POLYGONS AND HILBERT MODULAR GROUPS

Dedicated to Professor Shingo Murakami on his sixtieth birthday

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In [3], we described the structure of the icosahedral modular group in a geometric way. In this paper, we continue studying certain Hilbert modular groups through the same geometric means. Namely, we are able to determine the structures of the Hilbert modular groups of $Q(\sqrt{2})$, $Q(\sqrt{3})$ and $Q(\sqrt{5})$ by looking at the operations of the cubic, tetrahedral and icosahedral groups on $P_2(C)$. Here we shall treat the case of $Q(\sqrt{5})$ and $Q(\sqrt{2})$. Let G be a finite subgroup of $SO(3)$. Then G acts on $P_1(C)$ and we get the diagonal action of G on $P_1(C) \times P_1(C)$ which commutes with the transposition τ. We thus get the action of G over $P_2(C)$ which is isomorphic to $(P_1(C) \times P_1(C))/\tau$. The image of the diagonal in $P_2(C)$ is the invariant conic $[A] = P_1(C)$ for the G-action. The dual way of looking at this is as follows. We projectify the standard action of G on R^3 and get the action of G on $P_2(R)$. The inclusion $SO(3) \subset SO(3, C)$ then yields the operation of G over $P_2(C)$. From here on we consider $P_2(C)$ as a G-space through this G-action.

1. The structure of the Hilbert modular group of $Q(\sqrt{5})$.

We first study the action of the icosahedral group A_5 (the alternating group of five elements) over $P_2(C)$. We identify the invariant conic $[A]$ with the icosahedron. Let (x, x_a) be one of the six antipodal pairs of vertices in $[A]$, $p(x, x_a)$ the intersection point of the tangent lines to $[A]$ at x and x_a, and $[x, x_a]$ the polar line connecting x and x_a. In this way we get six points $p(x, x_a)$, which we call the poles, and a line configuration $[D]_6$ of six lines $[x, x_a]$. *The six poles form the unique minimal orbit of A_5* and come dually from the six lines in R^3 connecting six antipodal pairs of the twelve vertices. We can thus draw fifteen lines connecting two poles. Dually, the fifteen lines come from the fifteen planes determined by the antipodal pairs of the thirty edges. We write $[D]_{15}$ for this line configuration. Next, let (x, x_a) be one of the ten antipodal pairs of the barycenters of the faces in $[A]$ and we carry out the same procedure as above. As a result, we get ten polar lines $[D]_{10}$ and ten points $p(x, x_a)$ which come from the lines in R^3 connecting the antipodal pairs of the barycenters of the faces. But, in this case, the lines connecting these ten points are divided into two groups. One turns out to be $[D]_{10}$ consisting of the projectifications of planes containing exactly three lines through the origin and the barycenter of a face (see Figure 1).
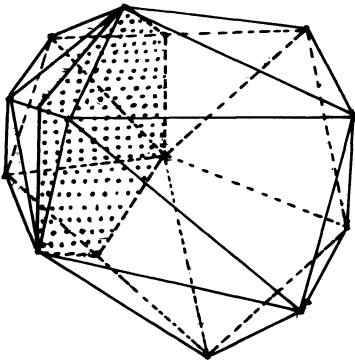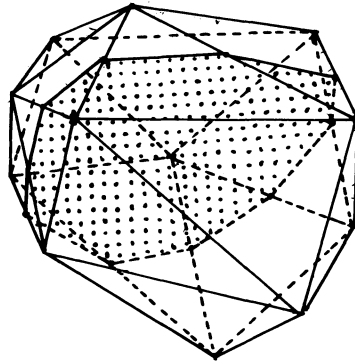
FIGURE 1.

FIGURE 2.

The other turns out to be $[D]_{15}$ consisting of the projectifications of planes containing exactly two lines through the barycenter of a face. Finally let (x, x_a) be one of the fifteen antipodal pairs of the middle points of the edges in $[A]$ and we do the same. Then we get fifteen polar lines which turn out to be $[D]_{15}$ and fifteen points $p(x, x_a)$ comming from the lines in R^3 through antipodal pairs of the middle points of the edges. The lines connecting these fifteen points are divided into two groups. One turns out to be $[D]_6$ consisting of the projectifications of six planes containing exactly five lines through the middle point of an edge (see Figure 2). The other again turns out to be $[D]_{15}$, the projectifications of planes containing excactly two of these lines.

All the above imply that the configuration $[D]_{15}$ consists of reflection lines of fifteen involutions in A_5 (note that A_5 is generated by three involutions) and the singularities of $[D]_{15}$ consist of *six* 5-ple points (poles), ten triple points and fifteen nodes all of which have appeared in the above arguments. See Figure 3.

