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# THE SPECTRAL DECOMPOSITION PROPERTY OF THE SUM AND PRODUCT OF TWO COMMUTING OPERATORS

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The sum and product of two commuting bounded spectral operators in a Hilbert space are spectral operators. This property, however, cannot be extended unconditionally to spectral operators in Banach spaces, as shown by an example of Kakutani [9]. In general, the stability problem under the sum and product of two commuting operators having in common a given spectral property, is subject to restrictive conditions. In extending the stability under sum and product to the class of decomposable operators [7], Apostol in [1] found that if $T$ and $S$ are commuting bounded operators one of which is decomposable as a multiplication operator, then $T+S$ and $TS$ are decomposable operators. Sun [10] substituted the extra condition of $T$ or $S$ being a multiplication operator, by requiring that $T$ be strongly decomposable relative to $S$, in terms of the following definition. If $T$, $S$ are commuting bounded operators such that, for every spectral maximal space $Y$ of $S$, both the restriction $T|Y$ and the coinduced $T/Y$ on the quotient space $X/Y$ are decomposable, then $T$ is said to be strongly decomposable relative to $S$.

In this paper, we shall determine sufficient conditions for two commuting operators $T$ and $S$ to preserve under sum and product the more general spectral decomposition property, by allowing one of the operators to be unbounded. The bounded operator techniques used by Apostol and Sun are not applicable to our case.

The terminology and notation are consistent with the ones used in [4]. For a Banach space $X$ over the complex field $C$, we denote by $C(X)$ the set of all closed operators $S$ with domain $D(S)$ and range $R(S)$ in $X$. $C_d(X)$ denotes the subset of $C(X)$ consisting of all densely defined operators in $C(X)$. $B(X)$ stands for the Banach algebra of bounded linear operators on $X$.

For a linear operator $A$ on $X$, three types of invariant subspaces are most frequently used: analytically invariant subspaces [8], spectral maximal spaces [7] and $A$-bounded spectral maximal spaces $\Xi(A, F)$, where $F$ is a compact subset of $C$. Their pertinent properties and relationships to each other are analyzed in the first chapter of the monograph [4]. The main theorems of this paper are anchored to two spectral-type analytic properties:

PROPERTY ($\kappa$): the given operator $A$ has the single valued extension property and, for every closed $F \subset C$, the spectral manifold $X(A, F)$ is closed.

PROPERTY ($\beta$) [2]: for any sequence $\{f_n : \omega \to D(A)\}$ of analytic functions on an open $\omega \subset C$, $(\lambda - A)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on every compact subset of $\omega$, implies that $f_n(\lambda) \to 0$ uniformly on every compact subset of $\omega$.

A third spectral-type property (Property ($\gamma$)) will be added later in this paper.

The spectral decomposition property (SDP) was extensively studied in the second chapter of the monograph [4].

Given $S \in C(X)$, $\{S\}'$ denotes the set of all bounded commutants of $S$. For a subset $E$ of $C$, we write $\mathrm{Co}(E)$ for the convex hull of $E$.

The following property, part of [5, Theorem 5.5] will be frequently referred to.

THEOREM A. *Given $T \in C_d(X)$, the following assertions are equivalent*:

(i) $T$ has the SDP;

(ii) for every pair of open disks $G$, $H$ with $\bar{G} \subset H$, there exist invariant subspaces $X_G$ and $X_H$ such that

$$X = X_G + X_H ; \quad X_H \subset D_T ; \quad \sigma(T|X_H) \subset H \quad \text{and} \quad \sigma(T|X_G) \subset G^c ;$$

(iii) both $T$ and $T^*$ have property ($\beta$).

**1.** In this section, we determine sufficient conditions for the sum $T+S$ of two operators $S \in C_d(X)$ and $T \in \{S\}'$ to possess the SDP.

1.1. DEFINITION. We say that a finite cover $\{G_i\}_{i=1}^n$ of a set $E \subset C$ is a convex open cover if each $G_i$ is both convex and open.

1.2. DEFINITION. $T \in B(X)$ is said to have the convex spectral decomposition property (abbrev. convex SDP) if, for every convex open cover $\{G_i\}_{i=1}^n$ of $\sigma(T)$, there exists a system $\{X_i\}_{i=1}^n$ of $T$-invariant subspaces with the following spectral decomposition

$$X = \sum_{i=1}^{n} X_i , \qquad \sigma(T|X_i) \subset G_i , \qquad 1 \leqq i \leqq n .$$

In particular, if $T$ is decomposable then it has the convex SDP.

1.3. THEOREM. *If $T$ has the convex* SDP, *then $T$ has the single valued extension property and, for each convex closed set $F$, $X(T, F)$ is closed.*

PROOF. First, we show that $T$ has the single valued extension property. Let $f : \omega \to X$ be analytic in an open $\omega \subset C$ and identically verify the equation

$$(1.1) \qquad (\lambda - T)f(\lambda) = 0 \quad \text{on} \quad \omega .$$

Without loss of generality, we may assume that $\omega$ is connected. For some $\lambda_0 \in C$ and $r > 0$, define $G = \{\lambda : |\lambda - \lambda_0| < r\}$ subject to $\bar{G} \subset \omega$. Let $\varepsilon > 0$ be sufficiently small for the sets $G_1 = \{\lambda : \mathrm{Re}\,\lambda < \mathrm{Re}\,\lambda_0 + \varepsilon\}$, $G_2 = \{\lambda : \mathrm{Re}\,\lambda > \mathrm{Re}\,\lambda_0 - \varepsilon\}$ to satisfy: $G - \bar{G}_i \neq \emptyset$, $i = 1, 2$.

