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A GENERALIZATION OF STOLL'S THEOREM FOR MOVING TARGETS

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1. Introduction. In [4], Stoll obtained the defect bound $n(n+1)$ of holomorphic map $f: C \to P^n(C)$ for slow moving targets, while it is well-known that the defect bound for constant targets is $n+1$ and this is best possible. In this paper we will show a generalization of Stoll's result which interpolates the above two results.

The author thanks S. Mori for notifing his simplification [3] of the original proof of Stoll's theorem. His proof was a great help to the author in deriving the result of this paper.

2. Preliminaries. First we introduce the notation and situations used throughout in this paper. We denote the homogeneous coordinates system of $P^n(C)$ by the notation $(w_0: \cdots : w_n)$. Let $f: C \to P^n(C)$ be a holomorphic map and let $(f_0, \cdots, f_n): C \to C^{n+1}$ be its reduced representation, i.e., $f_0, \cdots, f_n$ are holomorphic functions without common zeros and $f(z) = (f_0(z): \cdots : f_n(z))$ for all $z \in C$. We fix one reduced representation of f and define $\|f(z)\|^2 := |f_0(z)|^2 + \cdots + |f_n(z)|^2$ for all $z \in C$. For $j = 1, \cdots, q$, we give $n+1$ meromorphic functions $a_0^j, \cdots, a_n^j$ without common zeros and common poles, where $q \geq n+2$.

Here we define the characteristic functions.

DEFINITION 1. For a holomorphic map f of C into $P^n(C)$, the characteristic function is defined by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log\|f(re^{i\theta})\| d\theta , \qquad r > 0 .$$

For a meromorphic function a, the characteristic function of a is defined by the characteristic function $T_a(r)$ of the holomorphic map a of C into $P^1(C)$.

Next we introduce the counting functions and the defects of $a^j := (a_0^j, \cdots, a_n^j)$ for f. For this purpose, we set

$$F_j := a_0^j f_0 + \cdots + a_n^j f_n .$$

For a moment we impose the hypothesis that $F_j \not\equiv 0$. However, this is always true under the assumption (A2) below. Without loss of genrality, we may assume that any F_j has neither zero nor pole at $z = 0$.

DEFINITION 2. The counting function of a^j for f is defined by

$$N_{f,a^j}(r):=N_{F_j}(0,r):=\int_0^r \sum_{|z|\le t} \nu_{F_j}(z)\frac{dt}{t}, \qquad r>0,$$

where $\nu_{F_j}(z)$ denotes the zero multiplicity of F_j at z. The defect of a^j for f is defined by

$$\delta(f,a^j):=\liminf_{r\to\infty}\left(1-\frac{N_{f,a^j}(r)}{T_f(r)}\right).$$

REMARK. If we put

$$N_{F_j}(r):=\frac{1}{2\pi}\int_0^{2\pi}\log|F_j(re^{i\theta})|d\theta,$$

then

$$N_{f,a^j}(r)=N_{F_j}(r)+o(T_f(r)) \qquad \text{as} \quad r\to\infty$$

and

$$\delta(f,a^j)=\liminf_{r\to\infty}\left(1-\frac{N_{F_j}(r)}{T_f(r)}\right)$$

under the assumption (A1) below.

We make the following assumption:

(A1) $T_{a_i^j}(r)=o(T_f(r))$ as $r\to\infty$, for $j=1,\cdots,q$ and $i=0,\cdots,n$.

Let K be the field generated by the set $\{a_i^j \mid 1\le j\le q, 0\le i\le n\}$ over $\mathbf{C}$. Then every element a of K satisfies $T_a(r)=o(T_f(r))$ as $r\to\infty$ (cf. [4, Lemma 5.3]). Furthermore, we make the following assumptions:

(A2) f is linearly non-degenerate over K, i.e., $f_0,\cdots,f_n$ are linearly independent over K,

and

(G.P.) For all integers $j_0,\cdots,j_n$ with $1\le j_0<\cdots<j_n\le q$,

$$\det(a_i^{j_\mu})_{0\le\mu,i\le n}\not\equiv 0.$$

We now recall known results.

THEOREM A(cf. [1]). *If all a_i^j are constants, and the assumptions* (A2) *and* (G.P.) *are satisfied* (*the assumption* (A1) *is trivially true*), *then*

$$\sum_{j=1}^q \delta(f,a^j)\le n+1.$$

THEOREM B([4, Theorem 6.19]). *If all a_i^j are holomorphic and the assumptions* (A1), (A2) *and* (G.P.) *are satisfied, then*

$$\sum_{j=1}^{q} \delta(f, a^j) \leq n(n+1) .$$

3. Defect relations for moving targets. To state our result, let p be an integer with $0 \leq p \leq n-1$ and we make the following further assumptions:

(C(p)) a_i^j is constant for $j=1, \cdots, q$ and $i=0, \cdots, p$.