The lines in $[D]_6$ and $[D]_{10}$ are not the reflecting lines, for the intersection points with $[A]$ have odd isotropy. Let A, D be the defining polynomials for $[A], [D] = [D]_{15}$. These are invariant polynomials for the A_5-action. Klein [2] proved the following:

FACT. *The ring of invariants is generated by A, B, C and D where* $[A]$ *is the unique invariant conic,* $[B]$ *is the unique curve of degree 6 through six poles,* $[C]$ *is* Klein's curve *of degree 10, the unique curve of degree 10 through six poles,* $[D]$ *is the unique curve of degree 15, the fifteen lines* $[D]_{15}$. *There is a relation* $R(A, B, C, D) = 0$ *with*

(1) $R(A, B, C, D) = -144\, D^2 - 1728\, B^5 + 720\, ACB^3 - 80\, A^2C^2B$

$$+ 64A^3(5B^2 - AC)^2 + C^3 .$$

The intersections of these curves are described in the following.

(i) $[A]$ and $[B]$ (also $[D]_6$) intersect at twelve points, i.e., at the vertices of the icosahedron.
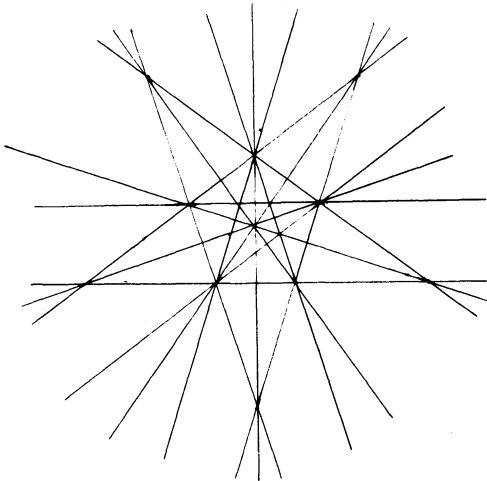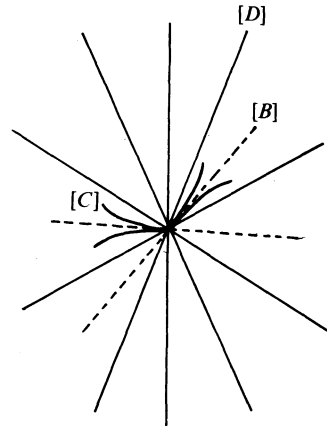
FIGURE 3.

FIGURE 4.

  (ii) $[A]$ and $[C]$ (also $[D]_{10}$) intersect at twenty points, i.e., at the barycenters of the faces.

  (iii) $[A]$ and $[D]$ intersect at thirty points, i.e., at the middle points of the edges.

  (iv) $[B]$ and $[C]$ intersect at six poles with multiplicity 10.

  (v) $[B]$ and $[D]$ intersect at six poles with multiplicity 10 and at the other thirty points on $[A]$ in (iii).

  (vi) $[C]$ and $[D]$ intersect at six poles with multiplicity 20 and at the other thirty points on $[A]$ in (iii).

Figure 4 explains how these curves intersect at one pole.

  Now we are ready to study the quotient $S = P_2(C)/A_5$. Firstly, we note that outside $[D]$ there are exactly two orbits of points at which A_5 has a non-trivial isotropy; they are the vertices and the centers of the faces on $[A]$. This implies that the quotient S has exactly two quotient singularities of type $C^2/\langle \xi, \xi^2 \rangle$, where $\xi = \exp(2\pi\sqrt{-1}/5)$ or $\xi = \exp(2\pi\sqrt{-1}/3)$. The graphs for the exceptional sets of the minimal resolutions are given by

$$\overset{E_1}{\underset{-2}{\bullet}} \rule{2cm}{0.4pt} \overset{E_2}{\underset{-3}{\bullet}} \qquad\qquad \overset{E'_1}{\underset{-2}{\bullet}} \rule{2cm}{0.4pt} \overset{E'_2}{\underset{-2}{\bullet}}$$

respectively. Here, E_2 and E'_2 cut the strict transform $[A]'$ of the image of $[A]$ transversally. If we slightly perturb $[A]'$ so that the perturbed 2-manifold intersects $[A]$ transversally at points outside E_2 and E'_2, then we have

$$(|A_5|)([A]' \cdot [A]') + 2(12 + 20) = [A] \cdot [A] .$$

This implies $[A]' \cdot [A]' = -1$, i.e., $[A]'$ is an exceptional curve of the first kind. If we

blow down $[A]'$, E_2', E_2, E_1 successively, we get a smooth compact complex surface $\bar{S}$. Then image L of E_1' has self-intersection number 1. Now we compute the Euler number of $\bar{S}$. The image $[D]'$ of $[D]$ in S is a rational curve with one node, one $(2, 3)$-cusp and one $(2, 5)$-cusp, and intersects $[A]'$ transversally at one point. It follows that the strict transform $[\bar{D}]$ of $[D]'$ is a rational curve with an additional $(2, 5)$-cusp at infinity at which $[\bar{D}]$ and L meet with multiplicity 5. This implies that the Euler number of $\bar{S}$ fulfills the relation:

