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RESILIENT LEAVES IN TRANSVERSELY AFFINE FOLIATIONS

Dedicated to Professor Akio Hattori on his sixtieth birthday

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1. Introduction. Let $\mathscr{F}$ be a codimension one foliation on a manifold M. We say that $\mathscr{F}$ is *transversely affine* if M is covered by a collection of $\mathscr{F}$-distinguished charts for which the coordinate transformations are affine (*i.e.*, of the form $x \mapsto ax+b$, $a \neq 0$) in the direction transverse to $\mathscr{F}$. See [6, I, Chap. III] for fundamental properties of codimension one transversely affine foliations.

The purpose of this paper is to study the problem of existence of resilient leaves in codimension one transversely affine foliations. Here, a leaf is said to be *resilient* if it is nonproper and with nontrivial holonomy. Resilient leaves are classified into two types—locally dense type and exceptional type. It is known that locally dense resilient leaves appear in some codimension one transversely affine foliations on closed manifolds (See, *e.g.*, [1] or Step 2 of §2 in this paper).

In [5], Furness and Fedida asserted that a codimension one transversely affine foliation cannot have exceptional leaves. But their proof seems to have a gap. In fact, it is rather easy to give a counterexample on an *open* manifold.

Now the first result of this paper is stated as follows:

THEOREM 1.1. *There exists a codimension one transversely affine foliation on a closed* 3*-manifold which contains an exceptional minimal set.*

By a classical theorem of Sacksteder [10], this exceptional minimal set contains a resilient leaf, necessarily of exceptional type.

REMARK. After circulating the earlier draft of this paper, the author received a letter from G. Hector to the effect that he constructed a similar example several years ago and will write his result up in the near future.

Let Aff($\boldsymbol{R}$) be the group of affine transformations of the real line. A codimension one transversely affine foliation $\mathscr{F}$ on a manifold M induces a holonomy homomorphism $h: \pi_1(M) \to \mathrm{Aff}(\boldsymbol{R})$. We call the image of h the *global holonomy group* of $\mathscr{F}$. The next result characterizes the existence of locally dense resilient leaves in terms of the global holonomy group.

THEOREM 1.2. *Let $\mathscr{F}$ be a codimension one transversely oriented, transversely affine foliation on a closed manifold and Γ its global holonomy group. Then either of the following* *holds*:

(1) *$\mathscr{F}$ is almost without holonomy, and Γ is abelian.*

(2) *$\mathscr{F}$ contains a locally dense resilient leaf, and Γ is nonabelian.*

Here $\mathscr{F}$ is said to be *almost without holonomy* if every noncompact leaf of $\mathscr{F}$ has trivial holonomy. Theorem 1.2 may be interpreted as follows: A codimension one transversely affine foliation cannot have a medium complexity—it is either so complicated as to contain resilient leaves or so simple as to be almost without holonomy. As an immediate corollary of Theorem 1.2, we have the following fact: If $\mathscr{F}$ has an exceptional leaf, then $\mathscr{F}$ also has a locally dense resilient leaf.

Throughout this paper, by the affine structure of an interval we mean the one which is induced from the standard affine structure of the real line $\boldsymbol{R}$ by the inclusion map, unless otherwise specified.

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2. Proof of Theorem 1.1. Let A be a matrix in $SL(2, \boldsymbol{Z})$ such that the trace of A is greater than 2. Then the smaller eigenvalue λ of A is positive and less than $1/2$.

Step 1. Let P be a twice punctured disk. Denote the boundary components of P by C_0, C_1 and C_2, which are assumed to have the orientations induced from a fixed orientation of P. Choose orientation reversing diffeomorphisms $f_1: C_0 \to C_1$ and $f_2: C_0 \to C_2$. Let I be the closed unit interval $[0, 1]$. Define two affine embeddings g_1, g_2 from I into itself by $g_1(t) = \lambda t$ and $g_2(t) = \lambda(t-1)+1$. Now define a manifold S with corner as the quotient space $P \times I/\sim$ where $\sim$ is the equivalence relation which identifies $(x, g_i(t))$ with $(f_i(x), t)$, $x \in C_0$, $t \in I$, $i = 1, 2$. The product foliation on $P \times I$ with leaves $P \times \{t\}$, $t \in I$, induces, by passing to the quotient, a foliation $\mathscr{F}_S$ on S. Note that $(S, \mathscr{F}_S)$ is the simplest *branched staircase* ([9], see also [3]). We summarize the properties of $(S, \mathscr{F}_S)$ in the following.