By the convex SDP, there are $T$-invariant subspaces $X_1$, $X_2$ such that

$$X=X_1+X_2\,, \qquad \sigma(T|X_i)\subset G_i\,, \qquad i=1,2\,. \tag{1.2}$$

In view of (1.2), there are analytic functions $f_i\colon G\to X_i$, $i=1,2$ such that

$$f(\lambda)=f_1(\lambda)+f_2(\lambda)\,.$$

By (1.1), we have

$$(\lambda-T)f_1(\lambda)=-(\lambda-T)f_2(\lambda)\in X_1\cap X_2\,. \tag{1.3}$$

For $\lambda\in G-\bar{G}_1$, the inverses $(\lambda-T|X_1)^{-1}$, $(\lambda-T|X_1\cap X_2)^{-1}$ exist. Therefore (1.3) gives rise to

$$f_1(\lambda)=(\lambda-T|X_1)^{-1}((\lambda-T)f_1(\lambda))=(\lambda-T|X_1\cap X_2)^{-1}((\lambda-T)f_1(\lambda))\in X_1\cap X_2\,.$$

Hence $f_1(\lambda)\in X_1\cap X_2$ for all $\lambda\in\omega$, by analytic continuation. Similarly, one obtains that $f_2(\lambda)\in X_1\cap X_2$ on $\omega$, and hence $f(\lambda)\in X_1\cap X_2$ on $\omega$.

Since $\lambda\in\omega$ subject to $|\mathrm{Re}\,\lambda-\mathrm{Re}\,\lambda_0|>\varepsilon$ implies that $\lambda\in\rho(T|X_1\cap X_2)$, it follows from (1.1) that $f(\lambda)=0$ on $\omega$, by analytic continuation. Thus, $T$ has the single valued extension property.

To show that $X(T,F)$ is closed for every convex closed set $F$, let $G_F$ be the family of all half open planes containing $F$. For given $G\in G_F$, let $H$ be another half open plane satisfying conditions $G\cup H=C$ and $F\cap\bar{H}=\varnothing$. By the convex SDP, there exist $T$-invariant subspaces $X_G$, $X_H$ such that

$$X=X_G+X_H \quad \text{with} \quad \sigma(T|X_G)\subset G\,, \qquad \sigma(T|X_H)\subset H\,.$$

Furthermore, $X/X_G$ and $X_H/X_G\cap X_H$ are topologically isomorphic, $\hat{T}=T/X_G$ and $\tilde{T}=(T|X_H)/X_G\cap X_H$ are similar. It follows from the convexity of $H$ that $\sigma(T|X_G\cap X_H)\subset H$. Then the following inclusion

$$\sigma(\tilde{T})\subset\sigma(T|X_H)\cup\sigma(T|X_G\cap X_H)$$

implies that $\sigma(\hat{T})=\sigma(\tilde{T})\subset H$.

Let $\Gamma$ be a simple positively oriented closed contour surrounding $\sigma(\hat{T})$ and leaving $F$ in its exterior. If $\lambda\notin F$, for every $x\in X(T,F)$, one has $(\lambda-T)x(\lambda)=x$ and hence

$$\hat{x}=\frac{1}{2\pi i}\int_\Gamma R(\lambda;\,\hat{T})\hat{x}d\lambda=\frac{1}{2\pi i}\int_\Gamma R(\lambda;\,\hat{T})(\lambda-\hat{T})\hat{x}(\lambda)d\lambda=\frac{1}{2\pi i}\int_\Gamma \hat{x}(\lambda)d\lambda=\hat{0}\,, \tag{1.4}$$

where $x(\cdot)$ is the local resolvent function of $x$ and $\hat{x}=x+X_G$ is the element of $X/X_G$ corresponding to $x$. In view of (1.4), $x\in X_G$ and since $G\in G_F$, one has

$$X(T,F)\subset\bigcap\{X_G : G\in G_F\}\,.$$

On the other hand, the convexity of $F$ implies that

$$F=\bigcap\{G : G\in G_F\}$$

and hence the opposite inclusion holds. Therefore

$$X(T, F) = \bigcap \{X_G : G \in G_F\}$$

and hence $X(T, F)$ is closed. □

1.4. DEFINITION. Given $S \in C(X)$ and $T \in \{S\}'$. If, for every spectral maximal space $Y$ of $S$, $T \mid Y$ has the convex SDP, then we say that $T$ has the strong convex SDP relative to $S$.

Clearly, if $T$ has the above property, then $T$ has the convex SDP.

1.5. PROPOSITION. *Given $S \in C(X)$ and $T \in \{S\}'$. Suppose that $T$ has the strong convex* SDP *relative to $S$, $S$ has the single valued extension property and, for every compact subset $F$ of $C$, $X(S, F)$ is closed. Then, for every $S$-bounded spectral maximal space $Y = \Xi(S, F)$, $T \mid Y$ has the convex* SDP.

PROOF. The $S$-bounded spectral maximal space $Y$ has the representation $Y = \Xi(S, \sigma(S \mid Y))$. Furthermore, it follows from [4, Theorem 4.34], that

$$X(S, \sigma(S \mid Y)) = \Xi(S, \sigma(S \mid Y)) \oplus X(S, \varnothing) .$$

Since, by hypothesis $T \mid X(S, \sigma(S \mid Y))$ has the convex SDP, it follows from the above decomposition that $T \mid Y = T \mid \Xi(S, \sigma(S \mid Y))$ has the convex SDP. □

1.6. THEOREM. *Given $S \in C_d(X)$ and $T \in \{S\}'$. If $S$ has the* SDP *and $T$ has the strong convex* SDP *relative to $S$, then $T^* + S^*$ has property* ($\beta$).

PROOF. Let $\{f_n : \omega \to X^*\}$ be a sequence of analytic functions in an open $\omega \subset C$ such that

$$\|(\lambda - (T^* + S^*) f_n(\lambda))\| \to 0 \qquad \text{as} \quad n \to \infty$$

uniformly in every compact subset of $\omega$. For $\lambda_0 \in \omega$ and $r > 0$, define the sets

$$G_0 = \{\lambda : |\lambda - \lambda_0| < r\} , \qquad G_1 = \{\lambda : |\lambda - \lambda_0| < 2r\}$$

such that $\bar{G}_1 \subset \omega$. Denote by $D$ a closed disk centered at the origin of radius $d$, satisfying inclusion $\sigma(T) \subset D$. Furthermore, suppose that $\{\sigma_j\}_{j=1}^n$ and $\{\delta_k\}_{k=0}^m$ are open covers of $\operatorname{Co}(\sigma(T))$ and $\sigma(S)$, respectively, with $\sigma_j$ $(1 \leqq j \leqq n)$, $\delta_k$ $(1 \leqq k \leqq m)$ open disks and $\delta_0$ the complement of a closed disk centered at the origin such that

$$\bar{G}_1 \cap (D + \bar{\delta}_0) = \varnothing . \tag{1.5}$$

By the sum $A + B$ of two sets $A, B \subset C$, we mean $A + B = \{a + b : a \in A, b \in B\}$. We may choose the disks $\sigma_j$ and $\delta_k$ such that the radius of the disk $\bar{\sigma}_j + \bar{\delta}_k$ is less than $r/2$ for $1 \leqq j \leqq n$ and $1 \leqq k \leqq m$. For all pairs $(j, k)$, the following two cases may hold:

(i) $(\bar{\sigma}_j + \bar{\delta}_k) \cap \bar{G}_0 = \varnothing$ for some $(j, k)$;

(ii) $(\bar{\sigma}_j+\bar{\delta}_k)\cap\bar{G}_0\neq\emptyset$, and hence $\bar{\sigma}_j+\bar{\delta}_k\subset G_1$ for some other pairs $(j,k)$.