$$(|A_5|)e(\bar{S} - L \cup [\bar{D}]) = e(P_2(C) - [D] \cup [A])$$

where $e(\bar{S} - L \cup [\bar{D}]) = e(\bar{S}) - 2$ and $e(P_2(C) - [D] \cup [A]) = 60$, and we have $e(\bar{S}) = 3$. Since $\bar{S}$ is birational to $P_2(C)$, $\bar{S}$ must be $P_2(C)$. It follows that L is a line in $\bar{S}$ and $\bar{S} - L = C^2$. We determine the equation for $[\bar{D}]$ in this affine part. To do so, we look at the commitative diagram:

$$
\begin{array}{ccc}
P_2(C) & \dashrightarrow{\delta} & \bar{S} = P_2(C) \\
\downarrow & & \uparrow{\scriptstyle\beta} \\
S = P_2(C)/A_5 & \xleftarrow{\pi} & S'
\end{array}
$$

where π is the minimal resolution, β is the successive blow down of $[A]'$, E_2', E_2, E_1 and δ is the rational map given by $(A^5 : A^2 B : C)$. Since δ is a holomorphic map of $P_2(C) - [A]$ to $\bar{S} - L$, we have

$$x = B/A^3, \qquad y = C/A^5$$

where x and y are the affine coordinates identifying $(1 : x : y)$ in $\bar{S} - L$ with (x, y) in C^2. The relation (1) then gives the equation for the affine curve $[\bar{D}]$:

$$(2) \qquad -1728\,x^5 + 720\,x^3 y - 80\,xy^2 + 64(5\,x^2 - y)^2 + y^3 = 0\,.$$

Clearly the equations for $[\bar{B}]$ and $[\bar{C}]$ are given by

$$(2)' \qquad\qquad\qquad x = 0$$

$$(2)'' \qquad\qquad\qquad y = 0$$

respectively. The important fact is that the G-space $P_2(C)$ has a natural real structure with respect to which the fifteen lines in $[D]$ are defined over the reals, i.e., the fifteen lines define the icosahedral arrangement of lines over the real part $P_2(R)$. Moreover the equations A, B, C also have real coefficients. Since the real part is disjoint from $[A]$, the rational map δ of $P_2(C)$ restricts to a morphism δ of $P_2(R)$. Hence all the singular points of $[\bar{D}]$ lie in the real part. We use this real property to compute $G = \pi_1(C^2 - [\bar{D}] \cup [\bar{C}])$. The real picture of $[\bar{C}]$ and $[\bar{D}]$ is as in Figure 5.

The vertical line denoted by l stands for a fixed real line in the Zariski pencil of curves through $(0 : 0 : 1)$ and a, b, c, d are the intersection points of l and $[\bar{D}] \cup [\bar{C}]$. The
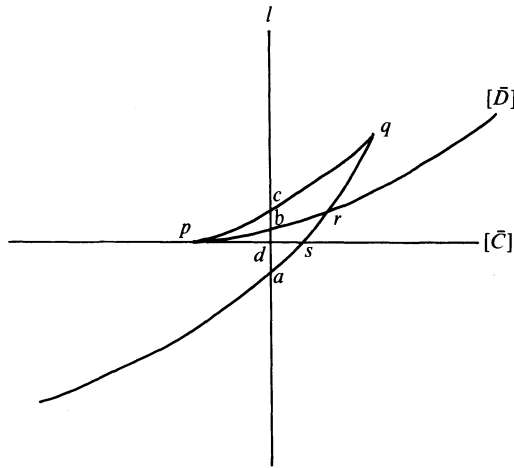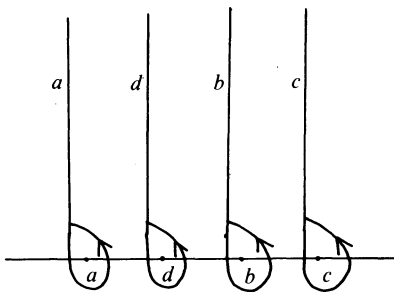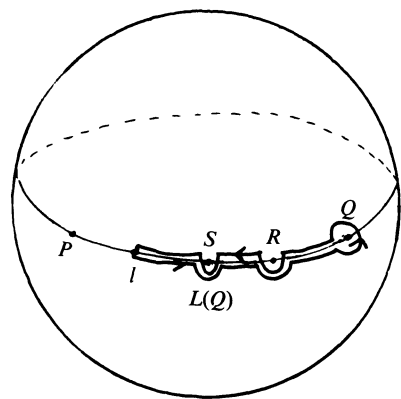
FIGURE 5.

FIGURE 6.