PROPOSITION 2.1. (1) *$\mathscr{F}_S$ is transversely affine.*

(2) *The corner of S is concave* (*see* [8, p. 107]) *with respect to $\mathscr{F}_S$.*

(3) *The boundary of S is divided by the corner into $\partial_{\tan} S$, the part tangent to $\mathscr{F}_S$, and $\partial_{tr} S$, the part transverse to $\mathscr{F}_S$.*

(4) *$\partial_{\tan} S$ is a disjoint union of a couple of once punctured tori, each of which has a linear contraction $t \mapsto \lambda t$ as a generator of its holonomy group.*

(5) *$\partial_{tr} S$ is a cylinder foliated by circles.*

(6) *$\mathscr{F}_S$ has an exceptional minimal set, which contains $\partial_{\tan} S$ and does not meet $\partial_{tr} S$.*

Step 2. First we recall a well-known example of a transversely affine foliation with dense resilient leaves. Let A be the matrix chosen at the beginning of this section. Denote by $\mathscr{G}$ the foliation on the torus $T^2 = \boldsymbol{R}^2/\boldsymbol{Z}^2$ defined by lines which are parallel to the eigenspace of A corresponding to the eigenvalue λ^{-1}. Let M_A be the manifold obtained from $T^2 \times I$ by identification of $T^2 \times \{0\}$ with $T^2 \times \{1\}$ as follows:

$(Ax, 0) \sim (x, 1)$. The foliation on $T^2 \times I$ whose leaves are of the form $G \times I$, with a leaf G of $\mathcal{G}$, induces a foliation $\mathcal{F}_A$ on M_A. We see that $\mathcal{F}_A$ is transversely affine and that all leaves of $\mathcal{F}_A$ are dense.

Next we will modify this example. We fix a transverse orientation of $\mathcal{F}_A$. Let $p: T^2 \times I \to M_A$ be the quotient map. Let L be the cylindrical leaf of $\mathcal{F}_A$ containing $p(\{0\} \times I)$. Denote by γ_0 the loop in L defined by $\gamma_0(t) = p(0, t)$. Then the holonomy along γ_0 is a contraction $t \mapsto \lambda t$. Let γ_+ (resp. γ_-^{-1}) be a closed transversal to $\mathcal{F}_A$ which is obtained from γ_0 by slight translation in the positive (resp. negative) transverse direction. We see that the orientations of γ_+ and γ_- are compatible with the transverse orientation of $\mathcal{F}_A$. Affine structures are induced on γ_+ and γ_- from the transverse affine structure of $\mathcal{F}_A$.

LEMMA 2.2. *γ_+ (resp. γ_-) is orientation preservingly and affinely isomorphic to the affine circle which is the quotient of the positive half line $(0, \infty)$ (resp. the negative half line $(-\infty, 0)$) by the action generated by the affine automorphism $t \mapsto \lambda t$.*

PROOF. We consider only the case of γ_+ (the case of γ_- being treated similarly). Let p be a point on γ_+ and q a point on γ_0. Then we can observe that there exist a transverse arc $\tau: [0, 1] \to M_A$ with $\tau(0) = q$ and $\tau(1) = p$ and a continuous map $P: [0, 1] \times (0, 1] \to M_A$ satisfying the following properties: (1) $P([0, 1] \times \{t\})$ is contained in a leaf of $\mathcal{F}_A$ for each t. (2) $P(\{s\} \times (0, 1])$ is transverse to $\mathcal{F}_A$ for each s. (3) $P | \{0\} \times (0, 1] = \tau | (0, 1]$. (4) $P(1, t)$ lies on γ_+ for all t. The existence of P implies that the negative half of the universal cover $\tilde{\gamma}_+$ is affinely isomorphic to $\tau((0, 1])$ in an orientation preserving manner. Since $\tau((0, 1])$ extends to a larger affine curve $\tau([0, 1])$, it follows that the negative end of $\tilde{\gamma}_+$ must be an incomplete end (or, equivalently, must be bounded when embedded affinely into the standard affine line $\boldsymbol{R}$). From this, the conclusion of Lemma 2.2 follows easily.

Now turbulize $\mathcal{F}_A$ along γ_+ and γ_-. The turbulization is possible by Lemma 2.2 above and [1, Theorem 2]. Note that the direction of the turbulization is uniquely determined since we require that the resulting foliation remains transversely affine. Indeed, we must choose a direction so that whenever a transverse curve approaches the new toral leaf, so does it with an incomplete end. Denote the resulting Reeb components by N_+ and N_-, which are tubular neighborhoods of γ_+ and γ_- respectively. Put $V = M_A - \operatorname{int} N_+ - \operatorname{int} N_-$ and denote by $\mathcal{F}_V$ the resulting foliation on V (Figure 1). Then the transverse orientation of $\mathcal{F}_V$ is directed inward at ∂N_+ and outward at ∂N_-. Furthermore, all leaves of $\mathcal{F}_V$ contained in $\operatorname{int} V$ are dense in V, because so were all leaves of $\mathcal{F}_A$. Thus we can choose two points p_+ and p_- in V satisfying the following properties: (1) p_+ and p_- lie in the same leaf, say F, of $\mathcal{F}_V$. (2) p_+ (resp. p_-) is sufficiently near ∂N_+ (resp. ∂N_-). Now first connect p_+ with p_- by a curve α contained in F. Next connect p_- with a point of ∂N_- by a transverse curve β_-, and a point of ∂N_+ with p_+ by a transverse curve β_+. By usual argument, the composite curve $\beta_- * \alpha * \beta_+$ can be