The SDP of $S$ implies the spectral decomposition

$$X=X(S,\bar{\delta}_0)+\sum_{k=1}^{m}\Xi(S,\bar{\delta}_k). \tag{1.6}$$

Set $Y_0=X(S,\bar{\delta}_0)$ and $Y_k=\Xi(S,\bar{\delta}_k)$ for $1\leqq k\leqq m$. Since $T\in\{S\}'$, for $0\leqq k\leqq m$, $Y_k$ is invariant under $T$ and

$$\sigma(T\,|\,Y_k)\subset \mathrm{Co}(\sigma(T)), \qquad 0\leqq k\leqq m. \tag{1.7}$$

In view of Theorem 1.3, $X_j=X(T,\bar{\sigma}_j)$ is closed for $1\leqq j\leqq n$ and (1.7) implies that, for $1\leqq k\leqq m$, $\{\sigma_j\}_{j=1}^n$ is also a convex open cover of $\sigma(T\,|\,Y_k)$. Then, Theorem 1.3 and Proposition 1.5 imply the spectral decomposition

$$Y_k=\sum_{j=1}^{n} Y_k(T\,|\,Y_k,\bar{\sigma}_j), \qquad 1\leqq k\leqq m.$$

Since $Y_k(T\,|\,Y_k,\bar{\sigma}_j)\subset Y_k\cap X_j$, $1\leqq j\leqq n$, $1\leqq k\leqq m$, one obtains

$$Y_k=\sum_{j=1}^{n} X_j\cap Y_k, \qquad 1\leqq k\leqq m. \tag{1.8}$$

Since $\bar{\sigma}_j$, $\bar{\delta}_k$ $(1\leqq j\leqq n,\ 1\leqq k\leqq m)$ are convex, we have

$$\sigma(T\,|\,X_j\cap Y_k)\subset\bar{\sigma}_j, \qquad \sigma(S\,|\,X_j\cap Y_k)\subset\bar{\delta}_k, \quad 1\leqq j\leqq n, \quad 1\leqq k\leqq m.$$

It follows from the inequality

$$r(A_1+A_2)\leqq r(A_1)+r(A_2) \tag{1.9}$$

on spectral radii of mutually commuting bounded operators $A_1$, $A_2$ that

$$\sigma((T+S)\,|\,X_j\cap Y_k)\subset\bar{\sigma}_j+\bar{\delta}_k, \qquad 1\leqq j\leqq n, \quad 1\leqq k\leqq m, \tag{1.10}$$

since $\bar{\sigma}_j$ and $\bar{\delta}_k$ are disks.

As regarding the subspace $Y_0$, let $\lambda_0\notin D+\bar{\delta}_0$. Then $\lambda_0\in\rho(S\,|\,Y_0)$, and it follows from the spectral mapping theorem that

$$r((\lambda_0-S\,|\,Y_0)^{-1})\leqq\frac{1}{\mathrm{dist}(\lambda_0,\bar{\delta}_0)}<\frac{1}{d}.$$

On the other hand, $\sigma(T\,|\,Y_0)\subset\mathrm{Co}(\sigma(T))\subset D$ implies $r(T\,|\,Y_0)\leqq d$. Therefore,

$$\lambda_0-(T+S)\,|\,Y_0=(\lambda_0-S\,|\,Y_0)(I-(\lambda_0-S\,|\,Y_0)^{-1}(T\,|\,Y_0))$$

is invertible and hence

$$\sigma((T+S)\,|\,Y_0)\subset D+\bar{\delta}_0. \tag{1.11}$$

Combining (1.6) and (1.8), one obtains

(1.12)
$$X=Y_0+\sum_{j=1}^{n}\sum_{k=1}^{m}X_j\cap Y_k\,.$$

There is $M>0$ such that, for every $x\in X$, there is a representation

$$x=x_0+\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk}\,,\quad x_0\in Y_0\,,\quad x_{jk}\in X_j\cap Y_k\,,$$

satisfying condition

(1.13)
$$\|x_0\|+\sum_{j=1}^{n}\sum_{k=k}^{m}\|x_{jk}\|\leqq M\|x\|\,.$$

For $\bar\sigma_j+\bar\delta_k$ satisfying (i) and $\lambda\in\bar G_0$, by virtue of (1.10), one obtains

(1.14)
$$\begin{aligned}|\langle x_{jk},f_n(\lambda)\rangle|&=|\langle(\lambda-(T+S))(\lambda-(T+S)\,|\,X_j\cap Y_k)^{-1}x_{jk},f_n(\lambda)\rangle|\\&=|\langle(\lambda-(T+S)\,|X_j\cap Y_k)^{-1}x_{jk},(\lambda-(T^*+S^*))f_n(\lambda)\rangle|\\&\leqq M_{jk}\|x_{jk}\|\;\|(\lambda-(T^*+S^*))f_n(\lambda)\|\,,\end{aligned}$$

where $M_{jk}=\sup\{\|(\lambda-(T+S)\,|\,X_j\cap Y_k)^{-1}\|:\lambda\in\bar G_0\}$.