FIGURE 7.

intersection of l and $[D]$ are a, b, c ordered from below. The fact that the equation (2) is of degree 3 with respect to y explains why l and $[\bar{D}]$ intersect at three points. Note that every line in the Zariski pencil meet $[D]$ at infinity with multiplicity at least two. We denote the loop issuing from some point in the upper half plane in the complex feature of the line l which surrounds a counterclockwise by the same letter a, etc. (see Figure 6).

Then we apply the well known method of Zariski and van Kampen to compute the required fundamental group G. Namely, G is generated by a, b, c, d and the adequate relations among them arise from the singular points of $[\bar{D}] \cup [\bar{C}]$. The Zariski pencil is parametrized by the 2-sphere minus a point corresponding to L. In Figure 7, the equator stands for the real lines and P, Q, R, S represent those through the singular

points p, q, r, s. For instance, if we move l along the loop $L(Q)$, we get the relation comming from q. See, for instance, [3]. We thus get the following relations:

(3) $ab=ba$, $aca=cac$, $ad=da$, $(dbc)^2d=d(dbc)^2$, $(dbc)^2bc=b(dbc)^2b$, $(dbc)^2b=c(dbc)^2$.

Now we recall Hirzebruch's result [1] on the description of the Hilbert modular surface of $Q(\sqrt{5})$. Let k be the real quadratic field $Q(\sqrt{5})$ and $\mathfrak{o}(k)=\mathfrak{o}$ the ring of integers in k and $\Gamma(\sqrt{5})$ the principal congruence subgroup of $SL(2, \mathfrak{o})$ associated with the prime ideal $(\sqrt{5})$ in $\mathfrak{o}$. Let $\check{Y}$ be the Hilbert modular surface $\check{Y}=(H \times H/\Gamma(\sqrt{5}))^-$ completed with the six cusps on which operates $SL(2, \mathfrak{o})/\Gamma(\sqrt{5}) \simeq PSL(2, F_5) \simeq A_5$. Each cusp is resolved by a 2-cycle of rational curves of self-intersection number -3. The transposition $\tau: (z_1, z_2) \rightarrow (z_2, z_1)$ induces an involution, also denoter by τ, on $\check{Y}$ and it is shown in [1] that there is an equivariant isomorphism of the A_5-spaces $\check{Y}/\tau$ and $P_2(C)$, and that the branch locus of $\check{Y} \rightarrow P_2(C)$ is Klein's curve of degree ten. The image of six cusps are the six poles. Since A_5 acts freely on $P_2(C) - [A] \cup [C] \cup [D]$ we have the following exact sequence:

(4) $1 \longrightarrow \pi_1(P_2(C) - [A] \cup [C] \cup [D]) \longrightarrow \pi_1(C^2 - [\bar{D}] \cup [\bar{C}]) \overset{h}{\longrightarrow} A_5 \longrightarrow 1$.

Here, h is given, for instance, by the following:

(5) $h(a)=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $h(b)=\begin{pmatrix} u & 0 \\ 0 & u+1 \end{pmatrix}$, $h(c)=\begin{pmatrix} u & 1 \\ 0 & u+1 \end{pmatrix}$, $h(d)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

where we identify A_5 with $PSL(2, F_5)$, $F_5 = \{0, 1, -1, u, 1+u \pmod{\sqrt{5}}\} = \mathfrak{o}/(\sqrt{5})$ with $u=(\sqrt{5}-1)/2$. If e is a loop in $P_2(C)$ surrounding $[A]$ and lying over a holomorphic curve segment which cut $[A]$ at a point with the isotropy of order 5 and is invariant under the action of the isotropy group, then the image loop $\bar{e}$ in $\bar{S}$ is a loop which surrounds L five times. It follows that if we fill $[A]$, then the following relations should be added to (3) for the fundamental group $\pi_1((P_2(C) - [D] \cup [C])/A_5 - \{$quotient singularities$\})$:

(6) $(abcd)^5 = 1$.

Since $[\bar{D}] \cup [\bar{C}]$ is essentially the branch locus of index 2 of the orbifold $(H \times H)/\langle PSL(2, \mathfrak{o}), \tau \rangle$, we must add the following relations to (3) and (6):

(7) $a^2 = b^2 = c^2 = 1$

(8) $d^2 = 1$

for the *fundamental group of the orbifold*. We have thus proved the following:

THEOREM 1. $\langle PSL(2, \mathfrak{o}), \tau \rangle$ *is a reflection group G generated by a, b, c, d with the relations* (3), (6), (7), (8) *among them.*

COROLLARY 1.   '*The fundamental group $\pi_1(P_2(C) - [C])$ is isomorphic to*

Ker $h: \bar{G} \rightarrow A_5$, where $\bar{G}$ is the group generated by a, b, c, d with the relations (3), (6), (7) and h is given by (5).