We introduce a pseudo-ordering $\leq$ among the leaves of $\mathscr{F}$ as follows: For leaves L, L' of $\mathscr{F}$, we set $L \leq L'$ if the closure of L is contained in the closure of L'. Let Ω be the union of all the leaves of $\mathscr{F}$ that are maximal with respect to this pseudo-order. Since $\mathscr{F}$ is transversely real analytic, by [2], Ω is an open dense subset of M and each connected component V of Ω is of one of the following types:

(1) V fibers over the circle and the fibers are the leaves.

(2) All leaves of V are dense and without holonomy.

(3) All leaves of V are dense and some leaves are resilient.

We call these components the *maximal components*.

This section is devoted to the proof of the following two assertions, which, in combination, are equivalent to Theorem 1.2.

ASSERTION 3.1. *If $\mathscr{F}$ does not contain locally dense resilient leaves, then $\mathscr{F}$ is almost without holonomy.*

ASSERTION 3.2. *$\mathscr{F}$ is almost without holonomy if and only if the global holonomy group Γ of $\mathscr{F}$ is abelian.*

Now we fix a nonsingular vector field X on M which is transverse to $\mathscr{F}$ and compatible with the transverse orientation of $\mathscr{F}$. Let $\Phi: \boldsymbol{R} \times M \to M$ be the flow generated by X. We need the following result due to Imanishi [7].

LEMMA 3.3. *Let p be a point of M and $\gamma: [0, 1] \to M$ a curve in the leaf through p such that $\gamma(0) = p$. Define a curve $\tau: [0, 1) \to M$ by $\tau(t) = \Phi(t, p)$. Suppose that every leaf which meets τ has trivial holonomy. Then there exists a continuous map $P: [0, 1] \times [0, 1) \to M$ with the following properties:* (1) *$P(s, 0) = \gamma(s)$ for all s.* (2)*$P(0, t) = \tau(t)$ for all t.* (3) *$P(s, t)$ and $P(0, t)$ are on the same leaf for all s, t.* (4) *$P(s, t)$ and $P(s, 0)$ are on the same Φ-trajectory for all s, t.*

PROOF OF 3.1. Suppose that $\mathscr{F}$ does not contain locally dense resilient leaves. Then maximal components of type (3) do not exist and hence every maximal component of $\mathscr{F}$ is without holonomy.

LEMMA 3.4. *Let V be a maximal component of $\mathscr{F}$. Then the transverse orientation of $\hat{\mathscr{F}}$ is directed either simultaneously outward on all the border leaves of $\hat{V}$ or simultaneously inward on them.*

PROOF. Suppose that there exist border leaves L_+ and L_- of $\hat{V}$ such that the transverse orientation of $\hat{\mathscr{F}}$ is outward on L_+ and inward on L_-. Let L be a leaf in V and p, p_+, p_- be distinct points of L. Let τ, τ_+, τ_- :$[0, 1] \to \hat{V}$ be positively oriented curves transverse to $\hat{\mathscr{F}}$ having mutually disjoint images such that $\tau(0) = \tau(1) = p$, $\tau_+(0) = p_+$, $\tau_+(1) \in L_+$, $\tau_-(0) \in L_-$ and $\tau(1) = p_-$. (The existence of τ is obvious. The existence of τ_+ and τ_- follows from the fact that L has L_+ and L_- in its closure.) We may assume that the images of τ, τ_+ and τ_- are on trajectories of the transverse flow

Φ, changing Φ if necessary. Let σ_+, $\sigma_- : [0, 1] \to L$ be curves such that $\sigma_+(0)=p$, $\sigma_+(1)=p_+$, $\sigma_-(0)=p$ and $\sigma_-(1)=p_-$. As is noted at the beginning of the proof of 3.1, $\hat{\mathscr{F}} \mid V$ is without holonomy. Therefore we can apply Lemma 3.3 to σ_+ and $\tau_+ \mid [0, 1)$ and then by the same argument as that in the proof of Lemma 2.2, we see that the positive end of the universal cover $\tilde{\tau}$ of τ is incomplete. If we apply Lemma 3.3 to σ_- and $\tau_- \mid (0, 1]$, we see that the negative end of $\tilde{\tau}$ is also incomplete. Consequently $\tilde{\tau}$ is affinely equivalent to a bounded open interval. But this contradicts the fact that $\tilde{\tau}$ covers an affine circle. Lemma 3.4 is proved.