For $\bar\sigma_j+\bar\delta_k$ satisfying (ii) and $\lambda\in G_1-(\bar\sigma_j+\bar\delta_k)$, one has

(1.15)
$$\begin{aligned}|\langle x_{jk},f_n(\lambda)\rangle|&=|\langle(\lambda-(T+S))(\lambda-(T+S)\,|\,X_j\cap Y_k)^{-1}x_{jk},f_n(\lambda)\rangle|\\&=|\langle(\lambda-(T+S)\,|\,X_j\cap Y_k)^{-1}x_{jk},(\lambda-(T^*+S^*))f_n(\lambda)\rangle|\\&\leqq M'_{jk}\|x_{jk}\|\;\|\lambda-(T^*+S^*))f_n(\lambda)\|\,,\end{aligned}$$

where $M'_{jk}=\sup\{\|(\lambda-(T+S)\,|\,X_j\cap Y_k)^{-1}\|:\lambda\in G_1-(\bar\sigma_j+\bar\delta_k)\}$.
By the maximum modulus principle, (1.15) remains valid for $\lambda\in\bar G_0$.

Finally, it follows from (1.5) and (1.11) that, for $\lambda\in\bar G_0$,

(1.16)
$$\begin{aligned}|\langle x_0,f_n(\lambda)\rangle|&=|\langle(\lambda-(T+S))(\lambda-(T+S)\,|\,Y_0)^{-1}x_0,f_n(\lambda)\rangle|\\&=|\langle(\lambda-(T+S))\,|\,Y_0)^{-1}x_0,(\lambda-(T^*+S^*))f_n(\lambda)\rangle|\\&\leqq M_0\|x_0\|\;\|(\lambda-(T^*+S^*))f_n(\lambda)\|\,,\end{aligned}$$

where $M_0=\sup\{\|(\lambda-(T+S)\,|\,Y_0)^{-1}\|:\lambda\in\bar G_0\}$.

Relations (1.13)–(1.16) imply the existence of a constant $K>0$ satisfying

(1.17)
$$|\langle x,f_n(\lambda)\rangle|\leqq K\|x\|\;\|(\lambda-(T^*+S^*))f_n(\lambda)\|\,,\qquad\lambda\in\bar G_0\,.$$

It follows from (1.17), that

(1.18)
$$\|f_n(\lambda)\|\to 0\qquad\text{as}\quad n\to\infty$$

uniformly on $\bar G_0$. By the Heine-Borel theorem, (1.18) remains true for every compact subset of $\omega$. Consequently, $T^*+S^*$ has property $(\beta)$. □

1.7. THEOREM. *Given $S\in C_d(X)$ and $T\in\{S\}'$. If $S$ has the* SDP, *$T$ and $T^*$ have the* *strong convex* SDP *relative to* $S$ *and* $S^*$, *respectively, then* $T+S$ *has the* SDP.

PROOF. $T^*+S^*$ has property ($\beta$), by Theorem 1.6. By a similar proof to that of Theorem 1.6, one can show that $T+S$ has property ($\beta$). Then, it follows from Theorem A that $T+S$ has the SDP. □

1.8. COROLLARY. *$T\in B(X)$ is decomposable if and only if both $T$ and $T^*$ have the convex* SDP.

PROOF. The "only if" part is evident. To show that the asserted conditions on $T$ and $T^*$ are sufficient, note that the only spectral maximal space of the zero operator $S=0$ is $X$ itself. Thus, the assumptions on $T$ and $T^*$ imply that $T$ and $T^*$ have the strong convex SDP relative to $S(=0)$ and $S^*$, respectively. Hence $T$ is decomposable, by Theorem 1.7. □

1.9. LEMMA. *If $A\in C(X)$ has property ($\kappa$) then, for every $A$-invariant subspace $Y$, $A\,|\,Y$ has property ($\kappa$).*

PROOF. Let $F\subset C$ be closed and let the sequence $\{x_n\}\subset Y(A\,|\,Y, F)$ converge to $x$. Then, since $\{x_n\}\subset X(A, F)$, it follows from

$$\lim_{n\to\infty} R(\lambda; A\,|\,X(A, F))x_n = \lim_{n\to\infty} x_{n,A}(\lambda) = \lim_{n\to\infty} x_{n,A|Y}(\lambda) \qquad \text{for} \quad \lambda\notin F,$$

and $x_{n,A|Y}(\lambda)\in Y$ for $\lambda\notin F$, that

$$x(\lambda) = \lim_{n\to\infty} x_{n,A|Y}(\lambda)\in Y.$$

Above, we wrote $R(\cdot\,; B)$ for the resolvent of an operator $B=A\,|\,X(A, F)$, and $x_{n,B}(\cdot)$ for the local resolvent of $x_n$ with respect to $B$, in cases $B=A$ and $B=A\,|\,Y$. For the limit point $x$, $x(\cdot)$ is its local resolvent. The operator $A$ being closed, at the limit as $n\to\infty$ equation

$$(\lambda-A)x_{n,A|Y}(\lambda)=x_n, \qquad \lambda\notin F$$

becomes

$$(\lambda-A)x(\lambda)=x, \qquad \lambda\notin F.$$

Thus it follows from $x(\lambda)\in Y$ ($\lambda\notin F$), that $x\in Y(A\,|\,Y, F)$ and hence $Y(A\,|\,Y, F)$ is closed. □

1.10. COROLLARY. *Given $S\in C_d(X)$ and $T\in\{S\}'$. Suppose that $S$ has the* SDP, *$T$ and $T^*$ have the strong convex* SDP *relative to $S$ and $S^*$, respectively. Then, for every spectral maximal space $Y$ (or $S$-bounded spectral maximal space), $T\,|\,Y$ has the* SDP. *The dual counterpart, i.e. for a ($S^*$-bounded) spectral maximal space $Y^*$ of $S^*$, $T^*\,|\,Y^*$ has the* SDP, *also holds.*

PROOF. It suffices to consider the case in which $Y$ is a spectral maximal space of $S$. By Corollary 1.8, $T$ has the SDP and hence, for every closed $F \subset C$, $X(T, F)$ is closed. Then, by Lemma 1.9, $Y(T|Y, F)$ is also closed.