COROLLARY 2. $\langle \Gamma(\sqrt{5}), \tau \rangle$ is isomorphic to Ker $h: G \rightarrow A_5$, where h is given by (5). In particular, $\langle \Gamma(\sqrt{5}), \tau \rangle$ is isomorphic to some quotient group of $\pi_1(P_2(C) - [C])$.

As the matrix representations for the generators a, b, c, d of G we have, for instance,

$$a = \begin{pmatrix} 0 & A \\ A* & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & B \\ B* & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & C \\ C* & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & D \\ D* & 0 \end{pmatrix},$$

where $*$ means the conjugation in $Q(\sqrt{5})$ and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} u & 0 \\ 0 & 1+u \end{pmatrix}, \quad C = \begin{pmatrix} u & 1 \\ 0 & 1+u \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with $u = (\sqrt{5} - 1)/2$. The action of a on $H \times H$ is given by $a(z_1, z_2) = (Az_2, A*z_1)$, etc.

2. The structure of the Hilbert modular group of $Q(\sqrt{2})$.

We first study the action of the cubic group S_4 (the symmetric group of degree 4) over $P_2(C)$. We carry out completely the same procedure as in the case of icosahedral group. The poles consist of four points from which we form the configuration $[D]$ of six lines, the projectified planes determined by the antipodal pairs of edges in R^3. From the six barycenters of the faces in $[A]$ (the invariant conic), we form three points. We then form three lines $[B]$, the projectified planes determined by the two antipodal pairs of the barycenters of the faces. From twelve middle points of the edges, we form six points. The lines connecting these points are divided into two groups. One group $[C]$ consists of four lines each of which is the projectification of the plane containing exactly three antipodal pairs of the middle potins of the edges. The other consists of three lines $[B]$. The lines in $[D]$ and $[B]$ are the reflecting lines for six and three involutions of S_4. These three involutions are squares of elements of order four. The equations for these configurations are given by (A corresponds to $[A]$, etc.)

$$A = X^2 + Y^2 + 2Z^2, \quad B = (X - Y)(X + Y)Z, \quad C = (X^2 - Z^2)(Y^2 - Z^2),$$
$$D = XY(X + Y + 2Z)(X + Y - 2Z)(X - Y + 2Z)(X - Y - 2Z)$$

in the homogeneous coordinates $(X: Y: Z)$. These except $[A]$ are realized in the real part $P_2(R)$ (see Figure 8).

Hirzebruch [1] proved the following:

FACT. The ring of invariants is generated by A, C, B^2, BD with the relation $R(A, C, B^2, BD) = 0$, where

(9) $R(A, C, B^2, BD) = 108(BD)^2 + [-(A^2 + 12C)^3 + (54B^2 - A^3 + 36AC)^2]B^2$.

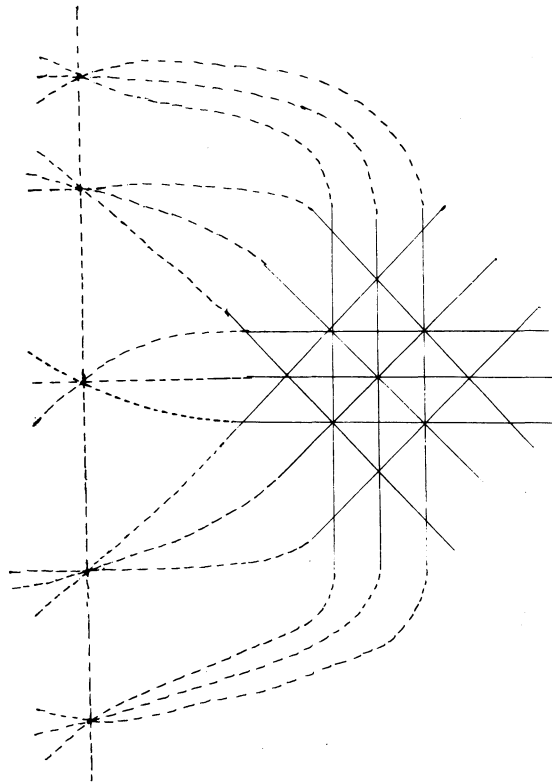Now we can study the quotient $S = P_2(C)/S_4$. As before, outside $[D]$ there are

FIGURE 8.

