ψ is given in the configuration (n) if $n \neq 15$, 18 and in the configuration (na) or (nb) if $n=15$, 18 (see Appendix). In particular, all components of D with at most three exceptions are contained in singular fibers of Ψ.*

(3) *All the cases $(2 \leq n \leq 27)$ are realizable.*

(4) *$V^0 := V - D$ is affine-ruled if $n=2$, 3, 4, 8, 9, 12, 13, 15, 18.*

The proof of Theorem 2.2 consists of the subsequent three lemmas. Throughout this section, we assume that (V, D) is not isomorphic to (Σ_3, M_3).

LEMMA 2.3. If (V, D) *is of type* (Ia) *then all the assertions in Theorem 2.2 with $n=3$, 4 hold.*

PROOF. Suppose that (V, D) is of type (Ia). Then there exist a curve C and a birational morphism $u: V \to V_*$ so that CASE (X) in Proposition 1.8 takes place. We use the notation D', D'', A_* and N_* there. Since $N_*=0$, for any connected component R of D' there exists a (-1)-curve E such that $E+R$ is contracted by u. In particular, D' consists of (-2)-rods. In view of Lemma 1.7, $C+D''$ is an SNC rational loop because $A_*=u_*(C+D'')$ is an NC divisor. Hence D consists of rods. Note that $t=\#(A_*)-\rho(V_*) \leq \#(A_*)-1 \leq 3$. For the configuration of the anti-canonical divisor A_* of normal crossing type, see Zhang [10; Lemma 2.6]. If $u=\mathrm{id}$ then $(V, D)=(\Sigma_3, M_3)$. Hence we assume that u contracts a (-1)-curve F of V. Then $0 < -(C, D^{\#}+K_V) \leq -(F, D^{\#}+K_V)=1-(F, D^{\#}) \leq 1$. Write the rod $D''=D_1+\cdots+D_i+\cdots+D_s$, where $(D_i^2)=-3$ and $(C, D_1)=(D_1, D_2)=\cdots=(D_s, C)=1$. By Lemma 1.4, one has $1 \geq -(C, D^{\#}+K_V)=2+(C^2)-(C, D^{\#})=2+(C^2)-(s-i+1)/(s+1+i(s-i+1))-i/(s+1+i(s-i+1)) > (C^2)+1$. Hence $(C^2)=-1$ by (2) of Lemma 1.1. Namely, C is a (-1)-curve.

Then, since each divisor $E+R$ of contraction type (B) does not meet C, we can find a birational morphism $w: V \to W_*$ such that w is a composite of smooth contractions of type (A) or (B), that w contracts C and that $(W_*, w_*(D))$ is a minimal quasi-Iitaka surface satisfying the assertion (3) of Proposition 1.8 (cf. [11; Lemma 3.5]). Then, by replacing u by w, one may assume that u contracts C.

We proceed according to the value of s.

The case $s=1$, *i.e., the* (-3)-*curve is isolated in* D. Since $(u_*(C+D''))^2=(K_{V_*})^2$ $(=8$ or $9)>1$, u contracts some $E+R$ (contraction of type (B)). Ler $\tau: V \to V_1$ be the contraction of C and let $g: V_1 \to \bar{V}_1$ be the contraction of $\tau(D')$. Note that C is disjoint from $E+D'$ by Lemma 1.7, (1) and Proposition 1.8, (2), and note that $\rho(\bar{V}_1)=1$ because $s=1$. Since $\tau(E)$ is a (-1)-curve and $\tau(R)$ is a (-2)-rod, let σ_1: $V_1 \to W_1$ be the contraction of $\tau(E+R)$. Then $(\sigma_1)_*((\tau(E)+\tau(D'))$ consists of admissible rational rods. This is a contradiction by Lemma 1.3, (1).

The case $s=2$. Since $(u_*(C+D''))^2>2$, for t defined in Proposition 1.8, one has $t \geq 1$. One also has $t=\#(A_*)-\rho(V_*) \leq 2-\rho(V_*) \leq 1$. Hence $t=1$, $\#(A_*)=2$ and $\rho(V_*)=1$, i.e., $V_* \cong \boldsymbol{P}^2$. But then the direct image in V_* of the component of D'', which $E+R$ (being the unique divisor of type (B) contracted by u) does not meet, has self-intersection number ≤ -1. This is absurd.

The case $s=3$. We may assume that $(D_1^2)=-2$. Note that $t=\#(A_*)-\rho(V_*) \leq 3-\rho(V_*) \leq 2$. If $t=2$, then $\#(A_*)=3$ and $V_* \cong \boldsymbol{P}^2$. However, the direct image in V_* of the component of D'', which the two divisors of type (B) contracted by u fail to meet, has self-intersection number ≤ -1, a contradiction. If $t=0$, then $(u_*(C+D''))^2=2$ or 3 according as whether $(D_3^2)=-2$ or not. This is absurd. Suppose that $t=1$. Let $E+R$ be the unique divisor of type (B) contracted by u. If $(D_2^2)=-2$ and E does not meet D_1, we reach a contradiction as in the case $s=1$, where we let τ be the contraction of all components of $C+D''$ except the component meeting E. If $(D_2^2)=-2$ and E meets D_1, then $(u_*D_2)^2=(u_*D_3)^2=-2$, again a contradiction. If $(D_2^2)=-3$, we let Φ: $V \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$, where $S_0=2C+D_1+D_3$. By Lemma 1.3, (2), all singular fibers $S_0, \cdots, S_k$ are of type (II). Since $D=D''+R$, one has $k=0$ or 1, and if $k=1$ then $\#(S_1)=\#(R)+1=4$. This is impossible in view of (3) of Lemma 1.5 with $H=D_2$.

Assume $s \geq 4$. Consider first the case where $(D_i^2)=-3$ for some $1<i<s$. Let Φ: $V \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$, where $S_0=2C+D_1+D_s$. By (3) and (2) of Lemma 1.3, one may assume that $i=2$ and that there exists exactly one singular fiber S_1 of type (I). If D_2 meets a (-1)-curve E_1 in S_1, then $-(E_1, D^\#+K_V)=1-2(s-1)/(3s-1)<1-(s-1)/(3s-1)-2/(3s-1)=-(C, D^\#+K_V)$ because $s \geq 4$ (cf. Lemma 1.4). This is absurd. Therefore, $s \geq 5$ and there are two (-1)-curves E_1 and E_2 with $(E_1, D_3)=(E_2, D_{s-2})=1$ such that $S_1=E_1+D_3+\cdots+D_{s-2}+E_2$. Then there are no other singular fibers because the cross-section D_2 meets only D_1 and D_3 in $D-D_2$. But then $-(E_1, D^\#+K_V)=1-2(s-2)/(3s-1)<1-(s-1)/(3s-1)-2/(3s-1)=-(C, D^\#+K_V)$ because $s>5$, a contradiction to the choice of C. Indeed, $s=9$ by

Lemma 1.5, (1) where $H_1:=D_{s-1}$ and $H_2:=D_2$.

Consider next the case where $(D_i^2)=-3$ for $i=1$ or s. We may assume $i=1$. Let $\Psi: V \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$, where $S_0=3C+2D_s+D_{s-1}+D_1$. Suppose $s=4$. Then D_2 is a 2-section. By Lemma 1.3, (2) and Lemma 1.5, (3), each singular fiber $(\neq S_0)$ is of type (II) and there are at most three singular fibers $S_0, S_1, \cdots, S_k$ $(k \leq 2)$. Since D consists of rods, $\#(S_j)=3$ or 4 $(j\neq 0)$. With the notation in (3) of Lemma 1.5 where $H:=D_2$, one has $\sum_{j=1}^{k}(\#S_j-1)-1=(v_*H)^2=4m$ for some $m\geq 1$. Thus, $m=1$ and $k=2$, and we may assume that $\#(S_1)=3$, $\#(S_2)=4$. Hence there are exactly three connected components R_r's in D' $(1\leq r\leq 3)$. Note that for each R_r there is a (-1)-curve E_r such that u contracts E_r+R_r. Let $w: V \to W$ be the contraction of C and (E_r+R_r)'s. Then $\rho(W)=\rho(V)-1-\sum_{r=1}^{3}\#(E_r+R_r)=(\#(D)+1)-1-8=1$, i.e., $W\cong \boldsymbol{P}^2$, while w_*D_1 does not meet w_*D_3. This is absurd.

Suppose $s\geq 5$. Then D_2 and D_{s-2} are cross-sections of Ψ. By (2) of Lemma 1.3, if one let $S_0, S_1, \cdots, S_k$ $(k\geq 1)$ be all singular fibers, we may assume that S_1 is of type (I) and S_j $(j\geq 2)$ is of type (II). By (1) of Lemma 1.5 where $H_1:=D_2$ and $H_2:=D_{s-2}$, one sees that D_2 and D_{s-2} do not meet the same (-1)-curve in any singular fiber and that $\{s, k; \#S_0, \cdots, \#S_k\}=\{5, 1; 4, 6\}$, $\{9, 1; 4, 6\}$ or $\{8, 2; 4, 3, 4\}$. The configuration of $C+D$ and S_i's is given in the configuration (2), (3) or (4) in Appendix. By Lemma 1.5, (2), where $H_1:=D_2$ and $H_2:=D_{s-2}$, V^0 is affine-ruled. We shall see in Remark 2.7 below, that the dP3-surface corresponding to the configuration (2) is of type (Ic). For the existence of the configurations (2), (3), and (4), we refer to the argument at the end of §2. q.e.d

LEMMA 2.4. *If (V, D) is of type (Ib) then all the assertions in Theorem 2.2 with $n=5, 6, 7$ hold.*

PROOF. Suppose (V, D) is of type (Ib). Then there exist a curve C and a birational morphism u so that CASE (Y) in Proposition 1.8 takes place. We use the notation D'', Φ in Proposition 1.8. In view of Lemma 1.7, $C+D''$ is an SNC rational loop because $u_*(C+D'')$ is an NC divisor. By the same argument as that in Lemma 2.3, one can prove that C is a (-1)-curve. The morphism u consists of contractions of type (A) by Proposition 1.8, (4). The 2-section H of Φ in A_* is not a (-1)-curve, for otherwise the contraction of a (-1)-curve H is a contraction of type (A) and this contradicts the minimality of the quasi-Iitaka surface (V_*, A_*+N_*). So, we can write $D''=D_1+D_2+D_3$ with $(C, D_1)=(D_1, D_2)=(D_2, D_3)=(D_3, C)=1$ and with $(D_2^2)=-3$. Let $S_0=2C+D_1+D_3$ and let $\Psi: V\to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Ψ is nothing but $\Phi\circ u$. By (2) of Lemma 1.3 and (3) of Lemma 1.5 where $\Phi:=\Psi$ and $H:=D_2$, all singular fibers $S_0, S_1, \cdots, S_k$ are of type (II) and $k\leq 2$. With the notation in (3) of Lemma 1.5, one has $-2+\sum_{j=1}^{k}(\#S_j-1)=(v_*H)^2=4m$. Note that $v_*(H+S_0)\in|-K_{\Sigma_n}|$, because $C+D''+K_V\sim 0$. Hence $(v_*H)^2=4$, and $\{k; \#S_0, \cdots, \#S_k\}=\{1; 3, 7\}$, $\{2; 3, 3, 5\}$ or $\{2; 3, 4, 4\}$. The configuration of $C+D$ and S_i's is given in the configuration (5), (6), or (7) in Appendix. For the existence of the configurations (5), (6) or (7), we refer to the argument at the end of § 2. q.e.d.

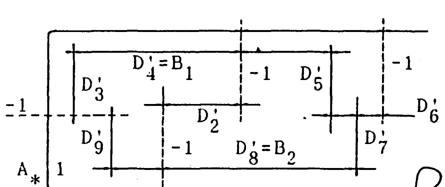
LEMMA 2.5. *If* (V, D) *is of type* (Ic) *then all the assertions in Thoerem* 2.2 *with* $n=2$ *and* $8 \le n \le 27$ *hold.*

PROOF. Assume that (V, D) is of type (Ic). Then there exist a curve C and a birational morphism $u: V \to V_*$ so that CASE (Z) in Proposition 1.8 takes place. We employ the notation D', D'', A_* and N_* there. Then $D'' \neq 0$, C is a (-1)-curve, u is a composite of the contractions of type (A), A_* is a rational nodal curve or a rational cuspidal curve, and C meets the (-3)-curve in D. We may (and shall) take u to be the composite of the contractions of all components of $C+D''$ except the (-3)-curve (cf. Lemma 1.7, (2)). Then $(K_{V_*}^2) = (u_*(C+D''))^2 = \#(D'')$ and $\#(D') = \#(N_*) = \rho(V_*) - 1 = 9 - (K_{V_*}^2) = 9 - \#(D'') \le 8$. Applying Lemmas 5.2 and 5.3 in [10] to the quasi-Iitaka surface $(V_*, A_* + N_*)$ we obtain:

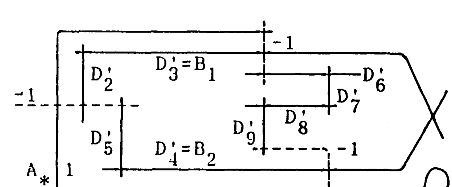
LEMMA 2.6. *Suppose that* A_* *is a nodal curve. Then there exists a* $\boldsymbol{P}^1$*-fibration* $\Phi: V_* \to \boldsymbol{P}^1$ *such that* A_* *is a* 2*-section of* Φ *and all components of* N_* *with at most two exceptions are contained in singular fibers of* Φ. *If there is an irreducible component of* N_* *not contained in singular fibers of* Φ, *it is a cross-section of* Φ. *We denote all these components by* B_i $(1 \le i \le m;\ m \le 2)$ (*see Figures* (5) *and* (6) *when* $m=2$). *Moreover, one of the following cases occurs.*

(ia) *We have* $m=1$. *All singular fibers* $f_1, \cdots, f_k$ *of* Φ *are of type* (II) *in Lemma* 1.3. *More precisely,* $\{k; \#f_1, \cdots, \#f_k\} = \{1; 4\}, \{1; 3\}, \{1; 5\}, \{2; 3, 3\}, \{2; 3, 4\}, \{1; 6\}, \{2; 3, 5\}, \{2; 4, 4\}, \{1; 7\}, \{2; 4, 5\}, \{2; 3, 6\}, \{1; 8\}$. *Hence the dual graph of D is one of No.* 2 *and No.* 8 ~ *No.* 18 *in the Table.*

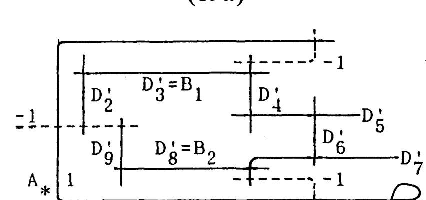
(ii$a$) *We have* $m=2$. $A_* + N_*$ *has one of the configurations* (19a)′ ~ (27a)′ *as shown in Figures* (5) *and* (6) *where* N_* *is written as* $N_* = \sum D_i'$ *and the fibers of* Φ *is given vertically. The dual graph of D is one of No.* 19 ~ *No.* 27 *in the Table.*



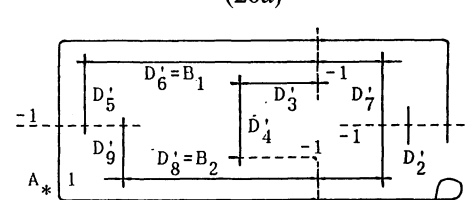
(19a)′



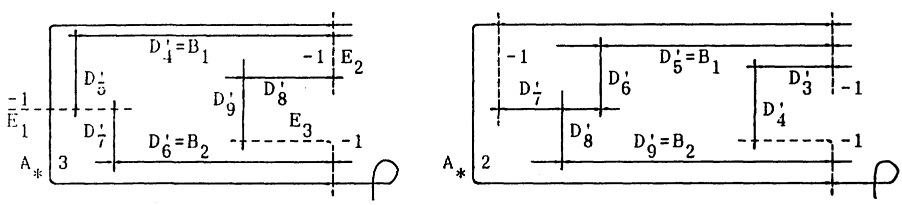
(20a)′



(21a)′

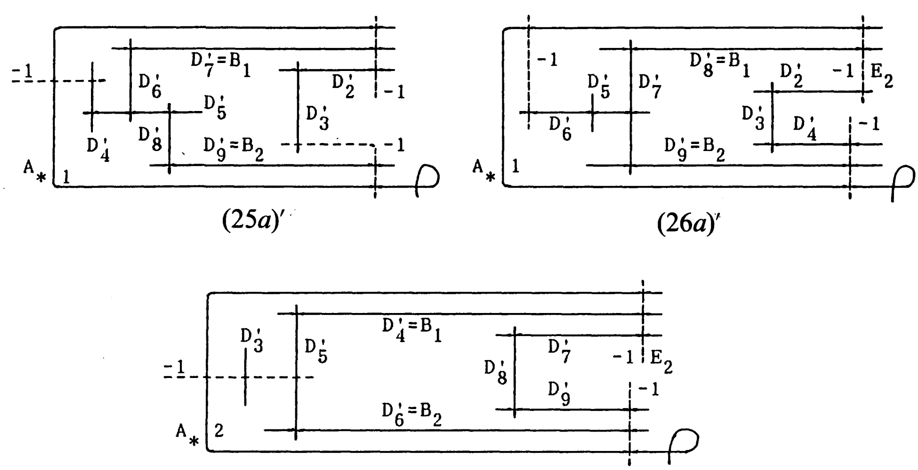


(22a)′



(23a)′ (24a)′

FIGURE (5)



(25a)′ (26a)′

(27a)′

FIGURE (6)

REMARK 2.7. (1) If the dual graph of D is given in No. 2 of the Table, we consider the $\boldsymbol{P}^1$-fibration $\psi\colon V \to \boldsymbol{P}^1$ defined by $|3C+2D_5+D_4+D_1|$ instead of $\Phi\circ u$, where D_1 is the (-3)-curve and $D''=D_1+\cdots+D_5$ with $(C, D_1)=(D_1, D_2)=\cdots=(D_4, D_5)=(D_5, C)=1$. Then we see that (V, D) is nothing but the one given in the proof of Lemma 2.3 with the same singularity type, and the assertions (2) and (4) in Theorem 2.2 for this case are verified there.

(2) By the arguments used in §6 to prove the impossibility of the configuration $(20b)'$, we can prove that in the configuration $(20a)'$, A_* meets the fiber of Φ passing through the point $D'_3\cap D'_4$ in two distinct points.

Now we continue our proof of Lemma 2.5 and consider the case where A_* is a rational cuspidal curve. Then either D'' is the (-3)-curve with $C\cap D''$ one point and with $(C, D'')=2$, or D'' consists of the (-3)-curve, say D_1, and a (-2)-curve, say D_2 with $C\cap D_1\cap D_2$ one point and with $(C, D_1+D_2)=2$ (cf. Lemma 1.7, (2)). Hence $u\colon V\to V_*$ is the contraction, of C in the first case and of C and the (-2)-curve D_2 in the second case. We can prove, by the same method as that in the proof of Lemmas

5.2 and 5.3 in [10], that two similar cases (ib) and (ii$b$) are possible whose statements are obtained from the corresponding cases (ia) and (ii$a$) in Lemma 2.6, respectively, by replacing the nodal curve A_* by a cuspidal curve. The configuration $(na)'$ in Figure (5) or (6) should be replaced by the same configuration $(nb)'$ $(19 \leq n \leq 27)$ but with a cuspidal curve A_*.

Suppose that the case (ib) occurs. Applying (3) of Lemma 1.5 to V_*, Φ and the 2-section A_*, one sees that there is exactly one singular fiber in Φ. Since $\#(N_*)=\rho(V_*)-1=9-(K_{V_*}^2)=9-\#(D'')=8$ or 7, N_* has the dual graph of Dynkin type E_8 or E_7, respectively, and D has the dual graph No. 18 or No. 15 in the Table, respectively.

We shall show that the case (ii$b$) does not occur. Since the $\boldsymbol{P}^1$-fibration Φ has at most one singular fiber of type (II) by Lemma 1.5, (3), where $H:=A_*$, the cases with the configurations $(19b)'$ and $(22b)'$ are impossible. In the case of the configuration $(26b)'$ (resp. $(27b)'$), we take a $\boldsymbol{P}^1$-fibration $\Phi_1: V \to \boldsymbol{P}^1$ defined by $|2E_2+D_2'+D_8'|$ (resp. $|2E_2+D_4'+D_7'|$) and get a contradiction by the same reasoning. Next we show that the configuration $(23b)'$ is impossible. This entails that the configurations $(21b)'$, $(24b)'$ and $(25b)'$ are impossible because they are obtained from the configuration $(23b)'$ by blowing up some of the points $E_1 \cap A_*$, $E_2 \cap A_*$ and $E_3 \cap A_*$ and their infinitely near points. Let $v: V_* \to \boldsymbol{P}^2$ be the contraction of E_1, D_5', D_4', E_3, D_9', D_8' in the configuration $(23b)'$. Then v_*D_7' and v_*D_6' are two inflectional tangents of a cuspidal cubic curve v_*A_* on $\boldsymbol{P}^2$. This is impossible (cf. Griffiths-Harris [5; p. 281]). The impossibility of the configuration $(20b)'$ will be proved in §6.