Let $D_0$ be an open disk and $D_1$ be the complement of a closed disk such that $D_0 \cup D_1 = C$. We may choose open disks $\{\delta_i\}_{i=1}^n$ such that $\{D_0, \delta_1, \delta_2, \cdots, \delta_n\}$ is a convex open cover of $\sigma(T)$ satisfying inclusions

$$\delta_i \subset D_1, \qquad 1 \leqq i \leqq n. \tag{1.19}$$

Since $T$ has the strong convex SDP relative to $S$, the following decomposition of $Y$ holds:

$$Y = Y(T, \bar{D}_0) + \sum_{i=1}^{n} Y(T, \bar{\delta}_i).$$

In view of (1.19), $Y(T, \bar{\delta}_i) \subset Y(T, \bar{D}_1)$, $1 \leqq i \leqq n$, one obtains

$$Y = Y(T, \bar{D}_0) + Y(T, \bar{D}_1).$$

By Theorem A, $T|Y$ has the SDP. □

1.11. DEFINITION. $T \in B(X)$ is said to be regularly decomposable with respect to the identity if, for every open cover $\{G_i\}_{i=1}^n$ of $\sigma(T)$, there is a system of $T$-invariant subspaces $\{X_i\}_{i=1}^n$ and a system of bounded linear operators $\{P_i\}_{i=1}^n$ such that each $P_i$ commutes with all closed commutants of $T$ and

$$\sigma(T|X_i) \subset G_i, \qquad 1 \leqq i \leqq n; \tag{1.20}$$

$$I = \sum_{i=1}^{n} P_i, \qquad R(P_i) \subset X_i, \qquad 1 \leqq i \leqq n. \tag{1.21}$$

The following theorem gives some examples of known operators which, as bounded commutants of $S \in C_d(X)$, satisfy the sufficient conditions of Theorem 1.7.

1.12. THEOREM. *Given $S \in C_d(X)$ and $T \in \{S\}'$. If any of the following conditions is satisfied then $T$ and $T^*$ have the strong convex* SDP *relative to $S$ and $S^*$, respectively*:

(i) *$\sigma(T)$ is totally disconnected*;

(ii) *$T$ is a spectral operator*;

(iii) *$T$ is boundedly decomposable* [6];

(iv) *$T$ is a generalized scalar operator and $S$ commutes with one of the spectral distributions of $T$* [3];

(v) *$T$ is regularly decomposable with respect to the identity.*

PROOF. Since the implications (i) ⇒ (v), (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) are evident, it suffices to prove that (v) implies that $T$ and $T^*$ have the strong convex SDP relative to $S$ and $S^*$, respectively. Let $\{G_i\}_{i=1}^n$ be a convex open cover of $\sigma(T)$. By hypothesis, $T$ is regularly decomposable with respect to the identity. There exists a system $\{X_i\}_{i=1}^n$

of $T$-invariant subspaces and a system $\{P_i\}_{i=1}^n$ of bounded operators with each $P_i$ commuting with all closed commutants of $T$ and satisfying conditions (1.20) and (1.21). By hypothesis, $S$ commutes with each $P_i$. Let $Y$ be a spectral maximal space of $S$. Then $Y$ is invariant under $T$ and $P_i$ $(1 \leqq i \leqq n)$. Relations (1.20), (1.21) and the convexity of $G_i$ imply

$$I|Y = \sum_{i=1}^{n} P_i|Y; \tag{1.22}$$

$$R(P_i|Y) \subset Y \cap X_i, \qquad \sigma(T|Y \cap X_i) \subset G_i \qquad \text{for} \quad 1 \leqq i \leqq n. \tag{1.23}$$

By (1.22) and (1.23), $T|Y$ has the convex SDP and hence $T$ has the strong convex SDP relative to $S$. Similarly, $T^*$ has the strong convex SDP relative to $S^*$. □

In view of Theorems 1.7 and 1.12, if $S$ has the SDP, then $T+S$ also has the SDP.

1.13. THEOREM. *If $S$, $T \in B(X)$ commute with each other and satisfy one of conditions* (i)–(v) *of Theorem* 1.12, *then $S+T$ is strongly decomposable.*

PROOF. Without loss of generality, we may assume that $S$ and $T$ are both regularly decomposable with respect to the identity. Theorems 1.7, 1.12 imply that $S+T$ is decomposable. Let $W$ be a spectral maximal space of $S+T$. Then $W$ is invariant under $S$, $T$ and $P_i$ $(1 \leqq i \leqq n)$.

First, we prove that $T|W$ is decomposable. Let $G \subset C$ be open and denote $Y = X(T, \bar{G})$. Since, by hypothesis $T$ is regularly decomposable, we may choose $P \in \{T\}'$ such that $Px = x$ for $x \in Y$. Since $S+T$ commutes with $T$, it follows that $(S+T)|Y$ commutes with $(\lambda - T|Y)^{-1}$ for $\lambda \notin \bar{G}$ and hence $S+T$ commutes with $(\lambda - T|Y)^{-1}P$. Consequently, $W$ is invariant under $(\lambda - T|Y)^{-1}P$ and hence $\sigma(T|W \cap Y) \subset \bar{G}$. By putting $Y_i = X(T, \bar{G}_i)$, $1 \leqq i \leqq n$, the above argument leads one to the following inclusions:

$$\sigma(T|W \cap Y_i) \subset \bar{G}_i, \qquad 1 \leqq i \leqq n. \tag{1.24}$$

Since we also have

$$I|W = \sum_{i=1}^{n} P_i|W, \tag{1.25}$$

(1.24) and (1.25) imply that $T|W$ is decomposable (actually decomposable with respect to the identity).

Next, assume that $Z$ is a spectral maximal space of $T|W$. Since $S$ is regularly decomposable, it can be shown by the routine applied above, that $S|Z$ is decomposable. Specifically, $S|W$ has the strong convex SDP relative to $T|W$.

Similar argument applied to $(S|W)^*$ and $(T|W)^*$ leads one to the conclusion that $(S|W)^*$ has the strong convex SDP relative to $(T|W)^*$. Thus it follows from Theorem 1.7 that $(S+T)|W$ is decomposable and hence $S+T$ is strongly decomposable. □

1.14. COROLLARY. *If $S$, $T \in B(X)$ commute with each other, $T$ is regularly* *decomposable with respect to the identity and $S$ is compact, then $S+T$ is strongly decomposable.*

PROOF. By the Riesz-Dunford functional calculus, $S$ is regularly decomposable with respect to the identity. Thus the assertion of the Corollary follows from Theorem 1.7. □

**2.** In this section, we obtain sufficient conditions for the product $ST$ of two operators $S\in C_d(X)$ and $T\in\{S\}'$ to have the SDP.