exactly two orbits of points at which S_4 has a non-trivial isotropy; they are the vertices and the centers of the faces in $[A]$. The action of the isotropy group at these points is of type (ξ, ξ^2) with $\xi = \exp(2\pi\sqrt{-1}/3)$ or $\xi = \exp(2\pi\sqrt{-1}/4)$. This implies that S has two quotient singularities of types

$$\begin{array}{ccc} E_1 & E_2 & E_1' \\ \bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet & & \bullet \\ -2 \quad\;\; -2 & & -2 \end{array}$$

and E_2, E_1' cut the proper transform $[A]'$ of $[A]$ transversally. Since

$$(|S_4|)([A]' \cdot [A]') + 2(8+6) = [A] \cdot [A],$$

we see that $[A]' \cdot [A]' = -1$, i.e., $[A]'$ is the exceptional curve of the first kind. The successive blow down of $[A]'$, E_2, E_1 yields a smooth compact complex surface $\bar{S}$ and a rational curve $L = E_1'$ with self-intersection number 1. Two curves L and $[\bar{D}]$ contact in order 3, and at the same point $[\bar{C}]$ cuts L transversally. The curve $[\bar{B}]$ meets L at another point transversally. Since

$$(|S_4|)e(\bar{S} - L \cup [\bar{D}] \cup [\bar{B}]) = e(P_2(C) - [D] \cup [B] \cup [A])$$

where $e(\bar{S} - L \cup [\bar{D}] \cup [\bar{B}]) = e(\bar{S}) - 2$, $e(P_2(C) - [D] \cup [B] \cup [A]) = 24$, we see that $e(\bar{S}) = 3$. As before, we conclude that $\bar{S} = P_2(C)$. Let δ be the rational map of $P_2(C)$ defined by $(A^3 : B^2 : AC)$. Then we get the following commutative diagram:

$$
\begin{array}{ccc}
P_2(C) & \overset{\delta}{\dashrightarrow} & P_2(C) = \bar{S} \\
\downarrow & & \uparrow \beta \\
S = P_2(C)/S_4 & \overset{\pi}{\longleftarrow} & S'
\end{array}
$$

If we identify (x, y) in C^2 with $(1 : x : y)$ in $P_2(C) - L$, the rational map δ is given by

$$x = B^2/A^3, \qquad y = C/A^2.$$

The relation (9) then implies that the equation for the affine curve $[D]$ is given by

(10) $$(1 + 12y)^3 = (54x + 36y - 1)^2$$

which has one $(2,3)$-cusp. Now we recall Hirzebruch's result on the Hilbert modular group of $Q(\sqrt{2})$. Let $k = Q(\sqrt{2})$, $\mathfrak{o} = \mathfrak{o}(k)$ the ring of integers, and Γ the group generated by the principal congruence subgroup $\Gamma(2)$ associated with the ideal (2) of $\mathfrak{o}$ and

$$D(u) = \begin{pmatrix} u & 0 \\ 0 & -u* \end{pmatrix}$$

where $u = -\sqrt{2} - 1$ is the fundamental unit. Let $\check{Y}$ be the Hilbert modular surface $(H \times H/\Gamma)^-$ completed with the six cusps. The transposition τ induces the involution τ on $\check{Y}$ and we get an S_4-space $\check{Y}/\tau = (H \times H/\langle \Gamma, \tau \rangle)^-$. Hirzebruch [1] proved that there is an equivariant isomorphism of the S_4-spaces $\check{Y}/\tau$ and $P_2(C)$ and that the branch locus of $\check{Y} \to P_2(C)$ is the curve $[R]$ of degree 10 defined by

$$R = C(AC + B^2) = (X^2 - Z^2)(Y^2 - Z^2)(X^2 + Y^2 - 2Z^2)(XY - Z^2)(XY + Z^2) = 0.$$

See Figure 9. We note that $[R]$ contains $[C]$. As before, we again have a morphism $\delta = (A^3 : B^2 : AC)$ of $P_2(R)$ and we draw the real picture of the image curves $[\bar{B}], [\bar{C}], [\bar{D}], [\bar{R}]$ in Figure 10.