Set $\Psi:=\Phi \circ u: V \to \boldsymbol{P}^1$ regardless of whether A_* is a nodal or cuspidal curve. Denote by D_1 the unique (-3)-curve. Let S_0 be the singular fiber of Ψ such that u_*S_0 is the nonsingular fiber of Φ passing through the double point of A_*. Note that in singular fibers $(\neq S_0)$ of Ψ, the 2-section D_1 of Ψ meets only (-1)-curves. One can write $S_0=E+D_2+\cdots+D_s+C$, where E is a (-1)-curve, $D''=D_1+\cdots+D_s$ and $(E, D_2)=(D_2, D_3)=\cdots=(D_{s-1}, D_s)=(D_s, C)=1$ ($D_2, \cdots, D_s$ might not exist). Let $f_1, \cdots, f_k$ be all singular fibers of Φ and let $S_i=u^*f_i$ $(1 \leq i \leq k)$. Then u^* modifies nothing on f_i and S_j $(0 \leq j \leq k)$ are all singular fibers of Ψ. $D-D_1-u'B_1$ (resp. $D-D_1-u'B_1-u'B_2$) are contained in the singular fibers of Ψ if the case (ia) or (ib) (resp. (ii$a$)) occurs. Suppose that D has the dual graph No. n in the Table for some $8 \leq n \leq 27$. If $n=15$, 18, the configuration of $C+D$ and S_j's is given in the configuration (na) or (nb) in Appendix according as A_* is a nodal curve or a cuspidal curve, respectively. If $n \neq 15$, 18, then A_* is a nodal curve and the configuration of $C+D$ and S_j's is given in the configuration (n) in Appendix.

Let $\bar{V}(a)$ (resp. $\bar{V}(b)$) be a dP3-surface such that its minimal resolution of singularity $(V(a), D(a))$ (resp. $(V(b), D(b))$) corresponds to the configuration $(18a)$ (resp. $(18b)$). We shall show that $\bar{V}(a)$ is not isomorphic to $\bar{V}(b)$. Suppose the contrary, and let σ: $\bar{V}(a) \to \bar{V}(b)$ be an isomorphism. Then σ induces an isomorphism $V(a) \to V(b)$, denoted also by σ, such that $\sigma(D(a))=D(b)$. Note that σ maps the unique (-3)-curve $D(a)_1$ on $V(a)$ to the unique (-3)-curve $D(b)_1$ on $V(b)$. We have (-1)-curves $C(a)$ and $C(b)$ such that $C(a)+D(a)_1+K_{V(a)}\sim 0$ and $C(b)+D(b)_1+K_{V(b)}\sim 0$. Then $\sigma C(a)\sim C(b)$, whence $\sigma C(a)=C(b)$. This is a contradiction because $C(a)\cap D(a)_1=$ two points and $C(b)\cap D(b)_1=$ one point. Similarly, the surfaces belonging to the configurations (15a) and (15b) are not isomorphic to each other.

As for the affine-ruledness of V^0, it remains to show it when $n=8, 9, 12, 13, 15, 18$. Employ the notation, $D_1=$ the (-3)-curve and $S_0=E+D_2+\cdots+D_s+C$ as in the above arguments. Let S_1 $(\neq S_0)$ be a singular fiber of Ψ such that $\#(S_1)$ is maximal among the singular fibers of Ψ other than S_0. Write $S_1=2(E_1+H_1+\cdots+H_{r-2})+H_{r-1}+H_r$, where E_1 is a (-1)-curve, H_i's are components of D and $(E_1, H_1)=(H_1, H_2)=\cdots=(H_{r-3}, H_{r-2})=(H_{r-2}, H_{r-1})=(H_{r-2}, H_r)=1$. Denote by H_{r+1} the unique cross-section of Ψ in D (hence $u(H_{r+1})=B_1$, cf. Lemma 2.6). Then one may assume that $(H_r, H_{r+1})=1$.

Consider the configuration (12) (resp. (15), or (18)) in Appendix. Then $r=5$ (resp. 6, or 7) and all components of D, except H_3 and D_2 (resp. H_3 and D_2, or H_3), are contained in the singular fibers of the $\boldsymbol{P}^1$-fibration $\varphi\colon V\to\boldsymbol{P}^1$ defined by $|3E_1+2H_1+H_2+D_1|$. Hence V^0 is affine-ruled by (2) of Lemma 1.5 where $\Phi:=\varphi$. Consider the configuration (8) (resp. (13)) in Appendix. Hence $s=6$ and $r=2$ (resp. $s=2$ and $r=4$). One can get the affine-ruledness of V^0 by applying (2) of Lemma 1.5 to the $\boldsymbol{P}^1$-fibration defined by $|3C+2D_6+D_5+D_1|$ (resp. $|2E_1+H_1+D_1+D_2+E|$) and its cross-sections D_2 and D_4 (resp. H_2 and H_5). Consider the configuration (9) in Appendix. Then $s=r=4$. Let $v\colon V\to\Sigma_2$ be a contraction of all (-1)-curves and consecutively contractible curves in the singular fibers of Ψ so that $(v_*H_5)^2=-2$. Take a nonsingular irreducible curve $\tilde{E}_2$ in $|v_*H_5+2v_*S_0|$ such that $\tilde{E}_2$ meets v_*D_1 with local intersection number $i(\tilde{E}_2, v_*D_1; v(C))=4$ at the node $v(C)$. Then the proper transform $E_2:=v'\tilde{E}_2$ is a (-1)-curve such that $(E_2, D_4)=(E_2, H_4)=(E_2, S_0)=1$. By considering the $\boldsymbol{P}^1$-fibration which is defined by $|2E_1+H_1+D_1+\cdots+D_4+E_2|$ and which has cross-sections H_2 and H_4 and by (2) of Lemma 1.5, the affine-ruledness of V^0 follows.

To complete the proof of Lemma 2.5, we must show the existence of the configurations (n) in Appendix. By contracting irreducible components of singular fibers of Ψ, we obtain a birational morphism $w\colon V\to\Sigma_m$ onto a relatively minimal ruled surface $\Sigma_m (m\leq 3)$. Looking at the configuration (n) in Appendix, we can easily find which curves should be contracted. Thus, we are reduced to proving the existence of the configuration of $w(C+D)$ on Σ_m. For $19\leq n\leq 27$, by the proof of Lemmas 3.5 and 4.2 in [10], we can take w as a composite of $u\colon V\to V_*$ (with the notation at the beginning of Lemma 2.5) and a blowing-down $w_*\colon V_*\to\Sigma_2$, and the configuration of $w(C+D)$ is one of Fig. (1), $\cdots$, Fig. (5) and Fig. (9) displayed in [10; pp. 418–419]. The existence of those figures was proved in [10; Lemma 5.3]. The other cases can be treated more easily.

3. dP3-surfaces of the second kind and type (II*a*). By [11; Lemma 2.2] and [6; Lemma 2.1.3] we obtain the following:

LEMMA 3.1 *Let (V, D) be a log del Pezzo surface of rank one with contractible boundary. Suppose that (V, D) is of the second kind and is not isomorphic to (Σ_m, M_m). Then there exists a (-1)-curve C such that $-(C, D^{\#}+K_V)$ attains the smallest positive value. Hence $|C+D+K_V|=\varnothing$. Furthermore, we have*:

(1) *Each component of D is a nonsingular rational curve, $C+D$ is an* SNC *divisor whose dual graph consists of trees, and $(C, D)\geq 1$ by Lemma* 1.3, (1).

(2) *One of the following cases takes place*:

CASE (α). *C meets at least two (-2)-components D_1 and D_2 of D.*

CASE (β). *C meets only one component D_1 of D.*

CASE (γ). *C meets only two components D_1 and D_2 of D, and D_1 is a (-2)-curve and D_2 is the (-3)-curve.*

Employing Lemma 3.1, we consider three types for dP3-surfaces of the second kind. Namely, we have the following:

DEFINITION 3.2. Let (V, D) be a dP3-surface of the second kind. (V, D) is said to be of type (II*a*), if there exists a (-1)-curve C so that CASE (α) in Lemma 3.1 occurs. (V, D) is said to be of type (II*b*) if there exists a (-1)-curve C so that CASE (β) in Lemma 3.1 occurs but (V, D) is not of type (II*a*). (V, D) is said to be of type (II*c*) if there exists a (-1)-curve C so that CASE (γ) in Lemma 3.1 occurs but (V, D) is neither of type (II*a*) nor type (II*b*).

In the present section, we consider dP3-surfaces of the second kind and type (II*a*). We shall prove the following:

THEOREM 3.3. *Let (V, D) be a* dP3*-surface of the second kind and type* (II*a*) *with a (-1)-curve C as in Lemma* 3.1. *Then we have*:

(1) *The dual graph of D is one of those given in the cases No. n $(28\leq n\leq 35)$ in the Table.*

(2) *There exist a $\boldsymbol{P}^1$-fibration $\Psi: V\to\boldsymbol{P}^1$ and a component H of D such that H is a cross-section of Ψ and the other components of D are contained in singular fibers of Ψ. Hence V^0 is affine-ruled.*

(3) *The configuration of $C+D$ and all singular fibers of Ψ is given in the configuration* (n) *for* $28\leq n\leq 35$ (see Appendix).

(4) *All the cases are realizable.*

PROOF. Let D_1 and D_2 be (-2)-components of D which C meets. Let $\Phi: V\to\boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|2C+D_1+D_2|$.

First, we consider the case where C meets a component of $D-D_1-D_2$. By Lemma 1.3, (3), we have $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ for every (-1)-curve E in a fiber of Φ. Hence $|E+D+K_V|=\varnothing$ because (V, D) is of the second kind. Thus, by the proof of [11; Lemma 5.2] and by (2) of Lemma 1.3 there exists another $\boldsymbol{P}^1$-fibration Ψ: $V \to \boldsymbol{P}^1$ ($\Psi = \Phi_1$ in [11]) such that the configuration (28) or (29) in Appendix shows the configuration of $C+D$ and all singular fibers of Ψ, where the components of D are written in solid lines. Hence V^0 is affine-ruled because every component of D, except one which is a cross-section of Ψ, is contained in a singular fiber of Ψ.

Next, we consider the case where C meets only D_1 and D_2 in D. Let $\Psi := \Phi$. By (3) of Lemma 1.3 and by (1) of Lemma 3.1, the (-3)-curve, say D_3, is a cross-section. Let ε_i be the number of all components of $D - D_i$ meeting D_i ($i=1, 2$). We may assume $\varepsilon_1 \geq \varepsilon_2$. As in the previous case, we have $|E+D+K_V| = \emptyset$ for any (-1)-curve E contained in a singular fiber of Ψ. Hence, by the arguments in the proof of [11; Lemma 5.3], we have $(\varepsilon_1, \varepsilon_2) = (1, 0)$, $(2, 0)$ or $(1, 1)$. Moreover, if $(\varepsilon_1, \varepsilon_2) = (2, 0)$ then the two cross-sections D_3 and D_4 contained in D meet two distinct (-1)-curves E_1 and E_2, respectively, in a singular fiber S_1 of type (I) and S_0 and S_1 exhaust all singular fibers. However, $-(E_1, D^\# + K_V) = 1 - 3/7 < 1 - 2/7 = -(C, D^\# + K_V)$, which contradicts the choice of C. So, $(\varepsilon_1, \varepsilon_2) \neq (2, 0)$.

CASE $(\varepsilon_1, \varepsilon_2) = (1, 0)$. Then V^0 is affine-ruled. By Lemma 1.3, (2), the configuration of $C+D$ and all singular fibers of Ψ is given in one of the configurations (n) ($30 \leq n \leq 35$) in Appendix.

CASE $(\varepsilon_1, \varepsilon_2) = (1, 1)$. By the proof of Lemma 5.3 in [11], all singular fibers of Ψ are as shown in Figure (7). We have $1 \leq s \leq 3$, $0 \leq b \leq a$ ($B_{a+1} := E_2$), and $(D_i^2) = -3$ or

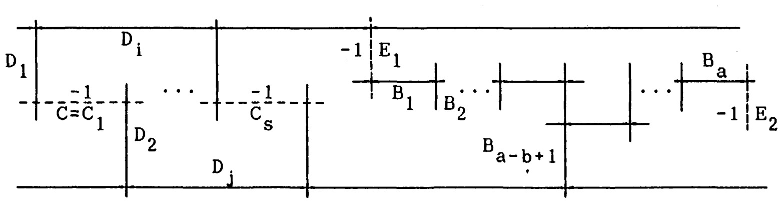


FIGURE (7)

$(D_j^2) = -3$. Suppose that $(D_i^2) = -3$. Then the coefficient of D_i in $D^\#$ is twice that of D_1 by Lemma 1.4. This leads to $-(E_1, D^\# + K_V) < -(C, D^\# + K_V)$, contradicting the choice of C. If $b=0$, we have a contradiction in a similar fashion. Thus, one may assume that $(D_j^2) = -3$ and $b \geq 1$. By (1) of Lemma 1.5, where $H_1 := D_i$ and $H_2 := D_j$, $2 = (v_* D_j)^2 = -3 + s + a - b + 1$, i.e., $a - b = 4 - s$. Since $g(D)$ are quotient singularities on $\bar{V}$, $\{s, a, b\} = \{1, 4, 1\}$ or $\{2, 3, 1\}$. But then $-(E_2, D^\# + K_V) = 1 - 8/17 < 1 - 5/17 = -(C, D^\# + K_V)$ if $s=1$, and $-(E_2, D^\# + K_V) = 1 - 3/5 < 1 - 2/5 = -(C, D^\# + K_V)$ if $s=2$ by Lemma 1.4. This is a contradiction to the choice of C.

For the existence of the configurations (n) ($28 \leq n \leq 35$) in Appendix, we refer to the argument at the end of § 2. q.e.d.

4. dP3-surfaces of the second kind and type (IIb). The purpose of the present section is to prove the following:

THEOREM 4.1. *Let (V, D) be a* dP3-*surface of the second kind and type* (IIb). *Then we have the following assertions*:

(1) *The dual graph of D is one of those given in the cases No. n $(36 \leq n \leq 62)$ in Table.*

(2) *There is a (-1)-curve C with which CASE (β) in Lemma* 3.1 *occurs and there is a $\boldsymbol{P}^1$-fibration $\Psi: V \to \boldsymbol{P}^1$ such the configuration of $C+D$ and all singular fibers of Ψ is given in the configuration (n) for $36 \leq n \leq 62$ in Appendix. In particular, all components of D, except one cross-section or two disjoint cross-sections, are contained in singular fibers of Ψ.*

(3) *All figures are realizable.*

(4) *V^0 is affine-ruled if $n \neq 60$, 62.*

PROOF. Suppose that (V, D) is a dP3-surface of the second kind and type (IIb) with a (-1)-curve C which meets only D_1 in D as in Lemma 3.1. By Lemma 3.1, (1), we have $(C, D_1)=1$ and hence $(D_1^2)=-2$ in view of (1) of Lemma 1.3.

(1) *Consider first the case where the connected component R of D containing D_1 is a rod.* By Lemma 1.3, (1), the intersection matrix of $C+D$ is not negative definite. So, either D_1 meet two (-2)-curves, say D_2 and D_3, or D_1 meets the (-3)-curve D_4 and a (-2)-curve D_2 which meets D_3 with $(D_2, D_3)=1$ for some $D_3 \leq D-D_1-D_2-D_4$. In the first case we let $S_0=2(C+D_1)+D_2+D_3$ and in the second case we let $S_0=3(C+D_1)+2D_2+D_3+D_4$. Denote by $\Psi: V \to \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Any component of R not contained in S_0 and meeting S_0 is a cross-section of Ψ and all other components of D are contained in the singular fibers of Ψ.

The case where $S_0=2(C+D_1)+D_2+D_3$. Then the (-3)-curve, which we denote by D_4, is a cross-section of Ψ by Lemma 1.3, (3) and Lemma 3.1, (1). Suppose that there is a singular fiber S_1 $(\neq S_0)$ with only one (-1)-curve E and with $\#(S_1)=3$. By Lemma 1.3, (3), we have $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$. Then, with the curve E, (V, D) is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (IIa) according as $|E+D+K_V| \neq \varnothing$ or $|E+D+K_V|=\varnothing$. This is a contradiction.

Suppose that every component of $D-D_4$ is contained in a singular fiber. Then V^0 is affine-ruled, and all singular fibers $S_0, S_1, \cdots, S_k$ of Ψ are of type (II) by Lemma 1.3, (2). Since R is a rod, we have $\{k; \#S_0, \cdots, \#S_k\}=\{0; 4\}$ or $\{1; 4, 4\}$, and the configuration of $C+D$ and S_i's is respectively given in the configuration (36) or (37) in Appendix.

Suppose next that there are two cross-sections D_4 and D_5 in $D-D_1$ with $(D_4, D_2)=(D_5, D_3)=1$. If D_4 or D_5 meets a component of $D-D_2-D_3-D_4-D_5$ in some singular fiber, say S_1, there is then a (-1)-curve E_1 in S_1 such that $R+E_1$ contains a loop and hence $|E_1+D+K_V| \neq \varnothing$. But then (V, D) is a dP3-surface of the first kind with the curve E_1 because $-(E_1, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ by Lemma 1.3, (3). So, D_4 and D_5 are tips of the rod R. Meanwhile, by Lemma 1.3, (2), there is a singular fiber

S_1 of type (I). Let E_2 be a (-1)-curve in S_1 meeting D_4. Then, $-(E_2, D^{\#}+K_V)=1-5/11<1-3/11=-(C, D^{\#}+K_V)$, which is a contradiction.

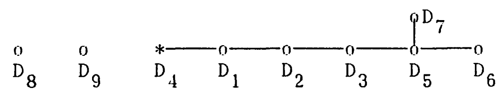
The case where $S_0=3(C+D_1)+2D_2+D_3+D_4$. Suppose that there is only one cross-section, say D_5, in D. Then V^0 is affine-ruled. All singular fibers $S_1, \cdots, S_k$ $(\neq S_0)$ are of type (II) by Lemma 1.3, (2). Since R is a rod, $\{k; \#S_0, \cdots, \#S_k\}=\{0; 5\}$, $\{1; 5, 3\}$ or $\{1; 5, 4\}$. The configuration of $C+D$ and S_i's is given in the configuration (n) for $38\leq n\leq 43$ in Appendix. Suppose next that there are two cross-sections D_5 and D_6 in $D-D_1-D_2$ with $(D_5, D_3)=(D_6, D_4)=1$. By Lemma 1.3 (2), there exists exactly one singular fiber S_1 of type (I) and all others $S_2, \cdots, S_k$ $(\neq S_0, S_1)$ are of type (II). By Lemma 1.5, (1), where $H_1:=D_5$ and $H_2:=D_6$, we have $\{\#R; k; \#S_0, \cdots, \#S_k\}=\{6; 1; 5, 4\}$, $\{9; 1; 5, 5\}$, $\{8; 2; 5, 3, 3\}$. The second case has two subcases according as whether D_6 is a tip of R or not. The configuration of $C+D$ and S_i's is given in the configuration (n) for $44\leq n\leq 47$ in Appendix. V^0 is affine-ruled by Lemma 1.5, (2) where $H_1:=D_5$ and $H_2:=D_6$.

We shall consider the case where the connected component of D containing D_1 is a fork F with a central component F_0 and three maximal twigs T_1, T_2 and T_3, i.e., $F=F_0+T_1+T_2+T_3$.

(2) *The case where* $D_1\leq T_1$ *and* $C+T_1$ *is not negative definite.* As in the previous case (1) where C meets a rod R, one can find a $\boldsymbol{P}^1$-fibration $\Psi:=\Phi_{|S_0|}: V\to\boldsymbol{P}^1$ and has two subcases. Using the notation there, we have:

The case where $S_0=2(C+D_1)+D_2+D_3$. Since (V, D) is neither a dP3-surface of the first kind nor a dP3-surface of the second kind and type (IIa), as in the case (1), there are no singular fibers S_1 of type (II) with $\#(S_1)=3$. Note that if there are two cross-sections in D then one of them meets a component of D in a singular fiber of Ψ, for F is a fork. As in the case (1), this leads to a contradiction because (V, D) is not of the first kind. By (3) and (2) of Lemma 1.3, $D-D_4$, with the (-3)-curve D_4, is contained in the singular fibers, whence V^0 is affine-ruled and there is a unique singular fiber S_1 $(\neq S_0)$ in view of the hypothesis that F is a fork. S_1 is of type (II) and $\{\#S_0, \#S_1\}=\{4, 5\}$. The configuration of $C+D$ and S_i's is given in the configuration (48) in Appendix.

The case where $S_0=3(C+D_1)+2D_2+D_3+D_4$. Suppose that there are two cross-sections D_5 and D_6 in D and that $S_0, \cdots, S_k$ exhaust all singular fibers of Ψ, where S_1 is of type (I) and $S_2, \cdots, S_k$ are of type (II) (cf. Lemma 1.3, (2)). Since F is a fork (of Dynkin type D_n), we have $\{k; \#S_0, \cdots, \#S_k\}=\{1; 5, 5\}$ or $\{2; 5, 3, 3\}$, and either one of D_5 and D_6 meets the central component F_0 or is F_0 itself. This is a contradiction by Lemma 1.5, (1). Suppose next there is exactly one cross-section D_5 in D. Then V^0 is affine-ruled. All singular fibers $S_1, \cdots, S_k$ $(\neq S_0)$ are of type (II). Since F is a fork, $\{k; \#S_0, \cdots, \#S_k\}=\{1; 5,5\}$ or $\{2; 5, 3, 3\}$. Then either D has the dual graph No. n $(49\leq n\leq 51)$ in the Table and the configuration of $C+D$ and S_i's is given in the configuration (n) in Appendix, or the dual graph of D is given in Figure (8).