For $S\in C(X)$ and $T\in B(X)$, the product $ST$ is clearly a closed operator.

2.1. LEMMA. *Given $S\in C_d(X)$ and $T\in\{S\}'$, the following inclusions hold*

$$T^*S^*\subset(ST)^*\subset S^*T^* .$$

PROOF. Let $x^*\in D(T^*S^*)$. For every $x\in D(ST)$, one has

$$\langle x, T^*S^*x^*\rangle=\langle STx, x^*\rangle=\langle x, (ST)^*x^*\rangle .$$

Consequently, we have $x^*\in D((ST)^*)$, $T^*S^*x^*=(ST)^*x^*$ and hence

$$T^*S^*\subset(ST)^* .$$

Next, let $x^*\in D((ST)^*)$. For every $x\in D(ST)$, one has

$$\langle x, (ST)^*x^*\rangle=\langle STx, x^*\rangle=\langle TSx, x^*\rangle=\langle x, S^*T^*x^*\rangle .$$

Consequently, we have $x^*\in D(S^*T^*)$, $(ST)^*x^*=S^*T^*x^*$ and hence

$$(ST)^*\subset S^*T^* .$$ □

2.2. LEMMA. *Suppose that $T_1, T_2\in B(X)$ commute with each other. If $\sigma(T_i)\subset D_i$, where $D_i=\{\lambda:|\lambda-\mu_i|<r_i\}$ for some $r_i>0$, $i=1, 2$, then $\sigma(T_1T_2)\subset D_{12}$ with $D_{12}=(D_1-\mu_1)(D_2-\mu_2)+\mu_1D_2+\mu_2D_1-\mu_1\mu_2$.*

PROOF. It follows from the inequality

(2.1) $$r(A_1A_2)\leqq r(A_1)r(A_2)$$

on spectral radii of mutually commuting bounded operators $A_1$, $A_2$ that

$$\sigma((T_1-\mu_1)(T_2-\mu_2))\subset(D_1-\mu_1)(D_2-\mu_2) .$$

This combined with

$$T_1T_2=(T_1-\mu_1)(T_2-\mu_2)+\mu_1T_2+\mu_2T_1-\mu_1\mu_2 ;$$

$$\sigma(\mu_1T_2)\subset\mu_1D_2, \qquad \sigma(\mu_2T_1)\subset\mu_2D_1$$

and property (1.9), gives

$$\sigma(T_1T_2)\subset(D_1-\mu_1)(D_2-\mu_2)+\mu_1D_2+\mu_2D_1-\mu_1\mu_2=D_{12} .$$ □

REMARK. It is easy to see that $D_{12}$, as defined above, is a disk centered at $\mu_1\mu_2$ of radius $r_1r_2+|\mu_1|r_2+|\mu_2|r_1$.

2.3. LEMMA. *Given $S\in C(X)$ and $T\in\{S\}'$. Suppose that $T$ has the* SDP *and $X(T,F)\subset D(S)$, for some $F\subset C$. Then $X(T,F)$ is invariant under $S$.*

PROOF. Let $X(T,F)\subset D(S)$. Since $S$ is closed, it follows that the restriction $S|X(T,F)$ is bounded. Let $x\in X(T,F)$ and let $x(\cdot)$ denote the local resolvent of $T$ at $x$. Since $Sx(\lambda)$ is analytic, it follows from

$$(\lambda-T)Sx(\lambda)=S(\lambda-T)x(\lambda)=Sx\,, \qquad \lambda\notin F\,,$$

that $\sigma_T(Sx)\subset F$ and hence $Sx\in X(T,F)$. Thus $X(T,F)$ is invariant under $S$. □

2.4. COROLLARY. *Given $S\in C_d(X)$ and $T\in\{S\}'$. Suppose that $T$ has the* SDP *and $X(T,F)\subset D(S)$, for some $F\subset C$. Then, for any closed neighborhood $K$ of $\infty$ with the property $F^0\cup K^0=C$, $X^*(T^*,K)$ is invariant under $S^*$.*

PROOF. The set $G=C-K$ is open and is contained in $F^0$. It follows, by a routine technique (used in the proof of Lemma 2.3), that $\overline{X(T,G)}\subset D(S)$ and $\overline{X(T,G)}$ is invariant under $S$. It follows from [4, Theorem 9.8 (ii)] that

$$X(T,G)^{\perp}=X^*(T^*,K)$$

and hence $X^*(T^*,K)$ is invariant under $S^*$. □

In a similar way, one can prove the following

2.5. LEMMA. *Given $S\in C_d(X)$ and $T\in\{S\}'$. Suppose that $T$ has the* SDP *and $X^*(T^*,F)\subset D(S^*)$, for some closed $F\subset C$. Then $X^*(T^*,F)$ is invariant under $S^*$. Furthermore, for every closed neighborhood $K$ of $\infty$ with the property $F^0\cup K^0=C$, $X(T,K)$ is invariant under $S$.*

2.6. DEFINITION. If $S\in C_d(X)$ and $T\in\{S\}'$ are such that $X(T,F)\subset D(S)$ and $X^*(T^*,F)\subset D(S^*)$, for some closed $F\subset C$ with $F^0\neq\varnothing$ and $0\in F^0$, then we say that $S$ and $T$ have property $(\gamma)$.

2.7. THEOREM. *Given $S\in C_d(X)$ and $T\in\{S\}'$. If*

(i) *$S$ and $T$ have property $(\gamma)$;*

(ii) *$S$ has the* SDP, *$T$ has the strong convex* SDP *relative to $S$ and $T^*$ has the strong convex* SDP *relative to $S^*$, then $ST$ has the* SDP.