(G.P.(p)) For all integers $j_0, \cdots, j_p$ with $0 \leq j_0 < \cdots < j_p \leq q$,

$$\det(a_i^{j_\mu})_{0 \leq \mu, i \leq p} \neq 0 .$$

In the above situation, our result is the following:

THEOREM. *If the assumptions* (A1), (A2), (G.P.), (C(p)) *and* (G.P. (p)) *are satisfied, then*

$$\sum_{j=1}^{q} \delta(f, a^j) \leq \frac{n(n+1)}{p+1} .$$

REMARK 1. This result can be extended to the case of meromorphic map f of C^m into $P^n(C)$ by the result of Vitter [5].

REMARK 2. (a) Theorem A is derived from this theorem for $p=n-1$. Indeed, we can regard a^j as representing a hyperplane in $P^n(C)$ under the assumption of Theorem A. Therefore the assumption (G.P. ($n-1$)) is satisfied after a suitable change of homogeneous coordinates, and (C($n-1$)) is obviously true.

(b) Theorem B is derived from this theorem for $p=0$. Indeed, we can regard a^j as a reduced representation of a holomorphic map into $P^n(C)$ under the assumption of Theorem B. Then we take a suitable homogeneous coordinates system such that any $a_0^j \not\equiv 0$, and it is enough to consider a_i^j/a_0^j in place of a_i^j.

4. Proof of Theorem. Let s be a positive integer. Then we define $L(s)$ to be the vector space over C spanned by the set

$$\{\prod_{\substack{1 \leq j \leq q \\ 0 \leq i \leq n}} (a_i^j)^{s_{ji}} \mid s_{ji} \text{ are non-negative integers and } \sum_{\substack{1 \leq j \leq q \\ 0 \leq i \leq n}} s_{ji} = s\} .$$

By the assumption (C(p)), the inclusion $L(s) \subset L(s+1)$ holds. Let $\{b_1, \cdots, b_k\}$ be a basis of $L(s)$ and let $\{c_1, \cdots, c_l\}$ be a basis of $L(s+1)$. We introduce the meromorphic functions

$$(1)\ J := W(b_1 f_0, \cdots, b_k f_0, \cdots, b_1 f_p, \cdots, b_k f_p, c_1 f_{p+1}, \cdots, c_l f_{p+1}, \cdots, c_1 f_n, \cdots, c_l f_n)$$

and

$$(2) \qquad J_{j_0 \cdots j_p} := W(b_1 F_{j_0}, \cdots, b_k F_{j_0}, \cdots, b_1 F_{j_p}, \cdots, b_k F_{j_p}, c_1 f_{p+1}, \cdots, c_l f_{p+1}, \cdots, c_1 f_n, \cdots, c_l f_n)$$

for all integers $j_0, \cdots, j_p$ with $1 \leq j_0 < \cdots < j_p \leq q$, where the notation $W(g_1, \cdots, g_m)$ denotes the Wronskian determinant of $g_1, \cdots, g_m$. Then it follows that $J \not\equiv 0$ by (A2) and the fact that $\{b_1, \cdots, b_k\}$ and $\{c_1, \cdots, c_l\}$ are bases of $L(s)$ and $L(s+1)$, respectively. Furthermore by (C(p)) and (G.P.(p)), there exist non-zero constants $C_{j_0 \cdots j_p}$ such that

$$J_{j_0 \cdots j_p} \equiv C_{j_0 \cdots j_p} J . \tag{3}$$

For an arbitrarily fixed $z \in \boldsymbol{C}$, we take distinct indices $\alpha_0, \cdots, \alpha_n = \beta_0, \cdots, \beta_{q-n-1}$ such that

$$|F_{\alpha_0}(z)| \leq \cdots \leq |F_{\alpha_n}(z)| \leq |F_{\beta_1}(z)| \leq \cdots \leq |F_{\beta_{q-n-1}}(z)| \leq \infty . \tag{4}$$

Then we have

$$\log \|f(z)\| \leq \log |F_{\beta_j}(z)| + \log^+ C(z) \tag{5}$$

for $j = 0, \cdots, q-n-1$, where

$$\int_0^{2\pi} \log^+ C(re^{i\theta}) d\theta = o(T_f(r)) \tag{6}$$

and $\log^+ x = \max(0, \log x)$ for $x > 0$. Indeed, $\gamma_0, \cdots, \gamma_n$ are distinct integers with $1 \leq \gamma_0, \cdots, \gamma_n \leq q$. Thus the equalities

$$F_{\gamma_i} = a_0^{\gamma_i} f_0 + \cdots + a_n^{\gamma_i} f_n \qquad \text{for} \quad i = 0, \cdots, n$$

and (G.P.) admit the representations

$$f_\mu = \sum_{i=0}^n A_{\mu,i}^\gamma F_{\alpha_i} \qquad \text{for} \quad \mu = 0, \cdots, n ,$$

where $A_{\mu,i}^\gamma \in K$ and γ is the multi-index $(\gamma_0, \cdots, \gamma_n)$. Therefore we have

$$|f_\mu(z)| \leq \sum_{i=0}^n |A_{\mu,i}^\alpha(z)| |F_{\beta_\nu}(z)| \qquad \text{for} \quad \mu = 0, \cdots, n \quad \text{and} \quad \nu = 0, \cdots, q-n-1$$

by (4), where $\alpha = (\alpha_0, \cdots, \alpha_n)$, and hence

$$\|f(z)\| \leq \sum_{0 \leq \mu, i \leq n} |A_{\mu,i}^\alpha(z)| \, |F_{\beta_\nu}(z)| \qquad \text{for} \quad \nu = 0, \cdots, q-n-1 .$$