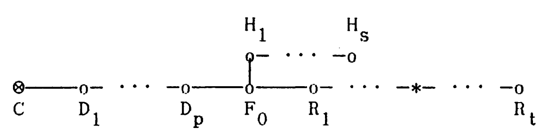
with $F_0 = D_5$.

FIGURE (8)

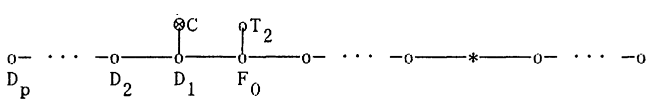
Consider the case with Figure (8). Let $v\colon V \to \Sigma_2$ be a contraction of all (-1)-curves and consecutively contractible curves in S_0, S_1 and S_2 so that $(v_*D_5)^2 = -2$. Take a nonsingular irreducible curve $\tilde{E}$ in $|v_*D_5 + 2v_*S_0|$ such that $\tilde{E}$ passes through three points $v(D_4)$, $v(D_8)$ and $v(D_9)$. Then the proper transform $E := v'\tilde{E}$ is a (-1)-curve satisfying $(E, D_4) = (E, D_8) = (E, D_9) = (E, S_0) = 1$ and $-(E, D^\# + K_V) = 1 - 1/2 = -(C, D^\# + K_V)$ by Lemma 1.4. Hence, with the curve E, (V, D) is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (II*a*) according as $|E + D + K_V| \neq \varnothing$ or $|E + D + K_V| = \varnothing$. This is a contradiction.

(3) *The case where $D_1 \leq T_1$ and $C + T_1$ is negative definite.* By applying (1) of Lemma 1.3 to the (-1)-curve C and by noting that $g(F)$ is a quotient singularity on $\bar{V}$, $C + F$ has one of those dual graphs shown in Figure (9).

(3*a*)

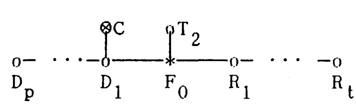


(3*b*)



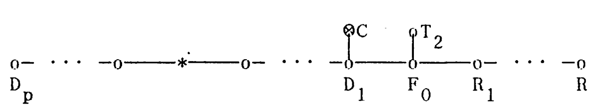
$(p \geq 2)$

(3*c*)



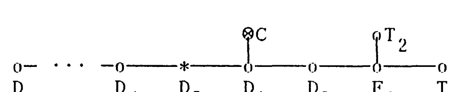
$(p \geq 2)$

(3*d*)



$(s = 1, 2)$

(3*e*)



(3*f*)

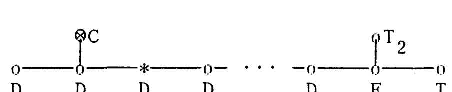


FIGURE (9)

If the case $(3f)$ occurs, then D is contained in singular fibers of the $\boldsymbol{P}^1$-fibration defined by $|4(C+D_1)+2(D_2+D_3+\cdots+D_p+F_0)+T_2+T_3|$. This is impossible by (2) of Lemma 1.3. Before investigating the remaining cases we prove the following results (due to M. Miyanishi):

LEMMA 4.2. *Let $\bar{V}$ be a log del Pezzo surface of rank one with contractible boundary. Suppose that there is a (-1)-curve C such that one of the following conditions is satisfied.*

(i) *C meets exactly one component D_1 of D.*

(ii) *C meets exactly two components D_1 and D_2 of D with $(C, D_2)=1$ and $(D_2^2)\leq -3$. Then the following assertions hold true.*

(1) *Let $\sigma: V\to W$ be the contraction of C, let $\tilde{C}=\sigma(D_1)$ and let $B=\sigma_*(D-D_1)$. Then B is contractible to quotient singularities on a projective normal surface $\bar{W}$. So, there exists a contraction $h: W\to\bar{W}$ such that $h: W-B\simeq\bar{W}-\mathrm{Sing}(\bar{W})$. Moreover, $\bar{W}$ is a log del Pezzo surface of rank one with contractible boundary.*

(2) *Suppose that the condition* (i) *above is satisfied and suppose furthermore that $-(C, D^{\#}+K_V)$ attains the smallest positive value. Then $0<-(\tilde{C}, B^{\#}+K_W)\leq-(\tilde{G}, B^{\#}+K_W)$ for every curve $\tilde{G}$ on W which is not a component of B. Moreover, if E is an irreducible curve on V such that $E\cap C=\emptyset$ and $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$, then $-(\sigma(E), B^{\#}+K_W)=-(\tilde{C}, B^{\#}+K_W)$.*

(3) *If $|C+D+K_V|=\emptyset$ then $|\tilde{C}+B+K_W|=\emptyset$.*

PROOF. The assertion (1) is proved in [11; Lemma 4.3].

(2) Let G be an irreducible curve not in $C+D$. Since $\rho(\bar{V})=1$, $G\equiv\alpha C+\beta D_1+\Gamma$ where $\alpha, \beta\in\boldsymbol{Q}$ and Γ is a $\boldsymbol{Q}$-divisor supported by $\mathrm{Supp}(D-D_1)$. Since $-(G, D^{\#}+K_V)\geqq-(C, D^{\#}+K_V)$, we have $\alpha\geq 1$. Since $(G, C)\geq 0$, we have $\beta\geq\alpha\geq 1$. On the other hand, since $\sigma(G)\equiv\beta\tilde{C}+\sigma_*(\Gamma)$ with $\mathrm{Supp}\,\sigma_*(\Gamma)\subseteq\mathrm{Supp}\,B$, one has $-(\sigma(G), B^{\#}+K_W)=-\beta(\tilde{C}, B^{\#}+K_W)\geq-(\tilde{C}, B^{\#}+K_W)$. Now we shall show the second assertion. Write $E\equiv aC+bD_1+\Delta$ where $a, b\in\boldsymbol{Q}$ and $\mathrm{Supp}\,\Delta\subseteq\mathrm{Supp}(D-D_1)$. Hence $\sigma(E)\equiv b\tilde{C}+\sigma_*(\Delta)$ with $\mathrm{Supp}\,\sigma_*(\Delta)\subseteq$ $\mathrm{Supp}\,B$. Since $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$, $a=1$. Moreover, $E\cap C=\emptyset$ implies that $0=(E, C)=-a+b$, i.e., $b=a=1$. Hence $-(\sigma(E), B^{\#}+K_W)=-(\tilde{C}, B^{\#}+K_W)$.

(3) *Note that $\sigma^*(\tilde{C}+B+K_W)=D+K_V$* or $C+D+K_V$ under the condition (i) or (ii), respectively. From this follows (3).

We shall return to the case (3).

CASE $(3a)$. Let $\sigma: V\to W$ be the contraction of $C, D_1, \cdots, D_{p-1}$, let $\tilde{C}=\sigma(D_p)$ and let $B=\sigma(D-D_p)$. By applying Lemma 4.2 successively, we know that (W, B) is a dP3-surface such that $|\tilde{C}+B+K_W|=\emptyset$ and $-(\tilde{C}, B^{\#}+K_W)$ is the smallest positive value. Then the argument in the case (1) works for (W, B) and $\tilde{C}$. So, if $(R_1^2)=-3$, then B has the dual graph No. m $(38\leq m\leq 47)$ in the Table. Since F is a fork, D has the dual graph No. n $(54\leq n\leq 56)$ in the Table and the configuration of $C+D$ and all singular fibers of the $\boldsymbol{P}^1$-fibration $\Psi: V\to\boldsymbol{P}^1$ which is defined by $|3(C+D_1+\cdots+D_p+F_0)+$

$2H_1+H_2+R_1|$ is respectively given in the configuration (n) in Appendix. In particular, all components of D, except for one cross-section, is contained in singular fibers of Ψ. Hence, V^0 is affine-ruled. Suppose $(R_1^2)=-2$. Consider the $\boldsymbol{P}^1$-fibration $\Psi: V\to\boldsymbol{P}^1$ defined by $|S_0|$, where $S_0=2(C+D_1+\cdots+D_p+F_0)+H_1+R_1$. By Lemma 1.3, (3), $(R_2^2)=-3$. If there are two cross-sections H_2 and R_2 in D, then $p=1$, $s=t=2$ for F is a fork. This contradicts the minimality of $-(\tilde{C}, B^{\#}+K_W)$ by the argument in the previous case (1). So, $s=1$ and $D-R_2$ is contained in the singular fibers, hence V^0 is affine-ruled. By Lemma 1.3, (2), all singular fibers $S_1, \cdots, S_k$ $(\neq S_0)$ are of type (II). Since (V, D) is neither a dP3-surface of the first kind nor a dP3-surface of the second kind and type (IIa), we have $\{k; \#S_0, \cdots, \#S_k\}=\{0; 5\}$, $\{0; 6\}$ or $\{1; 5, 4\}$ by the same argument as in the previous case (1). The configuration of $C+D$ and S_i's is given in the configuration (52), (53) or (48) in Appendix where the notation C and E should be interchanged. Note that $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ by Lemma 1.3, (3).

CASE (3b). Let $\sigma: V\to W$ be the contraction of C, let $\tilde{C}=\sigma(D_1)$ and let $B=\sigma(D-D_1)$. By Lemma 4.2, we may pass to the pair (W, B) which is a dP3-surface of the second kind and type (IIa) with the (-1)-curve $\tilde{C}$. Let $\Phi: V\to\boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defind by $|2(C+D_1)+D_2+F_0|$. For each (-1)-curve E in a singular fiber of Φ, we have $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ by Lemma 1.3, (3). Hence, $|E+D+K_V|=\varnothing$ because (V, D) is not a dP3-surface of the first kind. This, together with the minimality of $-(\tilde{C}, B+K_W)$, will lead to a contradiction by the argument in the proof of Theorem 3.3.

CASE (3c). Let $\sigma: V\to W$ be the contraction of C and D_1 and let $B=\sigma(D-D_2)$. By Lemma 4.2, (W, B) is a log del Pezzo surface of rank one with contractible boundary, on which there are no $(-n)$-curves with $n\geq 3$. Hence $-K_W=-(B^{\#}+K_W)$ is numerically effective and $1\leq(K_W^2)=10-\rho(W)=9-\#(B)\leq 7$. So, $\boldsymbol{P}^2$ is a relatively minimal model of W. There is a nonsingular elliptic curve A in $|-K_W|$ such that $(W, A+B)$ is an Iitaka surface with $\rho(W)=\#(\mathrm{Bk}(A+B))+1$ (cf. Demazure [4; III, Theorem 1, p. 39]). Note that B contains connected components of Dynkin type A_{p-2} and A_{t+2}.

Suppose $p\geq 4$. If $p=4$ then $t=1, 2$ (hence B contains connected components of Dynkin type A_2+A_3 or A_2+A_4) and if $p\geq 5$ then $t=1$ (hence B contains connected components of Dynkin type A_3+A_{p-2}), since $g(F)$ is a quotient singularity of $\bar{V}$. By [10; Lemmas 3.5, 4.2 and 4.3], B is of type A_1+2A_3 or $2A_1+2A_3$ (cf. Lemma 5.1 below). Hence $p=5$ and $t=1$. Note that if $\Psi: V\to\boldsymbol{P}^1$ is the $\boldsymbol{P}^1$-fibration defined by $|4(C+D_1)+2(D_2+F_0)+T_2+R_1|$ then $D-D_3$ is contained in the singular fibers. Then the singular fiber of Ψ containing D_4+D_5 must be of type (I) in Lemma 1.3, (2). This is impossible by the equality in Lemma 1.3, (2).

Suppose that $p=2$. Let S_0 and Ψ be the same as in the case $p=5$. Then $D-R_2$ is contained in the singular fibers and R_2 is a cross-section. Hence V^0 is affine-ruled. By Lemma 1.3, (2), all singular fibers $S_1, \cdots, S_k$ $(\neq S_0)$ are of type (II). Since $g(F)$ is a quotient singularity of $\bar{V}$, $\{k; \#S_0, \cdots, \#S_k\}=\{0; 6\}$ or $\{1; 6, 3\}$ and the configuration of $C+D$ and S_i's is given in the configuration (57) or (58) in Appendix.

Suppose that $p=3$. Let $S_0=3(C+D_1)+2D_2+D_3+F_0$ and let $\Psi: V \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Then $D-T_2-R_1$ is contained in the singular fibers. By Lemma 1.3, (2) and Lemma 1.5, (1) where $H_1:=T_2$ and $H_2:=R_1$, and by noting that F is a fork, we know that there is exactly one singular fiber S_1 $(\neq S_0)$ which is of type (I) and which has $\#(S_1)=5$, and that T_2 and R_1 meet two different (-1)-curves in S_1. The configuration of $C+D$ and S_i's is given in the configuration (59) in Appendix. In particular, V^0 is affine-ruled by Lemma 1.5, (2).

CASE (3*d*). Let $\sigma: V \to W$ be the contraction of C and all successively contractible curves in T_1, let $\tilde{F}_0:=\sigma(F_0)$, $\tilde{F}:=\sigma(F)$, $\tilde{T}_i:=\sigma(T_i)$ and let $B=\sigma(D)$. Then $\rho(W)=\#(\mathrm{Bk}(B))+1$ and $\bar{\kappa}(W-B)=-\infty$ for $\rho(V)=\#(D)+1$ and $\bar{\kappa}(V^0)=-\infty$. Indeed, (W, B) is a log del Pezzo surface of rank one with *non*-contractible boundary; for the proof, see [11; Lemma 6.4]. By Lemma 2.6 and Theorems 4 and 6 in Miyanishi-Tsunoda [8], B consists of $\tilde{F}$ and an admissible rational fork Q, and there is a $\boldsymbol{P}^1$-fibration $\Phi: W \to \boldsymbol{P}^1$ with exactly three singular fibers such that the support of each singular fiber f_i is a rod consisting of a (-1)-curve, a twig $\tilde{T}_i$ and a twig Q_i of Q, and that $\tilde{F}_0$ and the central component Q_0 of Q are cross-sections of Φ. Since $\mathrm{Bk}(B)$ consists of (-2)-rods in the present case, f_i is of type (II) and $\#(f_i)=3$. Hence $s=1$, $(D_p^2)=-3$ and Q has Dynkin type D_4. By Lemma 1.5, (1) where $H_1:=Q_0$ and $H_2:=\tilde{F}_0$, one sees that $(\tilde{F}_0^2)=-1$ and $p=2$. Let $\Psi=\Phi\circ\sigma$, $S_i=\sigma^*(f_i)$. Then $\{\#S_1, \#S_2, \#S_3\}=\{5, 3, 3\}$ and the configuration of $C+D$ and all singular fibers of Ψ is given in the configuration (60) in Appendix.

CASE (3*e*). By considering the $\boldsymbol{P}^1$-fibration $\Phi: V \to \boldsymbol{P}^1$ defined by $|2(C+D_1+D_2+F_0)+T_2+T_3|$, we can prove that $p=3$, as in Lemma 5.3, (2) below. Let $S_0=3(C+D_1)+2D_2+F_0+D_3$ and let $\Psi: V \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. By Lemma 1.3, (2) and Lemma 1.5, (1) with $H_1:=T_2$ and $H_2:=T_3$, the fibration Ψ has exactly one singular fiber S_1 $(\neq S_0)$ with $\#(S_1)=5$, and T_2 and T_3 meet two different (-1)-curves in S_1. Then V^0 is affine-ruled by Lemma 1.5, (2). The configuration of $C+D$ and S_i's is given in the configuration (61) in Appendix.

To finish the proof of Theorem 4.1 we shall consider the last case:

(4) *The case where D_1 is the central component F_0 of F.* Let $\sigma: V \to W$ be the contraction of C and let $B=\sigma(D)$. Then (W, B) is a log del Pezzo surface of rank one with *non*-contractible boundary (cf. [11; Lemma 6.4]). By the same reasoning as in the case (3*d*), we can prove that D has the dual graph No. 62 in the Table. More precisely, there is a $\boldsymbol{P}^1$-fibration $\Psi: V \to \boldsymbol{P}^1$, each of whose singular fibers, except S_0 which consists of C and another (-1)-curve E, consists of a (-1)-curve, a twig of F and a twig of another fork Q $(:=D-F)$, and the two central components F_0 and Q_0 of F and Q, respectively, are cross-sections of Ψ. The configuration of $C+D$ and all singular fibers of Ψ is given in the configuration (62) in Appendix.

For the existence of the configurations (n) $(36 \leq n \leq 62)$ in Appendix, we refer to the argument at the end of § 2.

5. dP3-surfaces of the second kind and type (IIc). Let (V, D) be a dP3-surface of the second kind and type (IIc) with a (-1)-curve C meeting a (-2)-curve D_1 and the (-3)-curve D_2 as in Lemma 3.1. Let $\sigma: V \to W$ be the contraction of C, let $\tilde{C} = \sigma(D_1)$ and let $B = \sigma_*(D - D_1)$. Then B consists of (-2)-rods and (-2)-forks and hence $B^\# = 0$. By Lemma 4.2, (1), $1 \leq (B^\# + K_W)^2 = (K_W^2) = 9 - \#(B) \leq 8$ and $\#(D) \leq 9$. There exist a (-1)-curve $\tilde{C}$ and a (-2)-curve $\sigma(D_2)$ on W with $(\tilde{C}, \sigma(D_2)) = 1$. This is absurd if $(K_W^2) = 8$. So, $(K_W^2) \leq 7$ and $\#(B) \geq 2$. As in the proof of Theorem 4.1 for the case (3c), there exists a nonsingular elliptic curve A in $|-K_W|$ such that $(W, A + B)$ is an Iitaka surface with $\rho(W) = \#(\mathrm{Bk}(A + \mathrm{B})) + 1$. B is contractible to rational double singular points on a Gorenstein log del Pezzo surface $\bar{W}$ of rank one. The dual graph of B is described (as $B = N$) in the following lemma which is proved in [10; Lemmas 3.5, 4.2 and 4.3].

LEMMA 5.1. *Let $(V, A + N)$ be an Iitaka surface where A is a nonsingular elliptic curve. Suppose that $\rho(V) = \#(N) + 1$. Then the dual graph of N is one of the following Dynkin graphs*:

$$A_1, A_1 + A_2, A_4, 2A_1 + A_3, D_5, A_1 + A_5, 3A_2, E_6, 3A_1 + D_4, A_7, A_1 + D_6, E_7,$$
$$A_1 + 2A_3, A_2 + A_5, D_8, 2A_1 + D_6, E_8, A_1 + E_7, A_1 + A_7, 2A_4, A_8, A_1 + A_2 + A_5,$$
$$A_2 + E_6, A_3 + D_5, 4A_2, 2A_1 + 2A_3, 2D_4 .$$

Our purpose is to prove the following:

THEOREM 5.2. *Let (V, D) be a* dP3-*surface of the second kind and type (IIc). Then the dual graph of D is one of those given in the cases No. n $(63 \leq n \leq 97)$ in the Table. Furthermore, there is a (-1)-curve C with which CASE (γ) in Lemma* 3.1 *occurs and there is a $\boldsymbol{P}^1$-fibration $\Psi: V \to \boldsymbol{P}^1$ such that the following assertions hold*:

(1) *The configuration of $C + D$ and all singular fibers of Ψ is given in the configuration (n) for $63 \leq n \leq 97$ in Appendix.*

(2) *All components of D, except at most two cross-sections, say H_1, H_k $(k \leq 2)$, are contained in singular fibers of Ψ, and H_1 and H_2 are disjoint provided $k = 2$ and $n \neq 83$.*

All the cases $(63 \leq n \leq 97)$ are realizable. Finally, V^0 is affine-ruled if $n \neq 93$.

Let C, D_1 and D_2 be as in the beginning of § 5. *Let Δ_i* be the connected component of D containing D_i for $i = 1, 2$, respectively. Since $|C + D + K_V| = \varnothing$, $\Delta_1 \neq \Delta_2$. We first prove:

LEMMA 5.3. *Assume that either Δ_1 is a rod and D_1 meets two (-2)-curves D_3 and D_4 in $D - D_1$, or Δ_1 is a fork with a central component G and three maximal twigs T_i's (hence $\Delta_1 = G + \sum T_i$). Then we have the following assertions*:

(1) *If Δ_2 is a fork then D_2 is one of the three tips of Δ_2.*

(2) *If Δ_2 is a rod then $\Delta_2 = D_2$.*

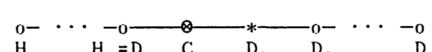
(3) *Suppose that Δ_1 is a fork. If $D_1 = G$ then Δ_1 is of Dynkin type D_4. If D_1 is a component of T_i then either D_1 is the tip of T_i or D_1 meets the tip of T_i.*

PROOF. We shall define a P^1-fibration $\Phi: V \to P^1$. If Δ_1 is a rod, let Φ be a P^1-fibration defined by a linear pencil $|2(C+D_1)+D_3+D_4|$. Suppose Δ_1 is a fork. Write $T_i = \sum_{j=1}^{n(i)} T_i(j)$ such that $T_i(j)$'s are irreducible components of T_i and $(T_i(j), T_i(j+1)) = (T_i(1), G) = 1$. If $D_1 = G$, we label three twigs so that $n(1) \geq n(2) \geq n(3)$ and let Φ be a P^1-fibration defined by $|2(C+D_1)+T_2(1)+T_3(1)|$. If $D_1 = T_i(j)$ for some i, j, we label twigs so that D_1 is a component of the twig T_1, and let Φ be a P^1-fibration defined by $|2(C+T_1(1)+\cdots+T_1(j)+G)+T_2(1)+T_3(1)|$. Note that D_2 is a 2-section of Φ in each case.