PROOF. First, we prove that $(ST)^*$ has property $(\beta)$. Let $\{f_n:\omega\to X^*\}$ be a sequence of analytic functions on an open $\omega\subset C$ such that

$$\|(\lambda-(ST)^*)f_n(\lambda)\|\to 0\,, \qquad \text{as } n\to\infty$$

uniformly in every compact subset of $\omega$. Let $\lambda_0\in\omega$ and set

$$G_0=\{\lambda:|\lambda-\lambda_0|<r\}, \qquad G_1=\{\lambda:|\lambda-\lambda_0|<2r\}$$

for some $r>0$ subject to $\bar{G}_1\subset\omega$. Let $\{\sigma_j\}_{j=1}^n$ and $\{\delta_k\}_{k=0}^m$ be open covers of $\mathrm{Co}(\sigma(T))$ and $\sigma(S)$, respectively, where $\delta_0$ is the complement of a closed disk centered at the origin and $\sigma_j$ $(1\leqq j\leqq n)$, $\delta_k$ $(1\leqq k\leqq n)$ are open disks. We assume that

$$\sigma_j=\{\lambda:|\lambda-\mu_j'|<r_j'\}\ (1\leqq j\leqq n); \qquad \delta_k=\{\lambda:|\lambda-\mu_k''|<r_k''\}\quad(1\leqq k\leqq m)$$

and let

$$\Delta_{jk}=(\sigma_j-\mu_j')(\delta_k-\mu_k'')+\mu_j'\delta_k+\mu_k''\sigma_j-\mu_j'\mu_k''.$$

By the Remark following Lemma 2.2, $\Delta_{jk}$ is a disk centered at $\mu_j'\mu_k''$ of radius $r_j'r_k''+r_j'|\mu_k''|+r_k''|\mu_j'|$. For the given bounded $T$ and fixed $\delta_0$, we may always assume that there is $K>0$ such that

$$|\mu_j'|\leqq K\quad(1\leqq j\leqq n); \qquad |\mu_k''|\leqq K\quad(1\leqq k\leqq m)$$

for any choices of $\sigma_j$ and $\delta_k$. For $r_j'$ and $r_k''$ sufficiently small, the radii of the disks $\Delta_{jk}$ $(1\leqq j\leqq n,\ 1\leqq k\leqq m)$ are less than $r/2$. Therefore, two cases may occur:

$$\bar{\Delta}_{jk}\cap\bar{G}_0=\emptyset \qquad \text{for some}\quad (j,k);$$

$$\bar{\Delta}_{jk}\cap\bar{G}_0\neq\emptyset \qquad \text{and hence}\quad \bar{\Delta}_{jk}\subset G_1\quad \text{for other pairs}\quad (j,k).$$

As in (1.12), the following decomposition holds

$$X=Y_0+\sum_{j=1}^{n}\sum_{k=1}^{m}X_j\cap Y_k, \tag{2.2}$$

where $Y_0=X(S,\bar{\delta}_0)$, $Y_k=\Xi(S,\bar{\delta}_k)$, $1\leqq k\leqq m$, $X_j=X(T,\bar{\sigma}_j)$, $1\leqq j\leqq n$. By Lemma 2.2, we have

$$\sigma(ST|X_j\cap Y_k)\subset\bar{\Delta}_{jk}, \qquad 1\leqq j\leqq n, \quad 1\leqq k\leqq m. \tag{2.3}$$

We investigate for the structure of $Y_0$. Let $G\supset\bar{\delta}_0$ be open. By [4, Theorem 9.8], one has $X(S,G)^{\perp}=\Xi(S^*,G^c)$. Since $T^*|\Xi^*(S^*,G^c)$ has the SDP by Corollary 1.10, the coinduced operator $T/\overline{X(S,G)}$ also has the SDP. In particular, $T/\overline{X(S,G)}$ has the single valued extension property and hence $\overline{X(S,G)}$ is analytically invariant under $T$. Furthermore, it follows from

$$Y_0=\bigcap_{G\supset\bar{\delta}_0}\overline{X(S,G)}$$

that $Y_0$ is analytically invariant under $T$. By virtue of property $(\gamma)$, one may choose a closed disk $D_0$ centered at the origin so that $X(T,D_0)\subset D(S)$ and $X^*(T^*,D_0)\subset D(S^*)$. Then Lemmas 2.3 and 2.5 imply that $X(T,D_0)$ is invariant under $S$ and $X^*(T^*,D_0)$ is invariant under $S^*$. Let $D_1$ be the complement of an open disk centered at the origin so that $D_0{}^0\cup D_1{}^0=C$. In view of Lemma 2.5, $X(T,D_1)$ is invariant under $S$ and

$X^*(T^*, D_1)$ is invariant under $S^*$. It follows from the hypotheses on $T$ and Corollary 1.10, that

$$Y_0 = Y_0(T, D_0) + Y_0(T, D_1) = Y_0 \cap X(T, D_0) + Y_0 \cap X(T, D_1) . \tag{2.4}$$

The second equality holds because $Y_0$ is analytically invariant under $T$. We may choose $D_0$ so that the spectral radius $r(ST \mid X(T, D_0))$ is sufficiently small to produce the relations

$$\mathrm{Co}(\sigma(ST \mid X(T, D_0))) \subset G_1 , \qquad \text{if } \lambda_0 = 0 ; \tag{2.5}$$

$$\mathrm{Co}(\sigma(ST \mid X(T, D_0))) \cap \bar{G}_0 = \emptyset , \qquad \text{if } \lambda_0 \neq 0 . \tag{2.6}$$

Using again the fact that $Y_0$ is analytically invariant, (2.5) and (2.6) can be rewritten as follows:

$$\sigma(ST \mid Y_0 \cap X(T, D_0)) \subset G_1 , \qquad \text{if } \lambda_0 = 0 ; \tag{2.7}$$

$$\sigma(ST \mid Y_0 \cap X(T, D_0)) \cap \bar{G}_0 = \emptyset , \qquad \text{if } \lambda_0 \neq 0 , \tag{2.8}$$

for any choice of $\delta_0$ (note that $Y_0 = X(S, \bar{\delta}_0)$). Since $S$ commutes with $T$, it follows that $R(\mu; S \mid Y_0)$ commutes with $T \mid Y_0$ for $\mu \notin \bar{\delta}_0$ and hence $Y_0 \cap X(T, D_1)$ is invariant under $R(\mu; S \mid Y_0)$ for $\mu \notin \bar{\delta}_0$. Then

$$\sigma(S \mid Y_0 \cap X(T, D_1)) \subset \bar{\delta}_0$$

and hence

$$\sigma((S \mid Y_0 \cap X(T, D_1))^{-1}) \subset (\bar{\delta}_0)^{-1} , \tag{2.9}$$

where $(\bar{\delta}_0)^{-1} = \{0\} \cup \{\lambda^{-1} : \lambda \in \bar{\delta}_0\}$.