It is essential that all the singularities of the configuration $[B] \cup [C] \cup [D] \cup [R]$ lie in the real part $P_2(R)$. This enables us to compute the fundamental group $\pi_1(\bar{S} - L \cup [\bar{B}] \cup [\bar{D}] \cup [\bar{R}])$ through Zariski-van Kampen's method. Pick a generic point in $L \cap P_2(R)$ and consider the Zariski pencil through this point. Here, "generic" means that the point is not on $L \cap ([\bar{B}] \cup [\bar{D}] \cup [\bar{R}])$. The Zariski pencil consists of L and lines parallel to the nameless line in Figure 10. The nameless line is fixed for the computation of the fundamental group. Let a_0, a, b, c, d, e be loops in the complex feature of it as in Figure 11. The fundamental group is generated by a_0, a, b, c, d, e and adequate
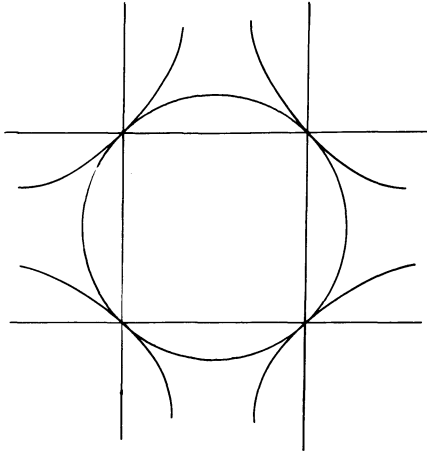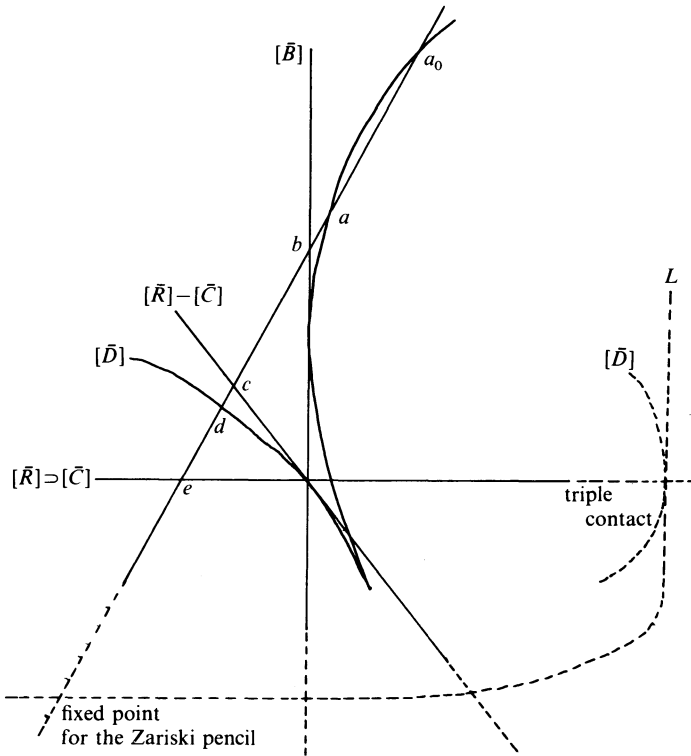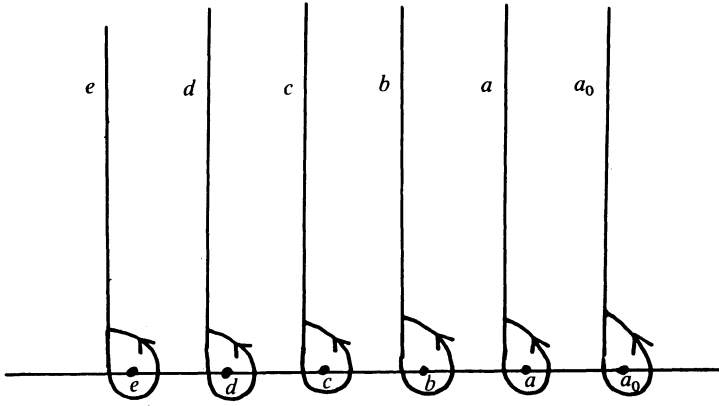
FIGURE 9.

FIGURE 10.

FIGURE 11.

relations arise from the singular points of the configuration. The required relations are as follows:

(11) $a_0 = a, (ab)^2 = (ba)^2, ae = ea, ac = ca, ada = dad, bdce = ebdc,$

$(bdce)dc = c(bdce)d, (bdce)dcd = d(bdce)dc, (bdce)b = b(bdce)$.

Since S_4 acts on $P_2(C) - [A] \cup [B] \cup [D] \cup [R]$ freely, we have the following exact sequence:

(12) $1 \longrightarrow \pi_1(P_2(C) - [A] \cup [B] \cup [D] \cup [R]) \longrightarrow$

$\pi_1(C - [\bar{B}] \cup [\bar{D}] \cup [\bar{R}]) \overset{h}{\longrightarrow} S_4 \longrightarrow 1$.

Here, h is given as follows: We regard S_4 as a quotient matrix group $PSL(2, \mathfrak{o}/(2))/\langle D(u) \rangle$, where $\mathfrak{o}/(2) = \{0, 1, u, 1 + u \pmod 2\}$ and $\langle D(u) \rangle$ is the center of $PSL(2, \mathfrak{o}/(2))$. Then

(13) $h(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad h(b) = \begin{pmatrix} 1 & u+1 \\ 0 & 1 \end{pmatrix}, \quad h(c) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad h(d) = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix},$

$h(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{considered mod } D(u)$.