Suppose the assertion (1) is false. Then $\Delta_2 - D_2$ is not connected and we let f_1 be a singular fiber of Φ which contains a connected component of $\Delta_2 - D_2$ and which does not contain the central component of Δ_2. Since D_2 is a 2-section of Φ, there is a (-1)-curve E in f_1 such that $E+D$ contains a loop and hence $|E+D+K_V| \neq \emptyset$. But then (V, D) is of the first kind with the curve E because $-(E, D^\# + K_V) = -(C, D^\# + K_V)$ by Lemma 1.3, (3). This is a contradiction. The assertions (2) and (3) can be proved in a similar way. Indeed, if the assertion (2) is false we let f_1 be a singular fiber of Φ containing a connected component of $\Delta_2 - D_2$ and if the assertion (3) is false we let f_1 be a singular fiber of Φ containing $T_1(n_1)$. q.e.d.

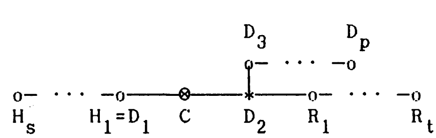
By Lemma 5.3 and by noting that $g(D)$ are quotient singularities on $\bar{V}$, the dual graph of $C+\Delta_1+\Delta_2$ is one of those cases as shown in Figures (10)$\sim$(12). Δ_1 and Δ_2 are rods in the cases (1)$\sim$(3); Δ_1 is a rod and Δ_2 is a fork in the cases (4)$\sim$(7); Δ_1 is a fork and Δ_2 is a rod in the cases (8)$\sim$(10); Δ_1 and Δ_2 are forks in the cases (11)$\sim$(13).

(1)



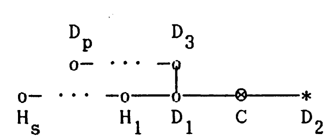
$(s \geq 1, p \geq 2)$

(2)



$(s \geq 1, t \geq 1, p \geq 3, t \geq p-2)$

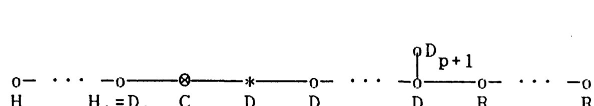
(3)



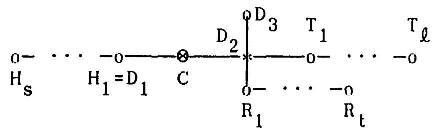
$(p \geq 3, s \geq 1, s \geq p-2)$

FIGURE (10)

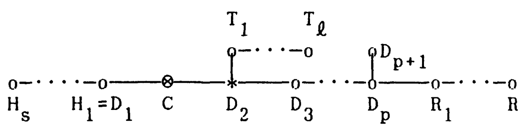
(4)



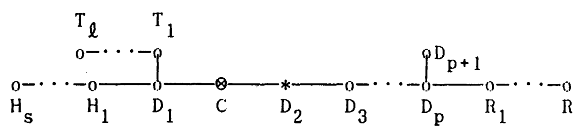
$(s \geq 1, t \geq 1, p \geq 3)$

(5) 

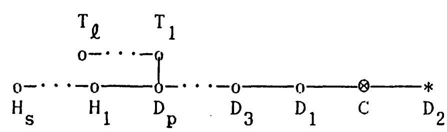
($s \geq 1, t \geq 1; l = 1, 2; t \geq l$)

(6) 

($s \geq 1, l \geq 1, p \geq 3; t = 1, 2$)

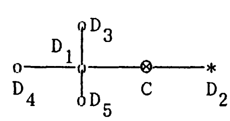
(7) 

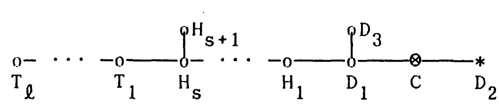
($l \geq 1, s \geq 1, t \geq 1, p \geq 3$)

(8) 

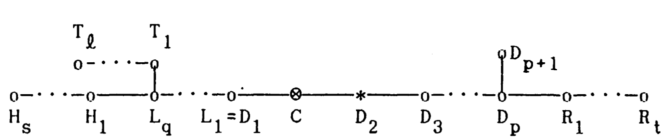
($p \geq 3, l \geq 1, s \geq 1, s \geq l$)

FIGURE (11)

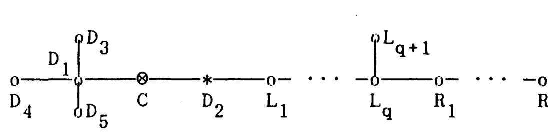
(9) 

(10) 

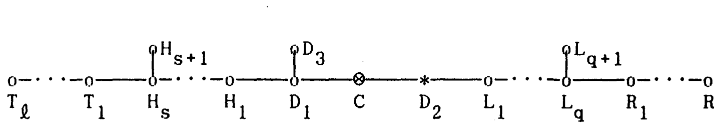
($l \geq 1, s \geq 1$)

(11) 

($l \geq 1, s \geq 1,$
$t \geq 1, q \geq 2, p \geq 3$)

(12) 

($q \geq 1, t \geq 1$)

(13) 

($l \geq 1, s \geq 1,$
$q \geq 1, t \geq 1$)

FIGURE (12)

Case (1). By Lemma 1.3, (1), $C+\Delta_1+\Delta_2$ is not negative definite, hence $s\geq 2$. Let $\Psi: V\to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$, where $S_0=3C+2D_1+H_2+D_2$. $D-H_3-D_3$, where we set $H_3=0$ if $s=2$ and set $D_3=0$ if $p=2$, is contained in the singular fibers. By Lemma 1.3, (2) and Lemma 1.5, (1), the configuration of $C+D$ and all singular fibers of Ψ is given in the configuration (n) for $63\leq n\leq 72$ in Appendix. By Lemma 1.5, (2), V^0 is affine-ruled.

Case (2). Assume $s\leq 2$. Let $\Psi: V\to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|4C+2(D_1+D_2)+D_3+R_1|$ if $s=1$ (resp. by $|3C+2D_1+H_2+D_2|$ if $s=2$). Then $D-D_4-R_2$ (resp. $D-D_3-R_1$) is contained in the singular fibers. For the same reason as in the case (1), the configuration of $C+D$ and all singular fibers of Ψ is given in the configuration (n) for $73\leq n\leq 81$ in Appendix, and V^0 is affine-ruled.

Let (W, B) be the pair given before Lemma 5.1. Then B contains connected components of Dynkin types A_{s-1} and A_{p+t-1}.

Assume $s=3$. By Lemma 5.1, B is of type A_2+A_5 or type $A_1+A_2+A_5$. Hence $p+t=6$ and $D-\Delta_1-\Delta_2=\varnothing$ or a (-2)-curve, respectively. We may assume that $t\geq 2$. Let $\Phi: V\to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|f_0|$, where $f_0=4C+2(D_1+D_2)+D_3+R_1$. Then H_2 is a 2-section, and R_2 and D_4 (if it exists) are cross-sections. Let f_1 be the fiber containing H_3, which is of type (I), and let E_1 be the (-1)-curve in the fiber f_1 with $(H_2, E_1)=(H_3, E_1)=1$. We can show that C and $E_1+H_2+H_3+K_V$ are numerically (hence linearly) equivalent. So, R_2 and D_4 (if it exists) meet another (-1)-curve E_2 in f_1 since $(C, D_4)=(C, R_2)=0$. By Lemma 1.5, (3) where $(v_*R_2)^2=-2$, we have $(v_*H_2)^2=8$. This, together with (2) of Lemma 1.3 and (1) of Lemma 1.5, implies that $p=t=3$, that B is of type $A_1+A_2+A_5$ (whence D has the dual graph No. 82 in the Table), and that R_3, together with one (-1)-curve and the curve $G:=D-\Delta_1-\Delta_2$, forms a singular fiber $(\neq f_0, f_1)$. Consider a $\boldsymbol{P}^1$-fibration $\Psi: V\to \boldsymbol{P}^1$ defined by $|S_0|$, where $S_0=C+D_1+H_2+H_3+E_2$. Then V^0 is affine-ruled by Lemma 1.5, (2). Indeed, there exist two (-1)-curves F_1 and F_2 such that S_0, $S_1:=2F_1+R_1+G$ and $S_2:=2F_2+D_3+R_3$ exhaust singular fibers of Ψ. The configuration of $C+D$ and S_i's is given in the configuration (82) in Appendix.

Assume $s=4$. By Lemma 5.1, B is of type A_1+2A_3 or $2A_1+2A_3$. Hence $p=3$ and $t=1$. Let f_0 and Φ be as in the case $s=3$. By Lemma 1.3, (2), the singular fiber of Φ containing H_3+H_4 is of type (II). This is impossible.

Assume $s\geq 5$. Then B is of type $2A_4$ by Lemma 5.1. Hence $s=5$ and one may assume $p=3$ and $t=2$. Then D has the dual graph No. 83 in the Table. Let f_0 and Φ be as in the case $s=3$. Let f_1 $(\neq f_0)$ be the singular fiber containing $H_3+H_4+H_5$. Let $v: V\to \Sigma_2$ be a contraction of curves in fibers as in (3) of Lemma 1.5 with $(v_*R_2)^2=-2$. Then $(v_*H_2)^2=8$ since H_2 is a 2-section with $(R_2, H_2)=0$. Hence, there are two (-1)-curves E_1 and E_2 in f_1 such that $(E_1, H_3)=(E_1, R_2)=(E_2, H_5)=(E_2, H_2)=1$. Let $\tilde{E}$ be a nonsingular irreducible curve in $|v_*R_2+2v_*f_0|$ such that $\tilde{E}$ meets v_*H_2 with local intersection number $i(v_*H_2, \tilde{E}; v(E_2))=4$ at the node $v(E_2)$. Then the proper transform $E:=v'\tilde{E}$ is a (-1)-curve with $(E, R_1)=(E, H_5)=(E, f_0)=1$. Let $S_0=C+$

$D_1+H_2+H_3+H_4+H_5+E$ and let Ψ: $V \to P^1$ be the P^1-fibration defined by $|S_0|$. By Lemma 1.3, (2), there is a (-1)-curve E_3 such that $S_1=2E_3+D_3+R_2$ is the unique singular fiber of Ψ other than S_0. The configuration of $C+D$ and S_i's is given in the configuration (83) in Appendix. V^0 is affine-ruled by Lemma 1.5, (2).

Case (3). Let (W, B) be the pair obtained from (V, D) as before Lemma 5.1. Then B contains the graphs of types A_1, A_{p-2} and A_s. By Lemma 5.1, B is of type $2A_1+A_3$ $(p=s=3)$, A_1+2A_3 $(p=5, s=3)$, $A_1+A_2+A_5$ $(p=4, s=5)$, $3A_1+D_4$ $(p=3, s=1)$ or $2A_1+2A_3$ $(p=3, 5; s=3)$. Let $f_0=3C+2D_1+D_3+D_2$ and let Φ: $V \to P^1$ be the P^1-fibration defined by $|f_0|$. Then H_1 is a 2-section and D_4 (if it exists) is a cross-section. By Lemma 1.3, (2) and Lemma 1.5, (3) where if D_4 exists we consider a contraction v: $V \to \Sigma_2$ such that $(v_*D_4)^2=-2$ and $(v_*H_1)^2=8$, we know that B is of type $3A_1+D_4$, $A_1+A_2+A_5$ or $2A_1+2A_3$ $(p=5, s=3)$ and that all singular fibers $f_0, f_1, \cdots, f_k$ are described as follows:

(i) B is of type $3A_1+D_4$. Hence D has the dual graph No. 84 in the Table, $k=1$, and $f_1=2(E+D_4+D_5)+D_6+D_7$ where E is a (-1)-curve and $D_4+D_5+D_6+D_7=D-\Delta_1-\Delta_2$ is a fork of Dynkin type (D_4) with D_5 as the central component. Let $S_0=2E+D_4+H_1$ and let Ψ: $V \to P^1$ be the P^1-fibration defined by $|S_0|$. Then there are (-1)-curves E_1 and E_2 such that S_0, $S_1:=2E_1+D_3+D_6$ and $S_2:=2E_2+D_7+D_2+C$ are all singular fibers of Ψ by Lemma 1.3, (2). The configuration of $C+D$ and S_i's is given in the configuration (84) in Appendix. V^0 is affine-ruled by Lemma 1.5 (2).

(ii) B is of type $A_1+A_2+A_5$. Then $k=1$ and there are two (-1)-curves E_1 and E_2 such that $(E_1, H_2)=(E_1, D_4)=(E_2, H_5)=(E_2, H_1)=1$ and $f_1=E_1+H_2+H_3+H_4+H_5+E_2$. As in the case (2) above, there exists a (-1)-curve E such that $(E, D_2)=(E, H_3)=(E, f_0)=1$. Let $S_0=C+D_1+H_1+H_2+H_3+E$ and let Ψ: $V \to P^1$ be the P^1-fibration defined by $|S_0|$. By Lemma 1.3, (2) and Lemma 1.5, (3) where we let v: $V \to \Sigma_2$ be a contraction of curves in fibers such that $(v_*D_3)^2=-2$ and $(v_*D_2)^2=8$, there are two (-1)-curves F_1 and F_2 such that F_1+F_2 is a singular fiber of Ψ and such that $(F_1, D_2)=2$. But then $-(F_1, D^\# + K_V)=1-2/3<1-1/3=-(C, D^\# + K_V)$, contradicting the minimality of $-(C, D^\# + K_V)$.

(iii) B is of type $2A_1+2A_3$ $(p=5, s=3)$. Then $k=2$ and there are three (-1)-curves E_1, E_2 and E_3 such that $(E_2, H_2)=(E_2, D_4)=(E_3, H_3)=(E_3, H_1)=1, f_1=2E_1+D_5+D_6$ $(D_6:=D-\Delta_1-\Delta_2)$ and $f_2=E_2+H_2+H_3+E_3$. We can find a (-1)-curve E, as in the case (2) above, such that $(E, D_2)=(E, D_6)=(E, H_2)=(E, f_0)=1$. By considering the P^1-fibration defined by $|2E+D_6+H_2|$, we reach a contradiction to the choice of C as in the case (ii) above.

Case (4). Assume $s=1$. Let $S_0=4C+2(D_1+D_2+\cdots+D_p)+D_{p+1}+R_1$ and Ψ: $V \to P^1$ the P^1-fibration defined by $|S_0|$. Then $t \geq 2$ by Lemma 1.3, (2) and $D-R_2$ is contained in the singular fibers. Hence V^0 is affine-ruled. By Lemma 1.3, (2), all singular fibers $S_1, \cdots, S_k$ $(\neq S_0)$ are of type (II). Since $g(D)$ are quotient singularities on $\bar{V}$, $\{k; \#S_0, \cdots, \#S_k\}=\{0; 6\}$, $\{0; 7\}$ or $\{1; 6, 3\}$. The configuration of $C+D$ and S_i's is given in the configuration (85), (86) or (87), respectively, in Appendix.

Assume $s \geq 2$. Let $S_0 = 3C + 2D_1 + H_2 + D_2$ and Ψ: $V \to \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Then D_3 and H_3 (if it exists) are cross-sections.

Assume further $t = 1$. By Lemma 5.1, B is of Dynkin type $3A_1 + D_4$ $(s = 2, p = 3)$, $A_1 + D_6$ $(s = 2, p = 5)$, $2A_1 + D_6$ $(s = 2, p = 5)$ or $A_3 + D_5$ $(s = p = 4)$. In the first (resp. third or remaining) case (s), $\mathrm{D} - \Delta_1 - \Delta_2$ consists of two (resp. one or none) isolated (-2)-curve (s). Suppose $s = 2$. Then $D - D_3$ is contained in the singular fibers of Ψ. Hence V^0 is affine-ruled. By Lemma 1.3, (2), the third case is impossible and there are two (-1)-curves E_1 and E_2 in the first case (resp. one (-1)-curve E in the second case) such that S_0, $S_1 := 2E_1 + D_4 + D_5$ and $S_2 := 2E_2 + R_1 + D_6$ (resp. S_0 and $S_1 := 2(E + R_1 + D_5) + D_4 + D_6$) are all singular fibers of Ψ, where $D_5 + D_6 := D - \Delta_1 - \Delta_2$ is a union of two isolated (-2)-curves in D. The configuration of $C + D$ and S_i's is given in the configuration (88) or (89) in Appendix. Suppose $(s, p) = (4, 4)$. Then H_4 and $R_1 + D_4 + D_5$ are in two distinct singular fibers of type (I). This contradicts Lemma 1.3, (2).

Next assume $t \geq 2$. Hence $(p, t) = (3, 2), (4, 2), (3, 3)$ or $(3, 4)$ for $g(D)$ are quotient singularities on $\bar{V}$. Thus, B contains a subgraph of type A_{s-1} and a subgraph of Dynkin type D_5, E_6, D_6 or D_7, respectively. By Lemma 5.1, B is of type $A_1 + D_6$ $(s = 2, p = t = 3)$, $2A_1 + D_6$ $(s = 2, p = t = 3)$, $A_2 + E_6$ $(s = 3, p = 4, t = 2)$ or $A_3 + D_5$ $(s = 4, p = 3, t = 2)$. $D - \Delta_1 - \Delta_2$ consists of a (-2)-curve in the second case and none otherwise. Consider the case where $(s, p, t) = (3, 4, 2)$ or $(4, 3, 2)$. By Lemma 1.3, (2) and Lemma 1.5, (1), there are (-1)-curves E_1 and E_2 (resp. E_1, E_2 and E_3) such that $(E_1, R_2) = (E_1, H_3) = (E_2, D_5) = 1$ (resp. $(E_1, H_3) = (E_1, R_2) = (E_2, R_1) = 1$) and that S_0 and $S_1 := E_1 + R_2 + R_1 + D_4 + D_5 + E_2$ (resp. S_0, $S_1 := E_1 + R_2 + R_1 + E_2$ and $S_2 := 2E_3 + H_4 + D_4$) are all singular fibers of Ψ. The configuration of $C + D$ and S_i's is given in the configuration (90) or (91) in Appendix. By Lemma 1.5, (2), V^0 is affine-ruled. Consider the case where $(s, p, t) = (2, 3, 3)$. Then $D - D_3$ is contained in the singular fibers of Ψ and V^0 is affine-ruled. By Lemma 1.3, (2), $D - \Delta_1 - \Delta_2$ consists of a (-2)-curve D_5, i.e., B is of type $2A_1 + D_6$, and there are two (-1)-curves E_1 and E_2 such that S_0, $S_1 := 2E_1 + D_4 + D_5$ and $S_2 := 2(E_2 + R_2) + R_1 + R_3$ are all singular fibers of Ψ. The configuration of $C + D$ and S_i's is given in the configuration (92) in Appendix.

Case (5). Assume $s = 1$. Let $f_0 = 4C + 2(D_1 + D_2) + D_3 + T_1$ and Φ: $V \to \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|f_0|$. R_1 is a 2-section and T_2 (if it exists) is a cross-section. Suppose $l = 1$. By Lemma 1.3, (2), Lemma 1.5, (3) and the fact $\#(D) \leq 9$, we know that $t = 1$, that there is a (-1)-curve E with $(E, R_1) = 1$ and that f_0, $f_1 := 2(E + D_4 + D_5) + D_6 + D_7$ are all singular fibers of Φ, where $D_4 + D_5 + D_6 + D_7 := D - \Delta_1 - \Delta_2$. Then D has the dual graph No. 93 in the Table. Let $S_0 = 2E + R_1 + D_4$ and Ψ: $V \to \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. By Lemma 1.3, (2), there are (-1)-curves E_1, E_2 and E_3 such that S_0, $S_1 := C + D_1 + E_1$, $S_2 := 2E_2 + D_3 + D_6$ and $S_3 := 2E_3 + T_1 + D_7$ are all singular fibers of Ψ. The configuration of $C + D$ and S_i's is given in the configuration (93) in Appendix. Suppose $l = 2$. By Lemma 1.3, (2) and by noting that the cross-section T_2 meets only T_1 in $D - T_2$, there are two (-1)-curves E_1 and E_2 such that S_0 and

$S_1:=E_1+R_2+\cdots+R_t+E_2$ are all singular fibers of Ψ. By Lemma 1.5, (3) where $(v_*T_2)^2=-2$, we must have $(v_*R_1)^2=8$. This is however impossible because $\#(D)\leq 9$ implies $t\leq 4$.

Assume $s\geq 2$. Then B is of type $3A_1+D_4$ $(s=2, l=t=1)$, A_1+D_6 $(s=2, l=1, t=3)$, $2A_1+D_6$ $(s=2, l=1, t=3)$, A_1+E_7 $(s=l=2, t=3)$, A_2+E_6 $(s=3, l=t=2)$ or A_3+D_5 $(s=4, l=1, t=2)$. Hence $D-\Delta_1-\Delta_2$ has two (resp. one, none) isolated (-2)-curves in the first (resp. third, remaining) case (s).