Now we will finish the proof of 3.1. Suppose $\mathscr{F}$ is *not* almost without holonomy. Then $\mathscr{F}$ has a noncompact, nonmaximal leaf, say L. Since L is noncompact, there exists a closed transversal σ which intersects L. Since the union of the maximal leaves is dense in M, there is a maximal component V which intersects σ. Since V is open, each connected component δ of the intersection of σ and V is an open proper subarc of σ. (The properness follows from the fact that L is not contained in V.) Thus δ is a transverse arc which enters and exists V. This contradicts Lemma 3.4. The proof of Assertion 3.1 is complete.

Let $\tilde{M}$ be the covering of M corresponding to the kernel of the holonomy homomorphism h and $p : \tilde{M} \to M$ the covering map. Then there is a *developing submersion* $D : \tilde{M} \to \boldsymbol{R}$ such that the lifted foliation $p^*\mathscr{F}$ is the pullback of the point foliation of $\boldsymbol{R}$ by D.

PROOF OF 3.2. The "if" part is proved by Bobo [1, Theorem 7 and Proposition 8]. We will prove the "only if" part. Suppose $\mathscr{F}$ is almost without holonomy. If $\mathscr{F}$ is without holonomy, then by [1, Theorem 8] Γ is abelian. Thus we may assume that $\mathscr{F}$ has finitely many compact leaves. Let K be the union of compact leaves of $\mathscr{F}$. We claim that $p^{-1}(K)$ is mapped by D to a single point, say 0, of $\boldsymbol{R}$. If this claim is true, then since $p^{-1}(K)$ is invariant under the action of $\pi_1(M)$, we see that 0 is a fixed point of Γ. In other words, Γ is contained in the group of affine transformations fixing 0 and is thus abelian.

Now suppose that there are points x and y of $p^{-1}(K)$ such that $D(x) \neq D(y)$. Join x with y by an arc τ in $\tilde{M}$ which is transverse to $p^{-1}(K)$. Then there must be points z and w on τ such that $D(z) \neq D(w)$ and that the subarc τ_1 of τ with endpoints z and w does not intersect $p^{-1}(K)$ except at the endpoints. Let V be the connected component of $M-K$ such that the completion $\hat{V}$ of V contains $p(\tau_1)$. By a small homotopy in $\hat{V}$, $p(\tau_1)$ is perturbed relative to the endpoints to the following form: $\tau_2 = \alpha_1 * \beta_1 * \cdots * \alpha_k * \beta_k * \alpha_{k+1}$, where each α_i is a curve transverse to $\hat{\mathscr{F}}$ and each β_i is contained in a leaf. Since $\mathscr{F}|_V$ is without holonomy, by using Lemma 3.3 repeatedly, we see that τ_2 is homotopic relative to the endopoints to $\tau_3 = \gamma * \beta * \alpha$, where α and $\gamma : [0, 1] \to \hat{V}$ are curves transverse to $\hat{\mathscr{F}}$ such that $\alpha(0)=p(z)$ and $\gamma(1)=p(w)$, and $\beta : [0, 1] \to V$ has its image in a single leaf. Since by Lemma 3.4 τ_3 is never homotopic relative to the endpoints to a transverse curve, reversing the transverse orientation if necessary, we may assume that α is negatively oriented and γ is positively oriented with respect to the transverse orientation of $\mathscr{F}$. By using Lemma 3.3 again, we obtain a continuous map $P: [0, 1]\times[0, 1)\to V$ such that the leaves of $P^*\mathscr{F}$ are of the form $[0, 1]\times\{s\}$, and that $P(t, 0)=\beta(t)$, $P(0, s)=\alpha(1-s)$ and $P(1, s)=\gamma(s)$ for all $0\leq t\leq 1$, $0\leq s<1$, reparametrizing α and γ if necessary. Now we can lift α, γ and P to maps $\tilde{\alpha}, \tilde{\gamma}: [0, 1]\to\tilde{M}$ and $\tilde{P}: [0, 1]\times[0, 1)\to\tilde{M}$ so that $\tilde{\alpha}(0)=z$, $\tilde{\gamma}(1)=w$, $\tilde{P}(0, s)=\tilde{\alpha}(1-s)$ and $\tilde{P}(1, s)=\tilde{\gamma}(s)$. Since $\tilde{P}([0, 1]\times\{s\})$ is mapped by D to a point for each $0\leq s<1$, considering the limit as s tends to 1, we have $D(z)=D(w)$. This contradicts the choice of z and w. Assertion 3.2 is proved, thereby completing the proof of Theorem 1.2.

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