Let f be a loop surrounding $[A]$ and lying over a holomorphic curve segment which cuts $[A]$ at a generic point. Then the image loop f' in S' surrounds $[A]'$ at a generic point. Since $S' - [A]' \cup E'_1 \cup E_2 \cup E_1 \cup [B]' \cup [D]' \cup [R]' = C^2 - [\bar{B}] \cup [\bar{D}] \cup [\bar{R}]$, the new relation arising by filling the generic portion of $[A]'$ is $f' = 1$. The image loop $\bar{f}$ in $\bar{S}$ lies over a holomophic curve segment tangent to L at the point p where L, $[\bar{D}]$ and $[\bar{C}]$ meet. If we pick a small 3-sphere S^3 surrounding the point p, we get Figure 12. The vertical line is $S^3 \cap [\bar{C}]$ and the other two loops winding around each other

are $S^3 \cap [\bar{D}]$ and $S^3 \cap L$. Looking at the fundamental group of $X = S^3 - S^3 \cap ([\bar{C}] \cup [\bar{D}] \cup L) = R^3 -$ (one line and two loops in Figure 12), we have

$$(14) \qquad\qquad esa^{-1}sa^{-1}sa^{-1} = \bar{f} \qquad \text{in} \quad \pi_1(X),$$

where s is a loop surrounding L in the positive direction. To see this, we note that X is homotopy equivalent to the space obtained by attaching two 2-tori along a loop $\bar{f}$ (drawn in thick line) as in Figure 13. The loop e is now realized as a loop drawn in thin line. We thus have (14). Since, in view of (16) below, $s^{-1} = a_0abcde = bcde$, the new relation $f' = 1$ implies

$$(15) \qquad\qquad e = (abcde)^3 .$$

Since $[\bar{B}] \cup [\bar{D}] \cup [\bar{R}]$ is essentially the branch locus of the orbifold $(H \times H)/\langle PSL(2, \mathfrak{o}), \tau \rangle$, we should add the following relations to (11) and (15):

$$(16) \qquad\qquad a^2 = b^2 = d^2 = 1 ,$$

$$(17) \qquad\qquad c^2 = e^2 = 1$$

for the *fundamental group of this orbifold*. Thus we have proved:

THEOREM 2. $\langle PSL(2, \mathfrak{o}), \tau \rangle$ *is a reflection group* $\tilde{G}$ *generated by* a, b, c, d, e *with the relations* (11), (15), (16), (17).

COROLLARY 3. *The fundamental group* $\pi_1(P_2(C) - [R])$ *is isomorphic to* $\mathrm{Ker}\, h \colon \bar{G} \to S_4$, *where* $\bar{G}$ *is the group generated by* a, b, c, d, e *with the relations* (11), (15), (16), *and* h *is given by* (13).

COROLLARY 4. $\langle \Gamma, \tau \rangle$ *is isomorphic to* $\mathrm{Ker}\, h \colon G \to S_4$. *In particular, it is isomorphic to some quotient group of* $\pi_1(P_2(C) - [R])$.

It is now not difficult to find the matrix lements in $PSL(2, \mathfrak{o})$ for the group G. For
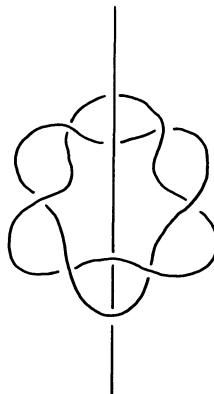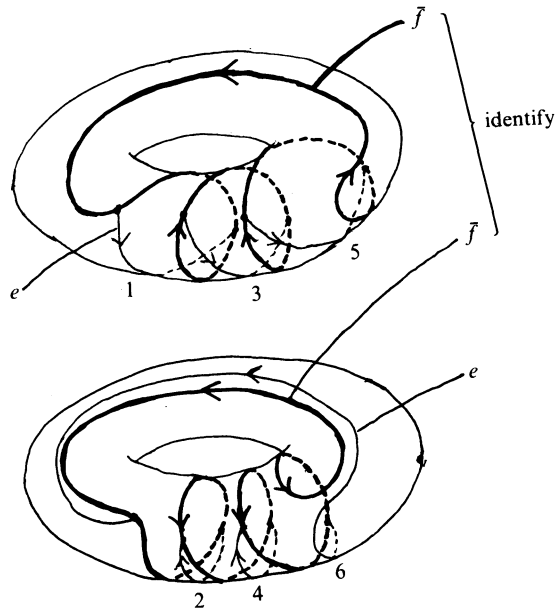

FIGURE 12.

FIGURE 13.

instance we have

$$a=\begin{pmatrix} 0 & A \\ A* & 0 \end{pmatrix}, \quad b=\begin{pmatrix} 0 & B \\ B* & 0 \end{pmatrix}, \quad c=\begin{pmatrix} 0 & C \\ C* & 0 \end{pmatrix}, \quad d=\begin{pmatrix} 0 & D \\ D* & 0 \end{pmatrix}, \quad e=\begin{pmatrix} 0 & E \\ E* & 0 \end{pmatrix},$$

where * means the conjugation in $Q(\sqrt{2})$ and

$$A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} 1 & u+1 \\ 0 & 1 \end{pmatrix}, \quad C=\begin{pmatrix} u & 0 \\ 0 & -u* \end{pmatrix}, \quad D=\begin{pmatrix} u & 1 \\ 0 & -u* \end{pmatrix}, \quad E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

on $PSL(2, \mathfrak{o})$-level, with $u=-\sqrt{2}-1$.

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