Consider the case $l=1$. Hence $s=2, 4$. Let f_0 and Φ be the same as in the case $s=1$. Suppose $t=1$. Then B is of type $3A_1+D_4$ and there are exactly two more singular fibers f_1 and f_2 $(\neq f_0)$ such that f_1 is of type (II) with $\#(f_1)=3$ and f_2 consists of two (-1)-curves E_1 and E_2 by Lemma 1.3, (2). Since $(R_1, E_1+E_2)=2$ we may assume $(R_1, E_1)\leq 1$. By (3) of Lemma 1.5 where we consider a contraction $v\colon V\to\Sigma_n$ which contracts E_1, we must have $(v_*R_1)^2\geq 4$, which is impossible. Suppose $t\geq 2$. Let f_1 be the singular fiber of type (I) containing $R_2+\cdots+R_t$. By Lemma 1.3, (2), all other singular fibers $(\neq f_0, f_1)$ are of type (II). Hence $s=2$ and $D=\Delta_1+\Delta_2$, i.e., B is of type A_1+D_6. By Lemma 1.5, (3) where v is a contraction which does not contract the (-1)-curve of f_1 meeting R_1, we must have $(v_*R_1)^2\geq 4$. This is impossible.

Consider the case $l\geq 2$. Then B is of type A_1+E_7 $(s=l=2,\ t=3)$ or A_2+E_6 $(s=3, l=t=2)$. Let $S_0=3C+2D_1+H_2+D_2$ and $\Phi\colon V\to\boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. D_3, T_1, R_1 and H_3 (if it exists) are cross-sections. Let S_1 and S_2 be singular fibers such that $S_1\geq T_2$ and $S_2\geq R_2$. Then $S_1\neq S_2$ and they are of type (I), for the cross-section D_3 meets only D_2 in $D-D_3$. If B is of type A_1+E_7 then S_0, S_1 and S_2 are all singular fibers by Lemma 1.3, (2). We then let $v\colon V\to\Sigma_2$ be the same as in Lemma 1.5, (1) where $H_1:=D_3$ and $H_2:=T_1$ or R_1. Then $(v_*T_1)^2=(v_*R_1)^2=(v_*T_1, v_*R_1)=2$, which is impossible by the construction of v. If B is of type A_2+E_6 then there is a singular fiber S_3 consisting of two (-1)-curves E_1 and E_2 by Lemma 1.3, (2). Hence $(D_3+T_1+R_1, E_i)\geq 2$ for $i=1$ or 2, say for $i=1$. But then $-(E_1, D^\#+K_V)\leq 1-3/7-4/7=0$ (cf. Lemma 1.4). This is not the case.

Case (6). Note that if $t=2$ then $l=1, p=3$, for $g(\Delta_2)$ is a quotient singularity of $\bar{V}$.

Assume $s=1$ and $l=1$. Let $S_0=4C+2(D_1+D_2)+T_1+D_3$ and $\Psi\colon V\to\boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. By Lemma 1.3, (2) and Lemma 1.5, (1) and by noting that $g(\Delta_2)$ is a quotient singularity on $\bar{V}$, we know that if $S_0, S_1, \cdots, S_k$ are all singular fibers, then $\{k; \#S_0, \cdots, \#S_k\}=\{1; 5, 5\}$ (where $p=6$), $\{2; 5, 3, 3\}$ or $\{1; 5, 5\}$ (where $p=3$). The configuration of $C+D$ and S_i's is respectively given in the configuration (94), (95) or (96) in Appendix. V^0 is affine-ruled by Lemma 1.5, (2).

Assume $s=1$ and $l\geq 2$. Then $t=1$. Let $S_0=4C+2(D_1+D_2+\cdots+D_p)+D_{p+1}+R_1$ and $\Phi\colon V\to\boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Then the singular fiber containing $T_2+\cdots+T_l$ must be of type (I), which contradicts Lemma 1.3, (2).

Assume $s\geq 2$. By Lemma 5.1, B is of type A_1+D_6 $((s, l, p, t)=(2, 1, 4, 1), (2, 2, 3, 1))$, type $2A_1+D_6$ $((s, l, p, t)=(2, 1, 4, 1), (2, 2, 3, 1))$, type A_3+D_5 $((s, l, p, t)=(4, 1, 3, 1))$ or type A_2+E_6 $((s, l, p, t)=(3, 1, 3, 2))$. Let $\Phi\colon V\to\boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$, where

$S_0=3C+2D_1+H_2+D_2$. T_1, D_3 and H_3 (if it exists) are cross-sections.

Consider the case $s=2$. By Lemma 1.3, (2), B is of type A_1+D_6, and we may assume that R_1 is in a singular fiber of type (I), and that if $l=2$ then T_2 and D_4 are in the same singular fiber of type (II). This leads to a contradiction by Lemma 1.5, (1) where $H_1:=T_1$ and $H_2:=D_3$.

Consider the case $s\geq 3$. If $(s, l, p, t)=(4, 1, 3, 1)$ (resp. $(3, 1, 3, 2)$), then H_4, D_4 and R_1 (resp. D_4 and R_1+R_2) are in distinct singular fibers of type (I) because the cross-section T_1 meets only D_2 in $D-T_1$. This contradicts Lemma 1.3, (2) (resp. Lemma 1.5, (1) where $H_1:=D_3$ and $H_2:=T_1$).

Case (7). *Let* $S_0=2(C+D_1)+T_1+H_1$ and Φ: $V\rightarrow \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Then D_2 is a 2-section. Let S_1 be the singular fiber containing Δ_2-D_2. As in Lemma 5.3, one can prove that S_1 is a fiber of type (II). Hence $t=1$ and there exists a (-1)-curve E meeting D_3 such that $S_1=2(E+D_3+\cdots+D_p)+D_{p+1}+R_1$. By Lemma 1.3, (3), we have $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$. Then (V, D), with the curve E, is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (II*b*) according as $|E+D+K_V|\neq\varnothing$ or $|E+D+K_V|=\varnothing$. This is a contradiction.

Case (8). Let $S_0=3C+2D_1+D_3+D_2$ and Ψ: $V\rightarrow \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Let $S_0, S_1, \cdots, S_k$ be all singular fibers of Ψ.

Assume $s=1$ (hence $l=1$). Let $f_0=2(C+D_1+D_3+\cdots+D_p)+T_1+H_1$ and Φ: $V\rightarrow \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|f_0|$. By Lemma 1.3, (2), all singular fibers $f_0, f_1, \cdots, f_b$ of Φ are of type (II). By Lemma 1.5, (3), $b\leq 1$ and $4m=(v_*D_2)^2=-3+p+1+\sum_{i=1}^{b}(\#f_i-1)\leq 5$ (for $\#(D)\leq 9$). Hence $m=1$ and $6=p+\sum_{i=1}^{b}(\#f_i-1)$. If $b=1$, then $\#(f_1)=4$ or 3. For the (-1)-curve E in f_1 we have $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ by Lemma 1.3, (3). Hence, if $\#(f_1)=4$, we are reduced to the previous case (3) by replacing C by E. If $\#(f_1)=3$, (V, D), with the curve E, is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (II*a*) according as $|E+D+K_V|\neq\varnothing$ or $|E+D+K_V|=\varnothing$. This is a contradiction. Thus we may assume $b=0$ and $p=6$, and D has the dual graph No. 97 in the Table. Since all components of D, except the cross-section D_4, are contained in the singular fibers of Ψ, V^0 is affine-ruled. By Lemma 1.3, (2), $k=1$ and there is a (-1)-curve E such that $(E, T_1)=1$ and $S_1=2(E+T_1+D_6)+H_1+D_5$. The configuration of $C+D$ and S_i's is given in the configuration (97) in Appendix.

Assume $s\geq 2$. By Lemma 1.3, (2), Lemma 1.5, (1) and by noting that $g(D)$ are quotient singularities on $\bar{V}$, we have one of the following cases:

(i) $(p, l, s, k)=(3, 1, 5, 1)$. Then there are (-1)-curves E_1 and E_2 such that $(E_1, H_2)=(E_2, T_1)=(E_2, H_5)=1$ and that $S_1=E_1+H_2+H_3+H_4+H_5+E_2$. Let τ: $Y\rightarrow W$ be the blowing-up of the point $A\cap\sigma(E_2)$, where σ, W, B and A are as given before Lemma 5.1. Let $\Delta=T_1+D_3+H_1+\cdots+H_5+E_2$. Then $-K_W\sim A\sim\sigma(\Delta)$ and $-K_Y\sim\tau'(A)\sim\tau'\sigma(\Delta)$. Hence $\varphi:=\Phi_{|\tau'(A)|}$: $Y\rightarrow \boldsymbol{P}^1$ is a minimal elliptic fibration. Hence there is a (-1)-curve E on V such that $(E, D_2)=2$, $(\tau'\sigma(E))^2=-2$ and $\tau'\sigma(E+D_2)$ is a singular fiber of φ. But then $-(E, D^{\#}+K_V)=1-2/3<1-1/3=-(C, D^{\#}+K_V)$, a contradiction.

(ii) $(p, l, s, k)=(3, 2, 2, 2)$. Then there are (-1)-curves E_1, E_2 and E_3 such that $(E_1, T_1)=(E_2, H_1)=1$ and that $S_1=E_1+D_4+D_5+E_2$ and $S_2=2E_3+T_2+H_2$, where $D_4+D_5:=D-\Delta_1-\Delta_2$ is a (-2)-rod with two components. Let $\Delta=T_2+T_1+D_3+H_1+H_2+E_3$. By the same argument as in the above case (i), we reach a contradiction to the minimality of $-(C, D^{\#}+K_V)$.

(iii) $(p, l, s, k)=(4, 1, 3, 2)$. Then there are (-1)-curves E_1 and E_2 such that $S_1=2E_1+T_1+D_5$ and $S_2=2(E_2+H_2)+H_3+H_1$, where $D_5:=D-\Delta_1-\Delta_2$ is an isolated (-2)-curve in D. As in the case (2), one can find a (-1)-curve E such that $(E, D_2)=(E, D_5)=(E, H_3)=(E, S_0)=1$. Note that $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)=1-1/3$. Then, with the curve E, (V, D) is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (II*a*) according as $|E+D+K_V|\neq\varnothing$ or $|E+D+K_V|=\varnothing$. This is absurd.

(iv) $(p, l, s, k)=(6, 1, 2, 1)$. Then there is a (-1)-curve E_1 such that $(E_1, H_2)=1$ and $S_1=2(E_1+H_2+H_1+D_6)+T_1+D_5$. Consider the $\boldsymbol{P}^1$-fibration $\psi\colon V\to\boldsymbol{P}^1$ defined by $|2(C+D_1+D_3+\cdots+D_6)+T_1+H_1|$. Let E be the (-1)-curve such that $E+E_1$ is a singular fiber of ψ. Then $(D_2, E)=(D_2, E+E_1)=2$ and $-(E, D^{\#}+K_V)=1-2/3<1-1/3=-(C, D^{\#}+K_V)$, contradiction.

Case (9). Then B is of type $4A_1$. This is impossible by Lemma 5.1.

Case (10). Consider the $\boldsymbol{P}^1$-fibration $\psi:=\Phi_{|f_0|}$, where $f_0=3C+2D_1+D_3+D_2$, of which H_1 is a 2-section. By Lemma 1.3, (2), the singular fibers of ψ $(\neq f_0)$ are of type (II). Hence $l=1$. Let $S_0=2(C+D_1+H_1+\cdots+H_s)+H_{s+1}+T_1$ and $\Phi\colon V\to\boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. D_2 and D_3 are 2-sections of Φ. By Lemma 1.3, (2), there is a singular fiber S_1 of type (I) with two (-1)-curves E_1 and E_2. If $(D_i, E_j)=2$ for some (i, j) with $i=2, 3$ and $j=1, 2$, then E_j+D contains a loop and hence $|E_j+D+K_V|\neq\varnothing$ for $j=1$ or 2. Then (V, D) is a dP3-surface of the first kind with the curve E_j because $-(E_j, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ by Lemma 1.3, (3). This is absurd. Thus, $(D_2, E_j)=(D_3, E_j)=1$ ($j=1$ and 2). If $\#(S_1)\geq 3$, (V, D), with the curve E_1, is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (II*a*) according as $|E_1+D+K_V|\neq\varnothing$ or $|E_1+D+K_V|=\varnothing$. This is a contradiction. If $\#(S_1)=2$, we are reduced to the previous case (8) by replacing C by E_1.

Case (11). Then B contains either two forks (if $q\geq 3$) or a fork and a rod of type A_{l+s+1} (if $q=2$). By Lemma 5.1, B is of type A_3+D_5 ($l=s=1, q=2; (p, t)=(3, 2)$ or $(4, 1)$) or $2D_4$ ($l=s=t=1, q=p=3$). In particular, $l=s=1$. Let $S_0=2(C+D_1+L_2+\cdots+L_q)+T_1+H_1$ and $\Phi\colon V\to\boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Then D_2 is a 2-section. Let S_1 be the singular fiber of Φ containing Δ_2-D_2. By Lemma 1.3, (2), S_1 is of type (II), hence $t=1$ and there is a (-1)-curve E with $(E, D_3)=1$ such that $S_1=2(E+D_3+\cdots+D_p)+D_{p+1}+R_1$. Then (V, D), with the curve E, is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (II*b*) according as $|E+D+K_V|\neq\varnothing$ or $|E+D+K_V|=\varnothing$ because $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ by Lemma 1.3, (3). This is a contradiction.

Case (12). Then B contains three isolated (-2)-curves and a fork. By Lemma

5.1, B is of type $3A_1+D_4$. Hence $q=t=1$. Let $S_0=2(C+D_1)+D_3+D_4$ and $\Phi\colon V\to \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. As in Lemma 5.3, one can prove that the singular fiber S_1 containing Δ_2-D_2 is of type (II). Then there is a (-1)-curve E such that $(E, L_1)=1$ and $S_1=2(E+L_1)+L_2+R_1$. Note that $(E, D_5)=1$. Then (V, D), with the curve E, is either a dP3-surface of the first kind or a dP3-surface of the second kind and type (IIa) according as $|E+D+K_V|\neq\varnothing$ or $|E+D+K_V|=\varnothing$ because $-(E, D^{\#}+K_V)=-(C, D^{\#}+K_V)$ by Lemma 1.3, (3). This is a contradiction.

Case (13). Since $\#(D)\leq 9$, $l=s=q=t=1$ and $D=\Delta_1+\Delta_2$. Hence B is of type $A_1+A_3+D_4$. This is impossible by Lemma 5.1.

To complete the proof of Theorem 5.2, we refer to the argument at the end of § 2 for the existence of the configurations (n) ($63\leq n\leq 97$) given in Appendix.

6. Quasi-universal coverings. Let (V, D) be a dP3-surface and let $V^0=V-\mathrm{Supp}(D)$. In this section, we shall look into the fundamental group $\pi_1(V^0)$ and the quasi-universal covering $\bar{U}$ of $\bar{V}$ (cf. the notation in the Introduction). First of all, we prove:

PROPOSITION 6.1. *Let $\pi\colon X\to Y$ be a finite morphism between normal algebraic surfaces. Suppose that Y has only quotient singularities and that $\pi^0:=\pi_{|X^0}\colon X^0\to Y^0$ is étale where $Y^0:=Y-\mathrm{Sing}(Y)$ and $X^0:=\pi^{-1}(Y^0)$. Then X has only quotient singularities. In particular, if Y is a log del Pezzo surface with contractible boundary, so is X.*

PROOF. Assume that the first assertion is proved. Note that $-K_{X^0}=\pi^{0*}(-K_{Y^0})$ since π^0 is étale. Hence $-K_X=\pi^*(-K_Y)$ and $-K_X$ is ample because π is finite. Thus, the second assertion is proved.

To show the first assertion, we may assume that $Y=\mathrm{Spec}\, S$ and $X=\mathrm{Spec}\, R$, where S is the local ring of a singular point of Y. Then R is a semi-local ring with maximal ideals $\mathfrak{M}_1, \cdots, \mathfrak{M}_r$. Let $\mathfrak{M}$ be the maximal ideal of S, let $J=\mathrm{Rad}(R)$ $(=\mathfrak{M}_1\cdots\mathfrak{M}_r)$, let $\hat{S}$ be the $\mathfrak{M}$-completion of S, let $\hat{R}$ be the J-completion of R and let $\hat{R}_i$ be the $\mathfrak{M}_i$-completion of R. Then $\hat{R}=\mathrm{proj}\,\lim_N(R/J^NR)\cong\mathrm{proj}\,\lim_N(R/\mathfrak{M}^NR)\cong\mathrm{proj}\,\lim_N(R\otimes_S S/\mathfrak{M}^N)\cong R\otimes_S\hat{S}$ since R is a finite S-module. By the Chinese remainder theorem, $R/J^NR\cong\prod_i(R_{\mathfrak{M}_i}/\mathfrak{M}_i^NR_{\mathfrak{M}_i})$. Thus, $R\otimes_S\hat{S}\cong\hat{R}\cong\prod_i\hat{R}_i$. Since π is finite and π^0 is étale, the induced map $\hat{\pi}_i\colon\mathrm{Spec}\,\hat{R}_i\to\mathrm{Spec}\,\hat{S}$ is a finite morphism and is étale outside the closed point of $\hat{S}$. Note that $X=\mathrm{Spec}\ R$ has only quotient singularities if and only if $\mathrm{Spec}\,\hat{R}=\bigcup_i\mathrm{Spec}\,\hat{R}_i$ has only quotient singularities. Instead of R and S we consider $\hat{R}_i$ and $\hat{S}$. Rewrite $R:=\hat{R}_i$, $S:=\hat{S}$, $X:=\mathrm{Spec}\,\hat{R}_i$, $Y:=\mathrm{Spec}\,\hat{S}$ and $\pi:=\hat{\pi}_i$ by abuse of notation.

Since the singularity of Y is a quotient singularity, there exists a finite group $G\subseteq GL(2; k)$ such that the G-invariant ring $(k[[X_1, X_2]])^G=S$, where X_1 and X_2 are indeterminates. Let $Z:=\mathrm{Spec}\,k[[X_1, X_2]]$ and $q\colon Z\to Y$ the quotient morphism. Note that $Z\times_Y X$ is finite over Z and étale outside the closed point of Z. Let $T=k[[X_1, X_2]]\otimes_S R$ be the coordinate ring of $Z\times_Y X$. Then T is a reduced ring with minimal prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$. Let $K_i = Q(T/\mathfrak{p}_i)$ and $\tilde{T}_i$ the integral closure of $T/\mathfrak{p}_i$ in K_i. Then $\tilde{T} := \prod_i \tilde{T}_i$ is the integral closure of T in the total quotient ring $Q(T)$ $(=\prod_i K_i)$ of T. Note that $k[[X_1, X_2]] \subsetneq T/\mathfrak{p}_i$ and hence $k[[X_1, X_2]] \subsetneq \tilde{T}_i$ are integral extensions. Note also that $\operatorname{Spec} T/\mathfrak{p}_i \to Z$ and hence $\operatorname{Spec} \tilde{T}_i \to Z$ are étale outside the closed point of Z. By the purity of branch locus, $\operatorname{Spec} \tilde{T}_i \to Z$ is étale everywhere. Hence $\operatorname{Spec} \tilde{T}_i \cong Z$. Let p be the composite of morphisms: $Z \xrightarrow{\sim} \operatorname{Spec} \tilde{T}_i \subsetneq \operatorname{Spec} \tilde{T} \to Z \times_Y X \to X$. Then p is a finite morphism and if we set $H = \operatorname{Gal}(k(Z)/k(X))$, then $R = (k[[X_1, X_2]])^H$. So, X has only quotient singularities. q.e.d.

The following corollary is useful in finding the quasi-universal covering $\bar{U}$ of $\bar{V}$.

COROLLARY 6.2. *Let $\bar{V}$ be a log del Pezzo surface with contractible boundary. We employ the notation $g: V \to \bar{V}$, D and V^0 in Definition* 1 *of the Introduction. Suppose that there exist an effective divisor $\Delta > 0$ supported by D, an integer $l \geq 2$ and an integral divisor F such that $\Delta \sim lF$ and that l is prime to the greatest common divisor of the coefficients of Δ. Let $\tau_1: X_1 \to V$ be a Z/lZ-covering defined by the relation $O(F)^{\otimes l} \cong O(\Delta)$ and a nonzero global section of $O(\Delta)$. Let $\tau_2: X \to X_1$ be the normalization and let $\tau := \tau_1 \circ \tau_2$. Then $\tau^{-1}(D)$ is contractible to points on a projective normal algebraic surface $\bar{X}$ and τ induces a finite morphism $\bar{\tau}: \bar{X} \to \bar{V}$ such that $\bar{\tau} = \tau$ on X^0 $(:= \tau^{-1}(V^0))$. Hence $\bar{X}$ is a log del Pezzo surface with contractible boundary.*

PROOF. Let $\bar{\sigma}: \bar{T} \to \bar{V}$ be the normalization of $\bar{V}$ in the function field $k(X)$. Then $\bar{\sigma} = \tau$ on V^0. Hence there is a rational map $\tilde{g}: X \cdots \to \bar{T}$ such that $\tilde{g}_{|X^0}$ is an isomorphism. Let Z be the normalization of the graph of $\tilde{g}$ contained in $X \times \bar{T}$ and let $\alpha: Z \to X$ be the canonical projection. Then $\alpha_{|\alpha^{-1}(X^0)}$ is the identity morphism and $\beta := \tilde{g} \circ \alpha$ is a morphism. Note that $g \circ \tau \circ \alpha = \bar{\sigma} \circ \beta$. So, every exceptional curve of α is contractible to a point under β. Thus $\tilde{g}$ is a morphism which contracts $\tau^{-1}(D)$ to points on $\bar{T}$. Set $\bar{X} := \bar{T}$ and $\bar{\tau} := \bar{\sigma}$. Now Corollary 6.2 follows from Proposition 6.1. q.e.d.