Here if we put $C(z) := \sum_\gamma \sum_{0 \leq \mu, i \leq n} |A_{\mu,i}^\gamma(z)|$, where γ ranges over the set $\{\gamma = (\gamma_0, \cdots, \gamma_n) \mid \gamma_0, \cdots, \gamma_n$ are distinct and $1 \leq \gamma_0, \cdots, \gamma_n \leq q\}$, then we have (6) by the fact $A_{\mu,i}^\gamma \in K$ and the concavity of $\log^+$. Now (5) is clearly true.

By considering (3), we obtain

$$\log \frac{|F_1 \cdots F_q|^{k\binom{n-1}{p}}}{|J|^{\binom{n}{p+1}}} \tag{7}$$

$$= \log |F_{\beta_0} \cdots F_{\beta_{q-n-1}}|^{k\binom{n-1}{n}} - \log \frac{\Pi |J_{j_0 \cdots j_p}|}{|F_{\alpha_0} \cdots F_{\alpha_{n-1}}|^{k\binom{n-1}{p}}} + c_1$$

$$=\log|F_{\beta_0}\cdots F_{\beta_{q-n-1}}|^{k\binom{n-1}{p}}-\sum\log\frac{|J_{j_0\cdots j_p}|}{|F_{j_0}\cdots F_{j_p}|^k|f_{p+1}\cdots f_n|^l}$$

$$-\binom{n}{p+1}l\log|f_{p+1}\cdots f_n|+c_1$$

for some constant c_1, where the range of signs of product and sum is the set $\{(j_0,\cdots,j_p)\,|\,j_0<\cdots<j_p,\ \{j_0,\cdots,j_p\}\subset\{\alpha_0,\cdots,\alpha_{n-1}\}\}$. We put

$$D_{j_0\cdots j_p}:=\frac{|J_{j_0\cdots j_p}|}{|F_{j_0}\cdots F_{j_p}|^k|f_{p+1}\cdots f_n|^l}.$$

Then we obtain

$$\int_0^{2\pi}\log^+D_{j_0\cdots j_p}(re^{i\theta})d\theta=S_f(r) \tag{8}$$

by the theorem of Milloux (cf. [2, Chapter 3]) and the concavity of $\log^+$, where $S_f(r)$ is a quantity which satisfies

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{S_f(r)}{T_f(r)}=0 \tag{9}$$

for some subset E of $(0,\infty)$ with finite Lebesgue measure. By (7) we have

$$\log|F_{\beta_0}\cdots F_{\beta_{q-n-1}}|^{k\binom{n-1}{p}}\le\log\frac{|F_1\cdots F_q|^{k\binom{n-1}{p}}}{|J|^{\binom{n}{p+1}}}+\sum_{1\le j_0<\cdots<j_p\le q}\log^+D_{j_0\cdots j_p} \tag{10}$$
$$+\binom{n}{p+1}l\log|f_{p+1}\cdots f_n|+c_1\,.$$

By (5) and (10) we get an inequality

$$k\binom{n-1}{p}(q-n)\log\|f\|\le\log\frac{|F_1\cdots F_q|^{k\binom{n-1}{p}}}{|J|^{\binom{n}{p+1}}}+\sum_{1\le j_0<\cdots<j_p\le p}\log^+D_{j_0\cdots j_p}$$
$$+\binom{n}{p+1}l\log|f_{p+1}\cdots f_n|+c_2\log^+C+c_3$$

on $\boldsymbol{C}$, for some constants c_2, c_3. By integrating this inequality over the circle $\{z\in\boldsymbol{C}\,|\,|z|=r\}$ $(r>0)$, we obtain

$$k\binom{n-1}{p}(q-n)T_f(r)\le k\binom{n-1}{p}\sum_{j=1}^q N_{f,a^j}(r)+S_f(r)+\binom{n}{p+1}l(n-p)T_f(r)\,.$$

Therefore we have

$$q-\sum_{j=1}^{q}\frac{N_{f,a^j}(r)}{T_f(r)}\leq\frac{\binom{n}{p+1}(n-p)}{\binom{n-1}{p}}\cdot\frac{l}{k}+n+\frac{S_f(r)}{T_f(r)},$$

and hence

$$\sum_{j=1}^{q}\delta(f,a^j)\leq\frac{\binom{n}{p+1}(n-p)}{\binom{n-1}{p}}\cdot\frac{l}{k}+n.$$

Steinmetz' lemma (cf. [4, Lemma 3.12]) says that

$$\liminf_{s\to\infty}\frac{l}{k}=1.$$

Thus we obtain the defect relation

$$\sum_{j=1}^{q}\delta(f,a^j)\leq\frac{\binom{n}{p+1}(n-p)}{\binom{n-1}{p}}+n=\frac{n(n+1)}{p+1}.$$

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