On the other hand, one has

$$\sigma(T \mid Y_0 \cap X(T, D_1)) = \sigma(T \mid Y_0(T, D_1)) \subset D_1$$

and hence

$$\sigma((T \mid Y_0 \cap X(T, D_1))^{-1}) \subset D_1^{-1} . \tag{2.10}$$

In view of property (2.1), it follows from (2.9) and (2.10) that

$$\sigma((ST \mid Y_0 \cap X(T, D_1))^{-1}) \subset (\bar{\delta}_0)^{-1} D_1^{-1} .$$

By the spectral mapping theorem, one obtains

$$\sigma(ST \mid Y_0 \cap X(T, D_1)) \subset \bar{\delta}_0 D_1 . \tag{2.11}$$

We may choose $\bar{\delta}_0$ so that

$$\sigma(ST \mid Y_0 \cap X(T, D_1)) \cap \bar{G}_0 = \emptyset . \tag{2.12}$$

Relations (2.2) and (2.4) give rise to the decomposition

$$X = Y_0 \cap X(T, D_0) + Y_0 \cap X(T, D_1) + \sum_{j=1}^{n} \sum_{k=1}^{m} X_j \cap Y_k . \tag{2.13}$$

By using (2.13), (2.3), (2.7) (or (2.8)) and (2.12), one can apply the routine expanded in the proof of Theorem 1.6, to show that

$$\|f_n(\lambda)\| \to 0 \quad \text{as} \quad n \to \infty$$

uniformly on $\bar{G}_0$. In terms of the Heine-Borel theorem, it follows that $(ST)^*$ has property $(\beta)$.

Next, we prove that $ST$ has property $(\beta)$. A decomposition, similar to (2.13), holds in the dual space:

$$X^* = Y_0^* \cap X^*(T^*, D_0) + Y^* \cap X^*(T^*, D_1) + \sum_{j=1}^{n} \sum_{k=1}^{m} X_j^* \cap Y_k^* , \tag{2.14}$$

where $Y_0^* = X^*(S^*, \bar{\delta}_0)$ is analytically invariant under $T^*$ and

$$Y_k^* = \Xi^*(S^*, \bar{\delta}_k) , \qquad 1 \leqq k \leqq m ; \qquad X_j^* = X^*(T^*, \bar{\sigma}_j) , \qquad 1 \leqq j \leqq n .$$

It follows from Lemma 2.1 and from inclusions $Y_k^* \subset D(S^*)$, $1 \leqq k \leqq m$, that

$$T^*S^* \mid Y_k^* = (ST)^* \mid Y_k^* = S^*T^* \mid Y_k^* .$$

Consequently, we have

$$T^*S^* \mid X_j^* \cap Y_k^* = (ST)^* \mid X_j^* \cap Y_k^* = S^*T^* \mid X_j^* \cap Y_k^* , \qquad 1 \leqq j \leqq n , \quad 1 \leqq k \leqq m .$$

Thus the dual counterpart of (2.3) is obtained:

$$\sigma((ST)^* \mid X_j^* \cap Y_k^*) = \sigma(S^*T^* \mid X_j^* \cap Y_k^*) \subset \bar{\Delta}_{jk} . \tag{2.15}$$

Quoting again Lemma 2.1 and noting that $X^*(T^*, D_0) \subset D(S^*)$, one obtains

$$T^*S^* \mid Y_0^* \cap X^*(T^*, D_0) = (ST)^* \mid Y_0^* \cap X^*(T^*, D_0) = S^*T^* \mid Y_0^* \cap X^*(T^*, D_0) .$$

Hence we may choose $D_0$ so that $\sigma((ST)^* \mid Y_0^* \cap X^*(T^*, D_0)$ satisfies conditions:

$$\sigma((ST)^* \mid Y_0^* \cap X^*(T^*, D_0)) \subset G_1 , \qquad \text{if} \quad \lambda_0 = 0 ; \tag{2.16}$$

$$\sigma((ST)^* \mid Y_0^* \cap X^*(T^*, D_0)) \cap \bar{G}_0 = \varnothing , \qquad \text{if} \quad \lambda_0 \neq 0 , \tag{2.17}$$

for any choice of $\bar{\delta}_0$.

Finally, applying the technique that lead us to (2.11), to the spectrum of $(ST)^* \mid Y_0^* \cap X^*(T^*, D_1)$, we obtain the inclusion

$$\sigma((ST)^* \mid Y_0^* \cap X^*(T^*, D_1)) \subset \bar{\delta}_0 D_1 .$$

As in the former case, we may choose $\bar{\delta}_0$ so that

$$\sigma((ST)^* \mid Y_0^* \cap X^*(T^*, D_1)) \cap \bar{G}_0 = \varnothing . \tag{2.18}$$

Now, with the help of (2.3), (2.15), (2.16) (or (2.17)) and (2.18), one can show that $ST$ has property ($\beta$).

Since both $ST$ and $(ST)^*$ have property ($\beta$), Theorem A implies that $ST$ has the SDP. □

2.8. COROLLARY. *If* $S \in C_d(X)$ *and* $T \in \{S\}'$ *satisfy the following conditions*:

(i) *either* $T$ *is invertible and* $S \in C_d(X)$ *or both* $T$ *and* $S$ *are bounded*;

(ii) $S$ *has the* SDP, $T$ *has the strong convex* SDP *relative to* $S$ *and* $T^*$ *has the strong convex* SDP *relative to* $S^*$,
*then* $ST$ *has the* SDP.

PROOF follows from Theorem 2.7 and the fact that (i) implies that $T$ and $S$ have property ($\gamma$). □

2.9. THEOREM. *If* $S$ *and* $T$ *commute with each other, and both* $S$ *and* $T$ *are regularly decomposable with respect to the identity, then* $ST$ *is strongly decomposable.*

PROOF. It follows from Theorem 2.7 that $ST$ is decomposable. Let $W$ be a spectral maximal space of $ST$. As in the proof of Theorem 1.13, using the hypothesis on regular decomposability of $T$, one can show that $T|W$ is decomposable and hence $(T|W)^*$ is decomposable. Moreover, $S|W$ and $(S|W)^*$ have the strong convex SDP relative to $T|W$ and $(T|W)^*$, respectively. Therefore, $ST|W$ is decomposable or, equivalently, $ST$ is strongly decomposable. □

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