In [9; Lemma 2] we proved the injectivity of the following composite homomorphism:

$$H_2(D; Z) \to H_2(V; Z) \xrightarrow{\text{Poincaré duality}} H^2(V; Z) \xrightarrow{\text{res}} H^2(D; Z),$$

and the assertion (1) of the following lemma.

LEMMA 6.3. *Let $\bar{V}$ be a* dP3-*surface. We employ the notation $g: V \to \bar{V}$, D and V^0 in Definition* 1 *of the Introduction. Then the following assertions hold true.*

(1) *We have $H_1(V^0; Z) \cong (H^2(D; Z)/H_2(D; Z))/(\mathrm{Cl}(\bar{V})/\mathrm{Pic}(\bar{V}))$.*

(2) *Let $\Gamma = \sum_{i=1}^n D_i$ be a connected component of D and let $G = H^2(\Gamma; Z)/H_2(\Gamma; Z)$. Then we have*:

(i) *$G \cong Z/(n+1)Z$, $(Z/2Z)^{\oplus 2}$, $Z/4Z$, $Z/3Z$, $Z/2Z$ or* (0), *according as whether the dual graph of Γ is of Dynkin type A_n $(n \geq 1)$, D_n $(n = ever, n \geq 4)$, D_n $(n = odd, n \geq 5)$,*

E_6, E_7 or E_8, respectively.

(ii) $G \cong \boldsymbol{Z}/(n+1+i(n+1-i))\boldsymbol{Z}$ *if the dual graph of Γ is given in Figure* (13).

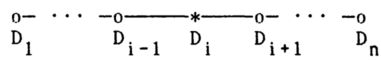


FIGURE (13)

(iii) $G \cong \begin{cases} \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2(i+1)\boldsymbol{Z} & \textit{if both } n \textit{ and } i \textit{ are even} \\ \boldsymbol{Z}/4(i+1)\boldsymbol{Z} & \textit{otherwise} \end{cases}$

if the dual graph of Γ is given in Figure (14) *and* $n \geq 4$.

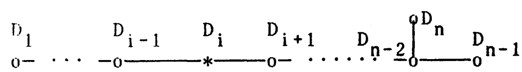


FIGURE (14)

(iv) $G \cong \boldsymbol{Z}/(n+4)\boldsymbol{Z}$ *if the dual graph of Γ is given in Figure* (15) *and* $n \geq 5$.

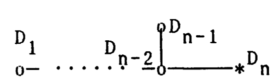


FIGURE (15)

(v) $G \cong \boldsymbol{Z}/7\boldsymbol{Z}$ *if the dual graph of Γ is given in Figure* (16).

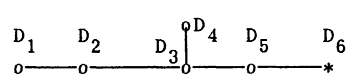


FIGURE (16)

(vi) $G \cong \boldsymbol{Z}/13\boldsymbol{Z}$ *if the dual graph of Γ is given in Figure* (17).

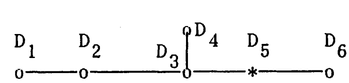


FIGURE (17)

(vii) $G \cong \boldsymbol{Z}/(5n-9)\boldsymbol{Z}$ *if the dual graph of Γ is given in Figure* (18) *and* $n \geq 5$.

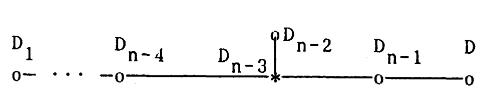


FIGURE (18)

(3) *Assume that $(V, D) \neq (\Sigma_3, M_3)$ and that there exist a $\boldsymbol{P}^1$-fibration Φ: $V \to \boldsymbol{P}^1$ and a (-2)-component H of D which is a cross-section of Φ. Then* $\mathrm{Cl}(\bar{V})$ *is generated by*

the direct images on $\bar{V}$ of all (-1)-curves E_i's $(i=1, \cdots, k)$ in the singular fibers of Φ. Moreover, if $\sum_{i=1}^{k} a_i\bar{E}_i$ is linearly equivalent to zero with $a_i \in \boldsymbol{Z}$ and $\bar{E}_i = g_*E_i$, then $\sum_{i=1}^{k} a_iE_i + \sum_{j=1}^{m} b_jD_j \sim 0$ on V with some $b_j \in \boldsymbol{Z}$ and some components D_j's of D.

(4) *Let P be a divisor on V such that $(P, D_i)=0$ for any component D_i of D. Suppose that (P, F_1) and (P, F_2) are coprime for some divisors F_1 and F_2 on V. Then g_*P generates* Pic$(\bar{V})$ *(cf. Lemma* 1.1, (3)).

PROOF. (2) Note that $G \cong (\boldsymbol{Z}\xi_1 + \cdots + \boldsymbol{Z}\xi_n)/\{\sum_{i=1}^{n}(D_i, D_j)\xi_i = 0;\ j=1, \cdots, n\}$, where ξ_i's form a $\boldsymbol{Z}$-basis of $H^2(\Gamma; \boldsymbol{Z})$. Then (2) follows from straightforward computation.

(3) Let $v: V \to \Sigma_2$ be a contraction of all (-1)-curves and consecutively contractible curves in the singular fibers of Φ such that $(v_*H)^2 = -2$. Note that $\mathrm{Cl}(\Sigma_2) = \boldsymbol{Z}[v_*H] \oplus \boldsymbol{Z}[v_*S]$, where S is a singular fiber of Φ. Therefore, $\mathrm{Cl}(V)$ is generated by H, S and all exceptional curves of v. Note that $g_*: \mathrm{Cl}(V) \to \mathrm{Cl}(\bar{V})$ is surjective. By (2) of Lemma 1.1, the first assertion is verified. The second one is obvious.

(4) By the condition that $(P, D_i)=0$ for any component D_i of D, P is linearly equivalent to a divisor disjoint from D (cf. [1; Cor. 2.6]). Hence g_*P is a Cartier divisor such that $P - g^*g_*P$ is linearly equivalent to zero. Write $g_*P \sim a\xi$ where $a \in \boldsymbol{Z}$ and ξ is a generator of Pic$(\bar{V})$. Since $(P, F_1) = a(g^*\xi, F_1)$ and $(P, F_2) = a(g^*\xi, F_2)$ are coprime we have $a = \pm 1$. q.e.d.

We shall treat only a dP3-surface $\bar{V}$ or (V, D) corresponding to the configuration (20) in Appendix and explain our method of computing $\pi_1(V^0)$ and constructing the quasi-universal covering. Let $v: V \to \Sigma_2$ be the contraction of C, E_2, D_9, D_8, D_7, D_6, E_3, D_5 which are displayed in the configuration (20). Let $\Psi: V \to \boldsymbol{P}^1$ be the vertical $\boldsymbol{P}^1$-fibration defined by $|S|$ where $S := C + E$. Then one has:

$$v^*(v_*D_3 + 3v_*S) \sim v^*v_*D_4 = D_4 + D_6 + 2D_7 + 3D_8 + 4D_9 + 5E_2 + D_5 + E_3$$

$$v^*(2v_*D_3 + 4v_*S) \sim v^*v_*D_1 = D_1 + 2C + D_6 + 2D_7 + 3D_8 + 4D_9 + 5E_2 + D_5 + 2E_3$$

$$S = E + C \sim E_1 + D_6 + \cdots + D_9 + E_2 \sim 2E_3 + D_2 + D_5 \,.$$

Hence $5(D_3 + S) = 4D_3 + (D_3 + 3S) + 2S \sim 4D_3 + D_4 + D_6 + 2D_7 + 3D_8 + 4D_9 + 5E_2 + D_5 + E_3 + 4E_3 + 2D_2 + 2D_5$ and if one lets $\Delta = 2D_2 + 4D_3 + D_4 + 3D_5 + D_6 + 2D_7 + 3D_8 + 4D_9$ and $F = S + D_3 - E_2 - E_3$ then $\Delta \sim 5F$. Let $P = -3C - 2D_1$. Then $(P, D_i) = 0$ for any component D_i of D, $(P, E) = -3$ and $(P, E_1) = -2$. By Lemma 6.3, (4), g_*P generates Pic$(\bar{V})$. Put $\bar{E} = g_*E$, $\bar{E}_1 = g_*E_1$ and so on. Then $g_*S = \bar{E} + \bar{C} \sim \bar{E}_1 + \bar{E}_2 \sim 2\bar{E}_3$, $5\bar{E}_2 + \bar{E}_3 = g_*v^*v_*D_4 \sim g_*(D_3 + 3S) \sim 6\bar{E}_3$, $2\bar{C} + 5\bar{E}_2 + 2\bar{E}_3 = g_*v^*v_*D_1 \sim g_*(2D_3 + 4S) \sim 8\bar{E}_3$ and they are all relations among $\bar{C}$, $\bar{E}$, $\bar{E}_i$'s which generate $\mathrm{Cl}(\bar{V})$ by Lemma 6.3. Hence $\mathrm{Cl}(\bar{V})/\mathrm{Pic}(\bar{V}) = (\boldsymbol{Z}[\bar{C}] + \boldsymbol{Z}[\bar{E}] + \boldsymbol{Z}[\bar{E}_1] + \boldsymbol{Z}[\bar{E}_2] + \boldsymbol{Z}[\bar{E}_3])/\sim$ (where "$\sim$" $= \{[\bar{E}] + [\bar{C}] = [\bar{E}_1] + [\bar{E}_2] = 2[\bar{E}_3],\ 5[\bar{E}_2] + [\bar{E}_3] = 6[\bar{E}_3],\ 2[\bar{C}] + 5[\bar{E}_2] + 2[\bar{E}_3] = 8[\bar{E}_3],\ 3[\bar{C}] = 0\}) \cong \boldsymbol{Z}/15\boldsymbol{Z}$. By Lemma 6.3, (1) and (2), $H_1(V^0; \boldsymbol{Z}) \cong ((\boldsymbol{Z}/5\boldsymbol{Z})^{\oplus 2} \oplus \boldsymbol{Z}/3\boldsymbol{Z})/(\boldsymbol{Z}/15\boldsymbol{Z}) \cong \boldsymbol{Z}/5\boldsymbol{Z}$.

Let $\sigma_1: U_1 \to V$ be the composite of the following morphisms in the given order:

the blowing-up τ of the point $P:=D_3\cap D_4$, the $Z/5Z$-covering defined by the relation $O(\tau^*F-\tau^{-1}(P))^{\otimes 5}\cong O(\tau'\Delta)$ and a nonzero global section of $O(\tau'\Delta)$, the normalization of the covering surface and the minimal desingularization of the isolated singularities on the normalized surface. Then $\sigma_1^{-1}(D)$ (written in solid lines) is given in Figure (19). Ψ induces a P^1-fibration $\Phi_1: U_1\to P^1$ of which all singular fibers are those four given in Figure (19). In particular, U_1 is rational and $(K_{U_1}^2)=-18$. Let $\sigma_2: U_1\to U$ be the contraction of $\sigma_1^{-1}(D_2+\cdots+D_9)$. Write $\sigma_1^{-1}(C)=\tilde{C}_1+\cdots+\tilde{C}_5$ and $\sigma_1^{-1}(D_1)=\tilde{H}_1+\cdots+\tilde{H}_5$ where $(\tilde{C}_1, \tilde{H}_1)=(\tilde{C}_1, \tilde{H}_4)=(\tilde{C}_2, \tilde{H}_2)=(\tilde{C}_2, \tilde{H}_5)=(\tilde{C}_3, \tilde{H}_3)=(\tilde{C}_3, \tilde{H}_5)=(\tilde{C}_4, \tilde{H}_2)=(\tilde{C}_4, \tilde{H}_4)=(\tilde{C}_5, \tilde{H}_1)=(\tilde{C}_5, \tilde{H}_3)=1$. Let $H_i=\sigma_2(\tilde{H}_i)$ and $C_i=\sigma_2(\tilde{C}_i)$. Let $B=\sigma_{2*}\sigma_1^{-1}(D)$ $(=H_1+\cdots+H_5)$ and let $g_2: U\to\bar{U}$ be the contraction of B. Then σ_1 induces a finite morphism $\bar{\sigma}_1: \bar{U}\to\bar{V}$ which is étale outside $\mathrm{Sing}(\bar{V})$, and $\bar{U}$ (or (U, B)) is a log del Pezzo surface with contractible boundary by Corollary 6.2. Note that $\rho(U)=10-(K_U^2)=10$ and $\rho(\bar{U})=\rho(U)-\#(B)=5$. Consider the P^1-fibration $\Phi_2: U\to P^1$ defined by $|T_0|$ where $T_0=3C_1+3C_4+2H_4+H_1+H_2$. By Lemma 1.1, (2), there are (-1)-curves F_1 and F_2 such that $(F_1, H_3)=(F_2, H_5)=1$ and that $T_1:=2C_3+F_1+F_2+H_3+H_5$ is a singular fiber of Φ_2. Let $u: U\to\Sigma_n$ be the contraction of C_3, F_2, H_5, H_3, C_1, C_4, H_4, H_2. Then $u(B)$ is a union of a single point and a fiber of the P^1-fibration $\Phi_2\circ u^{-1}: \Sigma_n\to P^1$. So, $\Sigma_n-u(B)$ and $U-B$ contain C^2. Hence $U-B$ is simply connected. Therefore, $\bar{U}$ is the quasi-universal covering of $\bar{V}$ and $\pi_1(V^0)\cong Z/5Z$.

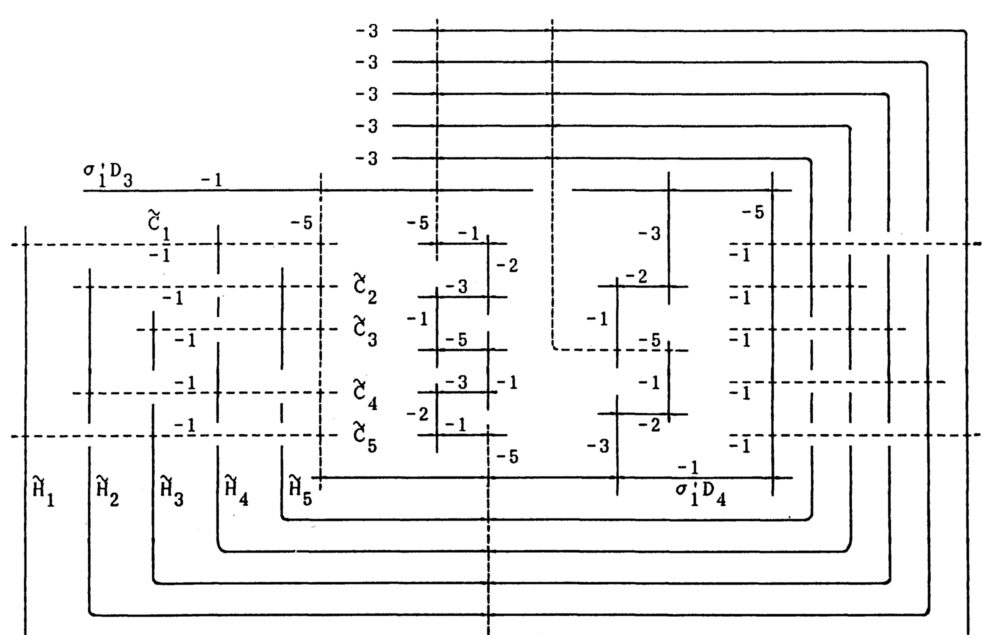


FIGURE (19)

We now prove the impossibility of the configuration (20b)′ which has the same configuration as the configuration (20a)′ in Lemma 2.6 but with the nodal curve A_* replaced by a cuspidal curve A_*. By blowing up the cusp of A_*, we can make a configuration (20b) from the configuration (20b)′ where the configuration (20b) has the same configuration as the configuration (20) (see Appendix) but with the exceptional curve C meeting the (-3)-curve D_1 with order of contact 2. In the case of the configuration (20b), using the same arguments and notation as those for the case of the configuration (20), we also have a $\boldsymbol{Z}/5\boldsymbol{Z}$-covering $\sigma_1: U_1 \to V$, a $\boldsymbol{P}^1$-fibration $\Phi_1: U_1 \to \boldsymbol{P}^1$ and Figure (19b) which has the same figure as Figure (19) but with every component $\tilde{H}_i$ of $\sigma_1^{-1}(D_1)$ meeting exactly one component of $\sigma_1^{-1}(C)=\sum \tilde{C}_i$ in a single point with order of contact 2. Let $v_1: U_1 \to \Sigma_1$ be the contraction of curves in the singular fibers of Φ_1 such that $(v_1\sigma'_1 D_3)^2=-1$. Then $(v_1\tilde{H}_i, v_1\sigma'_1 D_3)=0$ and $(v_1\tilde{H}_i)^2=6$ by the definition of v_1. This is absurd.

Similarly, one can compute $H_1(V^0; \boldsymbol{Z})$ for all cases. We also can get the quasi-universal covering $\bar{U}$ for each case with $H_1(V^0; \boldsymbol{Z}) \neq (0)$ and prove that, in this case, $\bar{U}$ is a rational log del Pezzo surface with contractible boundary, by taking successively morphisms like $\sigma_1: U_1 \to V$ in the case of the configuration (20), which are étale outside D. For the cases with $H_1(V^0; \boldsymbol{Z})=(0)$, we can check that $V^0 \supseteq \boldsymbol{C} \times \boldsymbol{C}^*$ where $\boldsymbol{C}^* := \boldsymbol{C} - \{0\}$, in the same fashion as the one given in the next paragraph for the case of the configuration (3). Hence $\pi_1(V^0)$ is a quotient group of $\pi_1(\boldsymbol{C} \times \boldsymbol{C}^*) \cong \boldsymbol{Z}$. So, $\pi_1(V^0)$ is an abelian group and $\pi_1(V^0) \cong H_1(V^0; \boldsymbol{Z})=(0)$. Since we know $H_1(V^0; \boldsymbol{Z})$ and $|\pi_1(V^0)|$, we can obtain $\pi_1(V^0)$ for all dP3-surfaces except for those with the configurations (6), (7), (27), (93) and (95) in Appendix. For the cases with the configurations (7), (93) and (95), we do not know which of D_2 and Q_3 the fundamental group $\pi_1(V^0)$ takes. For the cases with the configurations (6) and (27), we do not know what $\pi_1(V^0)$ is.

By treating the dP3-surface (V, D) corresponding to the configuration (3), we explain our method of investigating the affine-ruledness of $V^0 = V - D$. We employ the same notation $D=\sum_{i=1}^{9} D_i$, $\Psi: V \to \boldsymbol{P}$, S_0 and S_1 as in Lemma 2.3. Then $S_0=3C+2D_9+D_8+D_1$ and $S_1=E_1+D_3+\cdots+D_6+E_2$ where E_1 and E_2 are (-1)-curves with $(E_1, D_3)=(E_2, D_6)=1$. Let $\sigma_1: V_1 \to V$ be the blowing-up of the point $P := D_4 \cap D_5$ and let $F'_1=\sigma_1^{-1}(P)$. Let $\sigma_2: V_2 \to V_1$ be the blowing-up of the point $Q := \{\sigma'_1(D_5) \cap F'_1\}$ and let $\sigma := \sigma_1 \circ \sigma_2$. Denote by $F_1 := \sigma'_2(F'_1)$, $F_2 := \sigma_2^{-1}(Q)$, $\tilde{E}_i := \sigma'(E_i)$ and $\tilde{D}_i := \sigma'(D_i)$. Set $f_0 := 5(\tilde{E}_1+\tilde{D}_3)+3\tilde{D}_2+2\tilde{D}_4+F_1+\tilde{D}_1$ and $f_1 := 4(\tilde{E}_2+\tilde{D}_6)+3\tilde{D}_7+2\tilde{D}_8+\tilde{D}_9+\tilde{D}_5$. Then $|f_0|$ defines a $\boldsymbol{P}^1$-fibration $\varphi: V_2 \to \boldsymbol{P}^1$ and f_1 is the unique singular fiber of φ other than f_0. All components of $\sigma^{-1}(D)$, except F_2, are contained in the singular fibers of φ. Note that F_2 and $\sigma'(C)$ are cross-sections of φ. Let $\tau: V_2 \to \Sigma_0$ be the contraction of curves in f_0 and f_1 except F_1 and $\tilde{D}_9$. Then, $\tau(\sigma^{-1}D)$ is the union of $\tau(f_0)$, $\tau(f_1)$ and $\tau(F_2)$. We have, $V^0 - E_1 - E_2 = \Sigma_0 - \tau(\sigma^{-1}D) \cong \boldsymbol{C} \times \boldsymbol{C}^*$.

To complete the proof of the Main Theorem, we have only to verify the assertion (V). Suppose that $\pi_1(V^0) \neq (0)$ and the Picard number of the quasi-universal covering

$\bar{U}$ is equal to one. Then $\bar{V}$ is a surface corresponding to the configuration (n) for $n=23$, 28, 31, 34 or 88. For the case $n=23$, we see that $\boldsymbol{P}^2/\pi_1(V^0)\cong\bar{U}/\pi_1(V^0)\cong\bar{V}$ (cf. the Table). In the remaining cases, we see that $\bar{U}\cong\bar{\Sigma}_m$ ($m\geq 2$) which is the surface obtained by contracting the minimal section on the Hirzebruch surface Σ_m of degree m (cf. the Table). Since $\bar{\Sigma}_m$ is the quotient of $\boldsymbol{P}^2$ by a cyclic subgroup of $PGL(2, \boldsymbol{C})$ of order m, there are a finite subgroup H of $PGL(2, \boldsymbol{C})$ and a cyclic normal subgroup H_1 of H of order m such tht $H/H_1\cong\pi_1(V^0)$, $\boldsymbol{P}^2/H_1\cong\bar{\Sigma}_m\cong\bar{U}$ and $\boldsymbol{P}^2/H\cong\bar{V}$.

The "only if" part of the assertion (V) of the Main Theorem is a consequence of the following:

PROPOSITION 6.4. *Let $\bar{V}$ be a* dP3*-surface. Suppose that there is a finite morphism h: $\boldsymbol{P}^2\to\bar{V}$. Then $\rho(\bar{U})=1$.*

PROOF. Let π: $\bar{U}\to\bar{V}$ be the canonical finite morphism. Denote $U^0=\pi^{-1}(V^0)$ and $P^0=h^{-1}(V^0)$. Then U^0 and P^0 are simply connected. Consider $Z:=P^0\times_{V^0}U^0$. Since U^0 is finite and étale over V^0, so is Z over P^0. Since P^0 is simply connected, Z is a disjoint union of $\deg\pi$ copies of P^0. Let k^0: $P^0\to U^0$ be the restriction of the projection $Z\to U^0$ to a copy of P^0 in Z. Then k^0 is a finite morphism such that $\pi\circ k^0=h_{|P^0}$. Clearly, k^0 extends to a finite morphism k: $\boldsymbol{P}^2\to\bar{U}$ so that $\pi\circ k=h$. Therefore, $\rho(\bar{U})=1$ because $\rho(\boldsymbol{P}^2)=1$. q.e.d.

Appendix. Table and list of configurations. In the Table, we employ the following notation and convention:

Let f: $U\to\bar{U}$ be a minimal resolution of singularities on the quasi-universal covering $\bar{U}$ of a dP3-surface $\bar{V}$. The singularities of $\bar{V}$ (resp. $\bar{U}$) are described in terms of the dual graph of $D:=g^{-1}(\mathrm{Sing}\,\bar{V})\subseteq V$ (resp. $B:=f^{-1}(\mathrm{Sing}\,\bar{U})\subseteq U$).

- V^0, U^0: stand for $V-D$ and $U-B$, respectively; hence $U^0\supseteq\pi^{-1}(V^0)$
- $\boldsymbol{C}^*$, $\boldsymbol{C}^{**}$, $\boldsymbol{C}^2-P$: stand for $\boldsymbol{C}-\{0\}$, $\boldsymbol{C}-\{$two distinct points$\}$, and $\boldsymbol{C}^2-\{$one point $P\}$, respectively
- $\bar{\Sigma}_n$ ($n\geq 2$): the surface obtained by contracting the minimal section on the Hirzebruch surface Σ_n of degree n.

We employ the following notation for finite groups.

- D_2: the binary dihedral group of order 8
- Q_3: the quaternion group of order 8
- S_3: the symmetric group of degree 3 and of order 6

$\pi_1(V^0)=\langle x, y, z \mid x^3=y^3=z^2=1, xy=yx, yz=zy, xz=zx^2\rangle$ in No. 22

$\pi_1(V^0)=\langle a, b \mid a^3=b^4=1, ab=ba^2\rangle$ in No. 26.

In No. 7, No. 93 and No. 95, we do not know yet which of D_2 and Q_3 the fundamental group $\pi_1(V^0)$ takes.

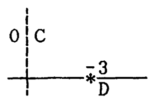
The No. *na* (resp. No. *nb*) row for $n=15$, 18 is the information concerning a dP3-surface corresponding to the configuration (*na*) (resp. (*nb*)).

No.	Sing. type of $\bar{V}$	$H_1(V^0;\mathbf{Z})$	$\pi_1(V^0)$	$\rho(\bar{U})$	Sing. type of $\bar{U}$	Ruledness of V^0, U^0
1	$*$	(0)	(0)	1	$\bar{U}=\bar{V}=\bar{\Sigma}_3$	$V^0\supseteq\mathbf{C}^2$
2	$A_4+(*\text{—}o^4)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
3	$*\text{—}o^8$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
4	$A_1+(*\text{—}o^7)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
5	$D_6+(o\text{—}*\text{—}o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$A_7+(-4)$	$U^0\supseteq\mathbf{C}^2$
6	$2A_1+D_4+(o\text{—}*\text{—}o)$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$\lvert\pi_1\rvert=16$	2	$\bar{U}=\Sigma_0$	$U^0=\Sigma_0$
7	$2A_3+(o\text{—}*\text{—}o)$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	D_2 or Q_3	4	$2A_1$	$U^0\supseteq\mathbf{C}^2$
8	$A_1+A_2+(*\text{—}o^5)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
9	$D_5+(*\text{—}o^3)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
10	$2A_1+A_3+(*\text{—}o^3)$	$\mathbf{Z}/2\mathbf{Z}$	S_3	3	$3A_1+2*$	$U^0\supseteq\mathbf{C}^2-\mathbf{P}$
11	$A_1+A_5+(*\text{—}o^2)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$A_2+2(*\text{—}o\text{—}o)$	$U^0\supseteq\mathbf{C}^2$
12	$E_6+(*\text{—}o^2)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
13	$A_1+D_6+(*\text{—}o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$D_4+2(*\text{—}o)$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
14	$A_7+(*\text{—}o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	3	$A_3+2(*\text{—}o)$	$U^0\supseteq\mathbf{C}^2$
15*a*	$E_7+(*\text{—}o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
15*b*	$E_7+(*\text{—}o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
16	D_8+*	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	3	D_5+2*	$U^0\supseteq\mathbf{C}^2-\mathbf{P}$
17	A_1+E_7+*	$\mathbf{Z}/2\mathbf{Z}$	S_3	4	D_4	$U^0\supseteq\mathbf{C}^2$
18*a*	E_8+*	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
18*b*	E_8+*	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
19	A_1+A_7+*	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	5	A_1+4*	$U^0\supseteq\mathbf{C}^2-\mathbf{P}$
20	$2A_4+*$	$\mathbf{Z}/5\mathbf{Z}$	$\mathbf{Z}/5\mathbf{Z}$	5	$5*$	$U^0\supseteq\mathbf{C}^2$
21	A_8+*	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	5	A_2+3*	$U^0\supseteq\mathbf{C}^2$
22	$A_1+A_2+A_5+*$	$\mathbf{Z}/6\mathbf{Z}$	$\lvert\pi_1\rvert=18$	4	smooth del Pezzo surface of deg 6	$U^0=\bar{U}$
23	$3A_2+(*\text{—}o^2)$	$\mathbf{Z}/3\mathbf{Z}$	$\lvert\pi_1\rvert=21$	1	$\bar{U}=\mathbf{P}^2$	$U^0=\mathbf{P}^2$
24	$A_2+A_5+(*\text{—}o)$	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	3	$A_1+3(*\text{—}o)$	$U^0\supseteq\mathbf{C}^2$
25	A_2+E_6+*	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	3	D_4+3*	$U^0\supseteq\mathbf{C}^2$
26	A_3+D_5+*	$\mathbf{Z}/4\mathbf{Z}$	$\lvert\pi_1\rvert=12$	6	smooth del Pezzo surface of deg 4	$U^0=\bar{U}$
27	$A_1+2A_3+(*\text{—}o)$	$\lvert H_1\rvert=4$	$\lvert\pi_1\rvert=20$	2	$\bar{U}=\Sigma_0$	$U^0=\Sigma_0$
28	$2A_1+D_5+*$	$\mathbf{Z}/2\mathbf{Z}$	S_3	1	$\bar{U}=\bar{\Sigma}_2$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^{**}$
29	$2A_1+(*\text{—}o^3\text{—}\overset{\overset{o}{\vert}}{o}\text{—}o)$	$\mathbf{Z}/2\mathbf{Z}$	S_3	5	$*\text{—}o\text{—}*$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^{**}$
30	$A_1+(*\text{—}o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}^2$
31	$2A_1+(o\text{—}*\text{—}o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	1	$\bar{U}=\bar{\Sigma}_4$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
32	$A_1+(o\text{—}*\text{—}o\text{—}o\text{—}o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$(-4)\text{—}o$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
33	$A_1+(o\text{—}*\text{—}o\text{—}\overset{\overset{o}{\vert}}{o}\text{—}o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$(-4)\text{—}\overset{\overset{o}{\vert}}{o}\text{—}o$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
34	$3A_1+(o\text{—}\overset{\overset{o}{\vert}}{*}\text{—}o)$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	1	$\bar{U}=\bar{\Sigma}_6$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^{**}$
35	$2A_1+(o\text{—}o\text{—}o\text{—}\overset{\overset{o}{\vert}}{*}\text{—}o)$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	$(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$	3	$o\text{—}(-6)\text{—}o$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^{**}$
36	$*\text{—}o\text{—}o\text{—}o$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}^2$
37	$o\text{—}o\text{—}o\text{—}*\text{—}o\text{—}o\text{—}o$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	3	$o\text{—}(-4)\text{—}o$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$
38	$*\text{—}o\text{—}o\text{—}o\text{—}o$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}^2$
39	$o\text{—}*\text{—}o\text{—}o\text{—}o$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}^2$
40	$A_1+(*\text{—}o^5)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbf{C}\times\mathbf{C}^*$

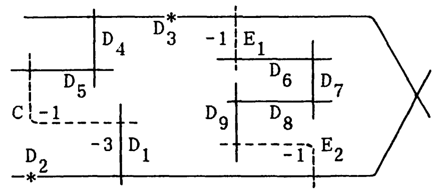
No.	Sing. type of $\bar V$	$H_1(V^0; \mathbf{Z})$	$\pi_1(V^0)$	$\rho(\bar U)$	Sing. type of $\bar U$	Ruledness of V^0, U^0
41	$A_1+(o-o-*-o^3)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
42	$*-o^7$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
43	$o-o-o-*-o^4$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
44	$A_2+(o-*-o^4)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
45	$o-*-o^7$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
46	$o-o-o-o-*-o^4$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
47	$A_1+(o-o-*-o^5)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
48	$o-o-o-*-o-\overset{o}{o}-o$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	3	$o-(-4)-\overset{o}{o}-o$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
49	$o-o-o-*-o-o-\overset{o}{o}-o$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
50	$*-o-o-o-o-o-\overset{o}{o}-o$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
51	$2A_1+(o-\overset{o}{o}-*-o^3)$	$\mathbf{Z}/2\mathbf{Z}$	S_3	5	$*-(-4)-*$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^{**}$
52	$*-o-\overset{o}{o}-o$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}^2$
53	$*-o-\overset{o}{o}-o-o$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}^2$
54	$*-\overset{o}{o}-o-o-o$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}^2$
55	$o-*-\overset{o}{o}-o-o$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}^2$
56	$A_1+(*-\overset{o}{o}-o^4)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
57	$o-o-\overset{o}{*}-o-o$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}^2$
58	$A_1+(o-o-\overset{o}{*}-o^3)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$o-o-\overset{o}{(-4)}-o-o$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
59	$A_3+(o-o-o-\overset{o}{*}-o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$A_1+(o-o-o-*-o^3)$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
60	$D_4+(*-o-\overset{o}{o}-o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$A_3+(*-o^3-*)$	$U^0\supseteq \mathbf{C}^2$
61	$A_3+(*-o-o-\overset{o}{o}-o)$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	2	$A_1+(*-o^5-*)$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
62	$D_5+(*-\overset{o}{o}-o)$	$\mathbf{Z}/2\mathbf{Z}$	S_3	6	$A_1+(-4)$	$U^0\supseteq \mathbf{C}^2$
63	$A_2+(*-o)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}^2$
64	$A_1+A_2+(*-o-o)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
65	$A_2+(*-o-o-o-o)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
66	A_3+*	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}^2$
67	A_1+A_4+*	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
68	A_6+*	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
69	$A_1+A_4+(*-o-o)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
70	$A_2+A_3+(*-o)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
71	$A_6+(*-o)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$
72	$A_3+(*-o-o-o-o)$	(0)	(0)	1	$\bar U=\bar V$	$V^0\supseteq \mathbf{C}\times\mathbf{C}^*$

No.	Sing. type of $\bar{V}$	$H_1(V^0; \boldsymbol{Z})$	$\pi_1(V^0)$	$\rho(\bar{U})$	Sing. type of $\bar{U}$	Ruledness of V^0, U^0
73	A_1+(o—*—o—o)	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}^2$
74	$2A_1+$(o—*—o—o—o)	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	2	$2A_1+$((−4)—o)	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
75	A_1+(o—*—o^5)	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	3	$2A_1+$((−4)—o—o)	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
76	A_1+A_2+(o^2—*—o^2)	$\boldsymbol{Z}/3\boldsymbol{Z}$	$\boldsymbol{Z}/3\boldsymbol{Z}$	3	$3A_1+(-5)$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
77	A_1+(o—o—*—o^5)	$\boldsymbol{Z}/3\boldsymbol{Z}$	$\boldsymbol{Z}/3\boldsymbol{Z}$	5	$3A_1+$((−5)—o)	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
78	$2A_1+$(o^3—*—o^3)	$(\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus 2}$	$(\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus 2}$	5	$4A_1+(-6)$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
79	A_2+A_3+(o—*—o)	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	2	$A_1+2A_2+(-4)$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
80	A_2+(o—*—o^5)	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	3	$2A_2+$((−4)—o—o)	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
81	$2A_2+$(o—o—*—o—o)	$\boldsymbol{Z}/3\boldsymbol{Z}$	$\boldsymbol{Z}/3\boldsymbol{Z}$	3	$3A_2+(-5)$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
82	A_1+A_3+(o—*—o^3)	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	2	$2A_3+$(o—(−4))	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
83	A_5+(o—*—o—o)	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
84	A_3+D_4+*	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	2	A_1+A_3+2*	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
85	A_1+(*—$\overset{\mathrm{o}}{\mathrm{o}}$—o—o)	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}^2$
86	A_1+(*—o—$\overset{\mathrm{o}}{\mathrm{o}}$—o—o)	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}^2$
87	$2A_1+$(*—$\overset{\mathrm{o}}{\mathrm{o}}$—o—o—o)	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	2	$2A_1+$((−3)—$\overset{\mathrm{o}}{\mathrm{o}}$—(−3))	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
88	$2A_1+A_2+$(*—$\overset{\mathrm{o}}{\mathrm{o}}$—o)	$\boldsymbol{Z}/2\boldsymbol{Z}$	S_3	1	$\bar{U}=\bar{\Sigma}_4$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^{**}$
89	A_2+(*—o—o—$\overset{\mathrm{o}}{\mathrm{o}}$—o)	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
90	A_3+(*—o—$\overset{\mathrm{o}}{\mathrm{o}}$—o—o)	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
91	A_4+(*—$\overset{\mathrm{o}}{\mathrm{o}}$—o—o)	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
92	A_1+A_2+(*—$\overset{\mathrm{o}}{\mathrm{o}}$—$\mathrm{o}^3$)	$\boldsymbol{Z}/2\boldsymbol{Z}$	S_3	4	o—$\overset{\mathrm{o}}{(-4)}$—o	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^{**}$
93	A_1+D_4+(o—$\overset{\mathrm{o}}{*}$—o)	$(\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus 2}$	D_2 or Q_3	4	*	$U^0 \supseteq \boldsymbol{C}^2$
94	A_1+(o—*—o^3—$\overset{\mathrm{o}}{\mathrm{o}}$—o)	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	3	$2A_1+$((−4)—o—$\overset{\mathrm{o}}{\mathrm{o}}$—o)	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
95	$3A_1+$(o—*—o—$\overset{\mathrm{o}}{\mathrm{o}}$—o)	$(\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus 2}$	D_2 or Q_3	3	o—(−4)—o	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^{**}$
96	A_1+A_3+(o—*—$\overset{\mathrm{o}}{\mathrm{o}}$—o)	$\boldsymbol{Z}/4\boldsymbol{Z}$	$\boldsymbol{Z}/4\boldsymbol{Z}$	4	$4A_1+((-4)-(-4))$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$
97	D_7+*	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0 \supseteq \boldsymbol{C}\times\boldsymbol{C}^*$

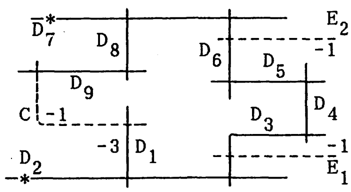
In the following list of configurations, the numbers in brackets coincide with the classifying numbers in the Table; a solid line stands for a component of D; the self-intersection number -2 of a (-2)-component of D is omitted; a line with $(*)$ on it is not contained in any fiber of the vertical $\boldsymbol{P}^1$-fibration $\Psi\colon V \to \boldsymbol{P}^1$.



(1)



(2)



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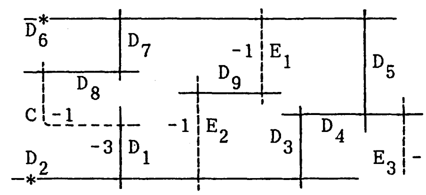

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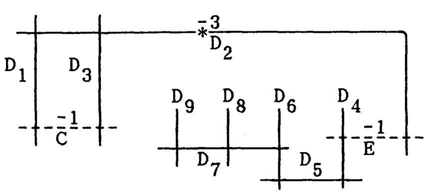

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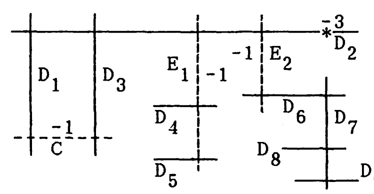

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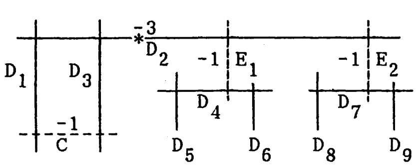

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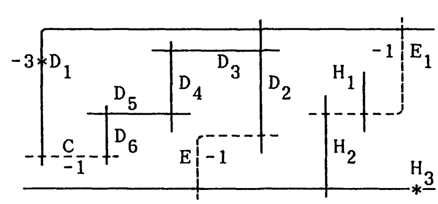

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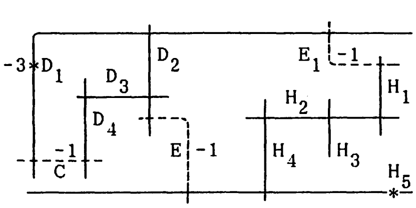

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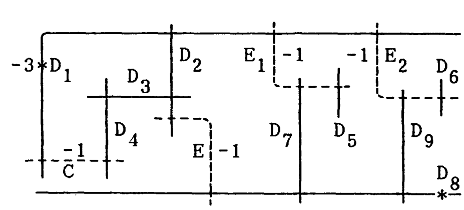

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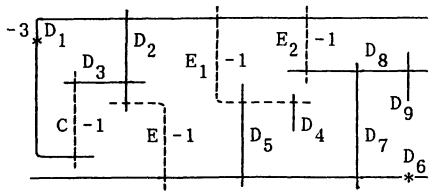

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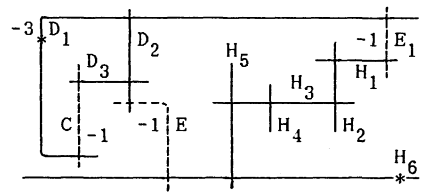

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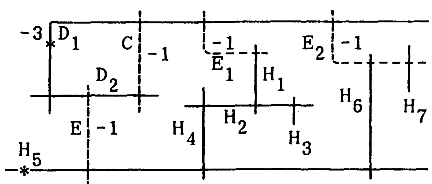

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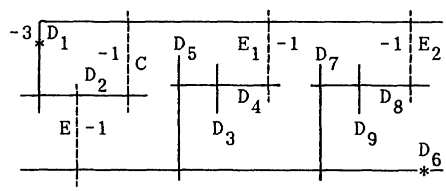

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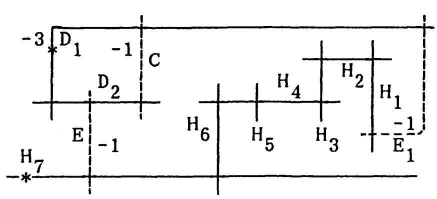
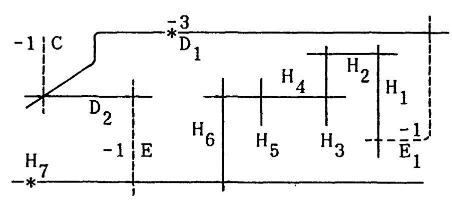
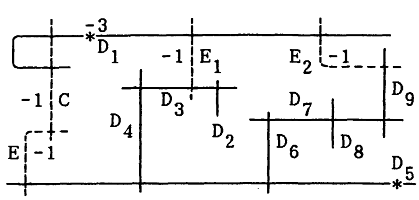


(15a)



(15b)



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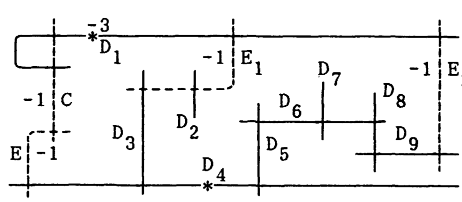

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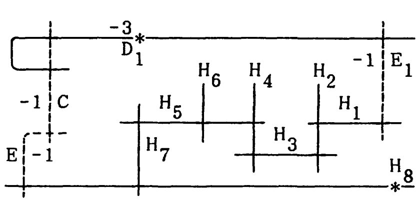

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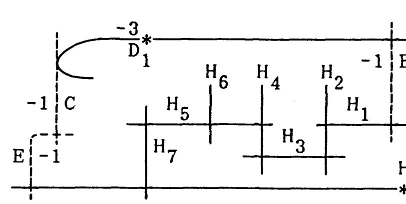
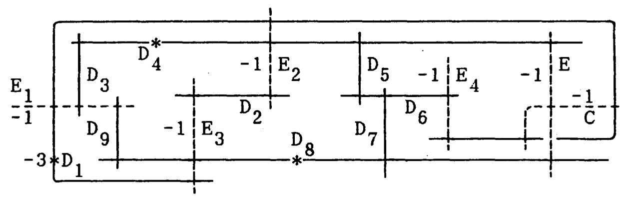


(18b)


E1
-1
D3
D4*
-1
E2
D5
-1
E4
-1
E
D9
-1
E3
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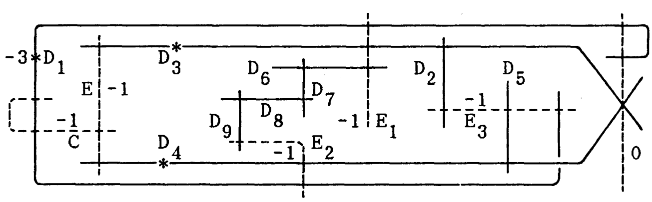
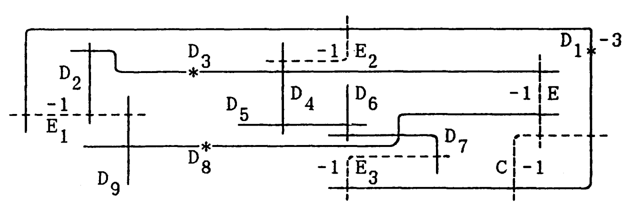
D8*
D7
D6
-1
C
-3
D1

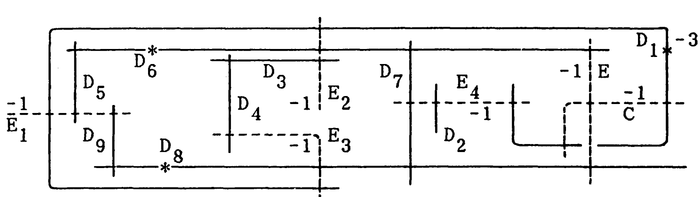
(19)


-3
D1
D3*
D6
D2
D5
E
-1
D7
-1
E3
-1
C
D9
D8
-1
E1
D4*
-1
E2
0

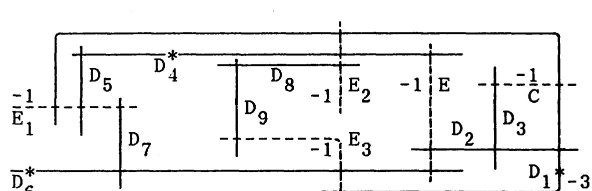
(20)


D3*
-1
E2
D1
-3
D2
-1
E1
D5
D4
D6
-1
E
D8*
D7
D9
-1
E3
C
-1

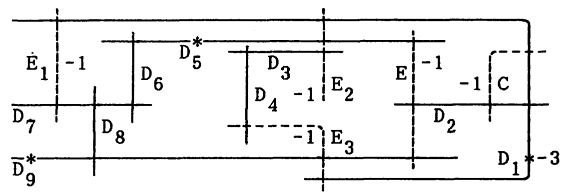
(21)


D6*
D1
-3
D5
D3
D7
E4
-1
E
-1
E1
D4
-1
E2
-1
-1
C
D9
D8*
-1
E3
D2

(22)


D5
D4*
D8
-1
E2
-1
E
-1
C
-1
E1
D9
D2
D3
D7
-1
E3
D6*
D1
-3

(23)



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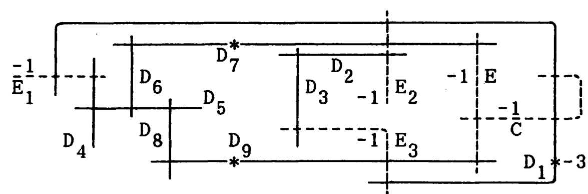

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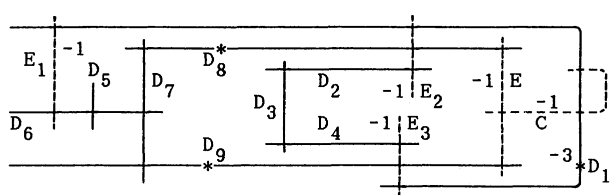

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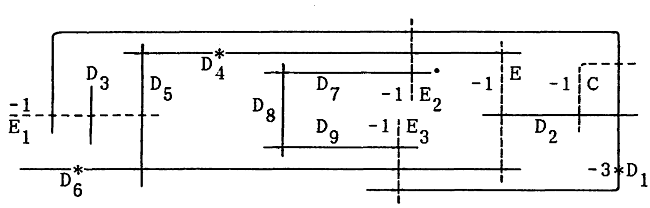

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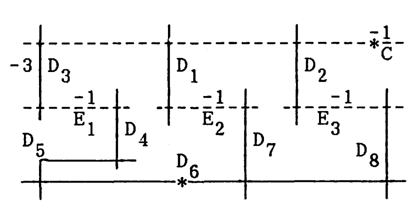

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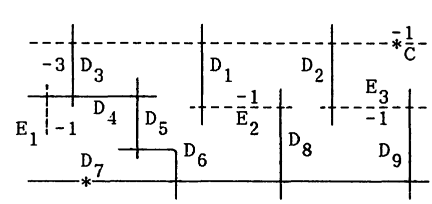

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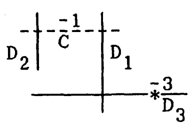

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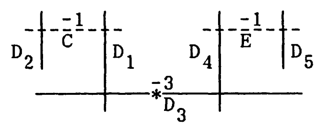

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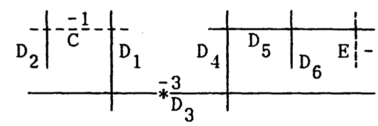

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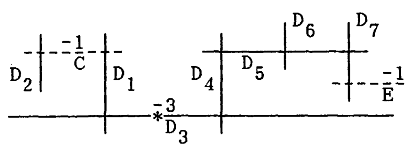

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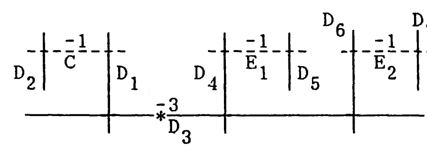

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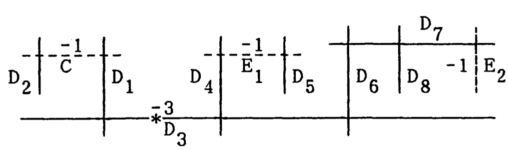

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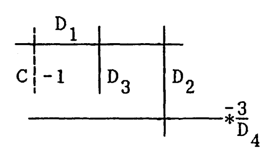

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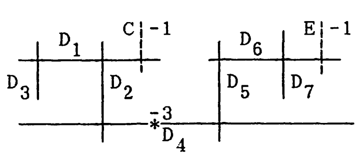

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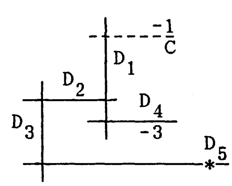

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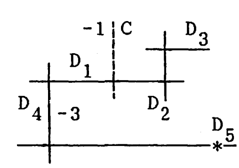

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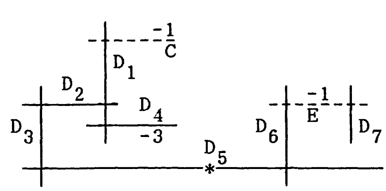

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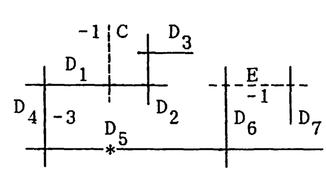

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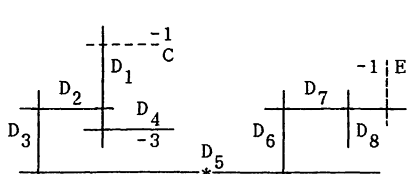

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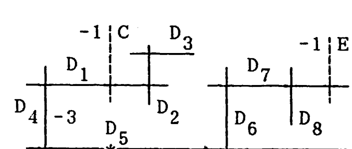

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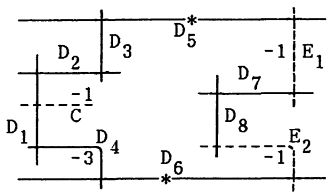

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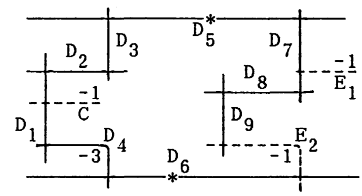

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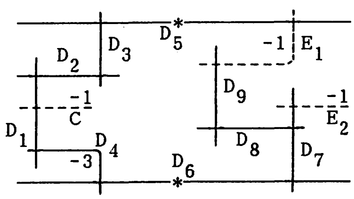

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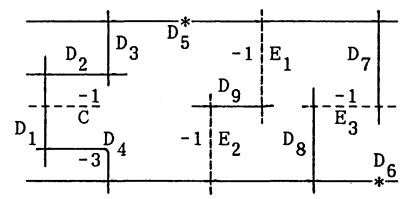

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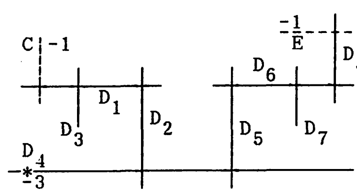

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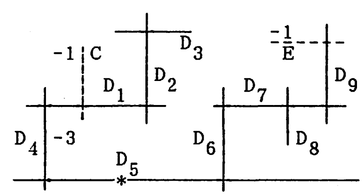

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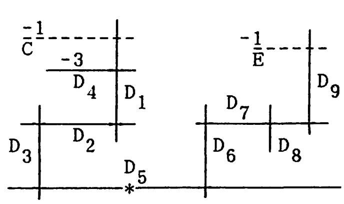

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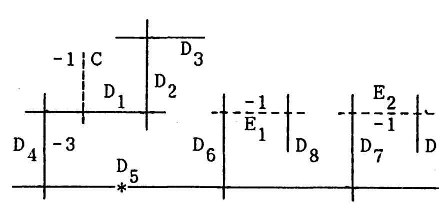

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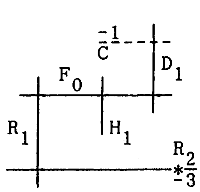

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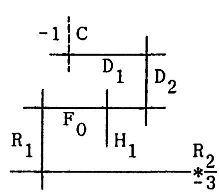

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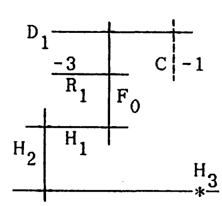

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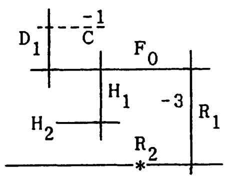

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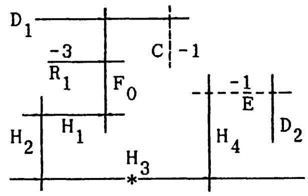

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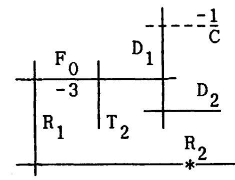

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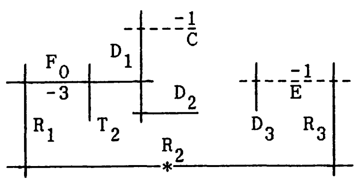

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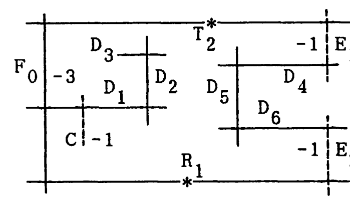

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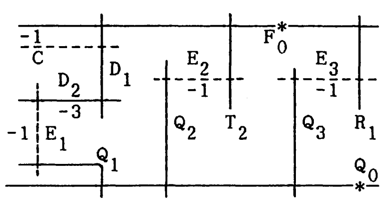

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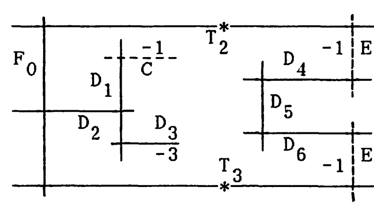

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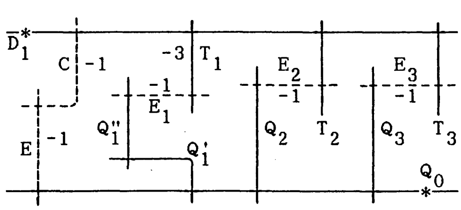

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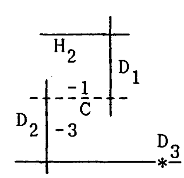

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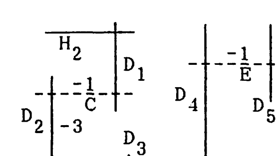

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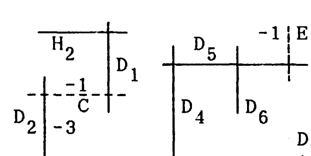

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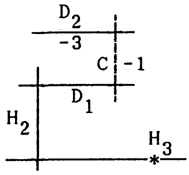

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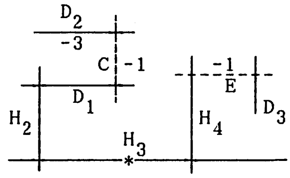

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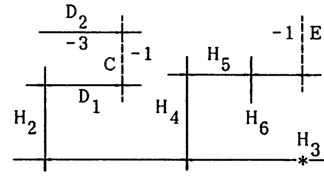

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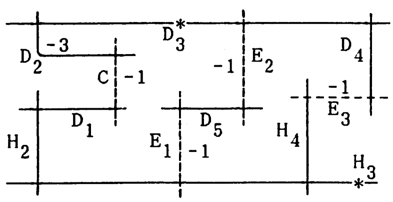

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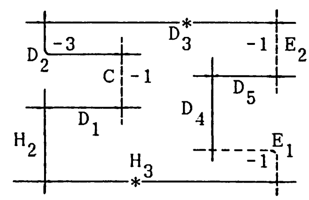

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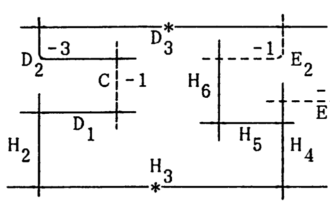

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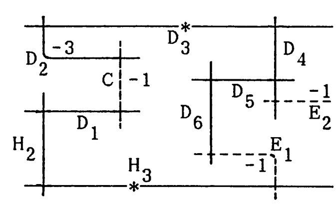

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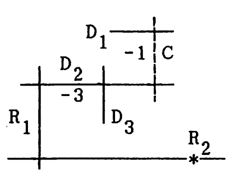

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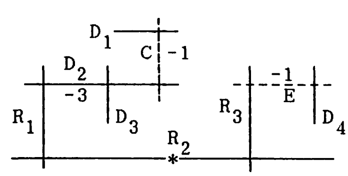

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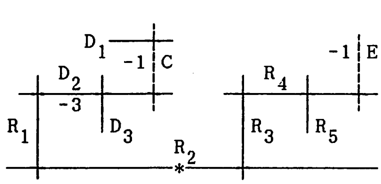

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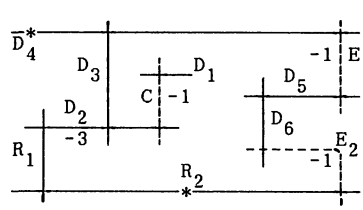

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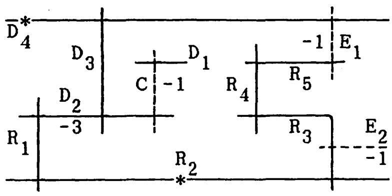

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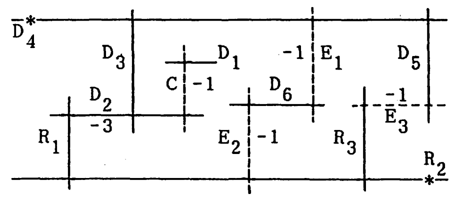

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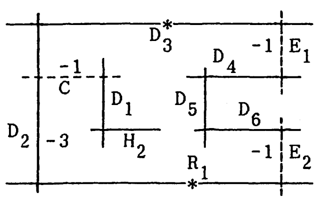

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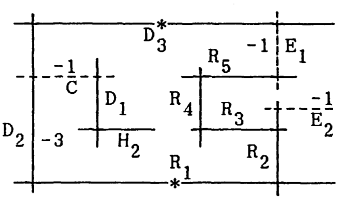

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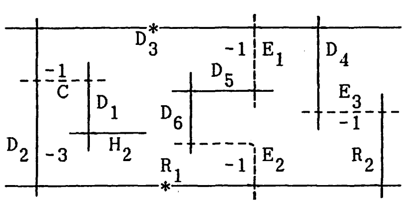

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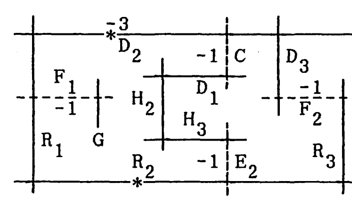

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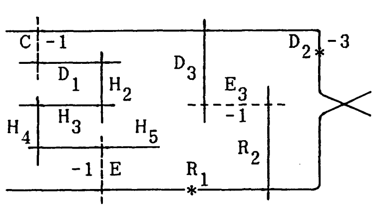

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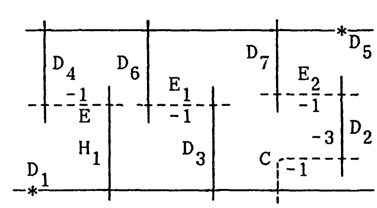

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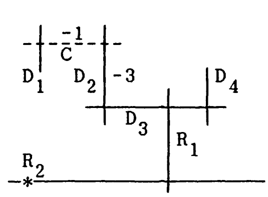

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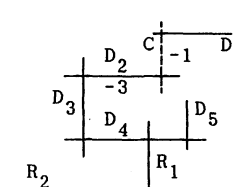

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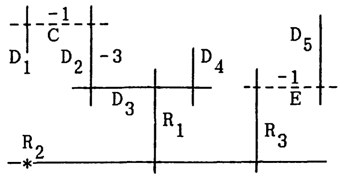

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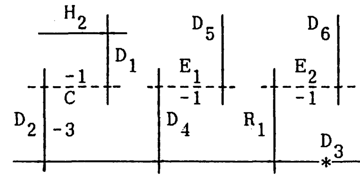

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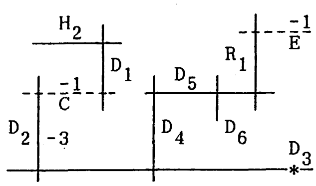

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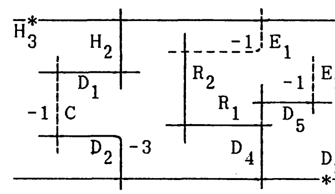

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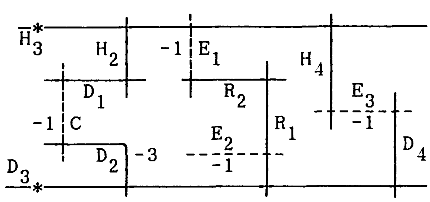

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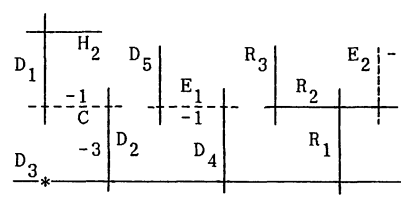

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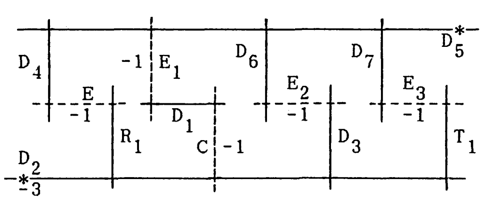

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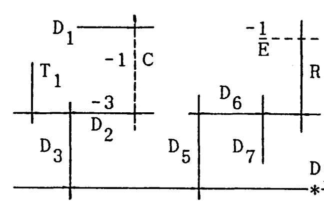

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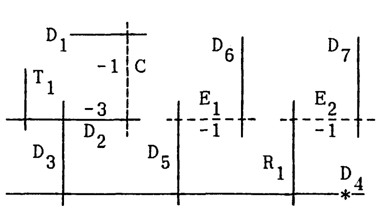

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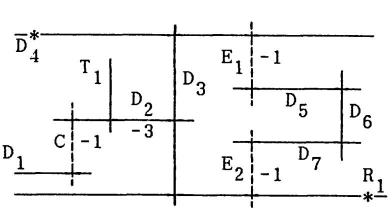

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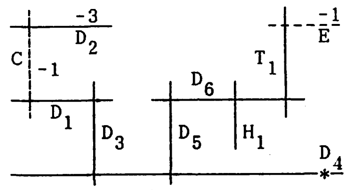

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