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ANOTHER PROOF OF STOLL'S THEOREM FOR MOVING TARGETS

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1. Introduction. In 1929, Nevanlinna [5] asked whether his defect relation remains valid for mutually distinct meromorphic target functions $g_1, \cdots, g_q$ on C which grow more slowly than a given meromorphic function f on C, that is, the Nevanlinna characteristic functions of those functions satisfy $T_{g_j}(r)=o(T_f(r))$ as $r\to\infty$, $(j=1, \cdots, q)$. We say that g is a slow moving target function for f if $T_g(r)=o(T_f(r))$ as $r\to\infty$.

Nevanlinna [5] proved the conjecture for $q=3$. Dufresnoy [3] proved a defect relation with defect bound $d+2$ for polynomials of degree $\leqq d$ as target functions. Chuang [2] obtained a defect relation with bound $p(1-\delta_f(\infty))+1$ for slow moving target functions which span a vector space of dimension p over C. Thus Nevanlinna's conjecture is valid for an entire function f on C. In 1986, Steinmetz [8] proved Nevanlinna's conjecture with an elegant short proof. On the other hand, in higher dimension, Shiffman [6], [7] proved Nevanlinna's conjecture for a meromorphic function f on C^n if

$$(*) \qquad \operatorname{rank}\{\{f\}\cup\Phi\}=1+\operatorname{rank}\Phi\,,$$

where Φ is a set of slow moving target functions ϕ_j on C^n for f. If $n=1$, then the condition $(*)$ implies that all elements of Φ are constant. Stoll [9] and Mori [4] discussed the problem for holomorphic mappings of C^m into $P^n(C)$. Stoll [10] proved an analogous defect relation with a defect bound $n(n+1)$ for holomorphic mappings of C into $P^n(C)$, but this bound is much bigger than $n+1$ when n is large. We expect that the bound $n(n+1)$ is replaced by $n+1$. (cf. Mori [4])

In this note, we give a short proof of Stoll's theorem ([10, Theorem 6.19]).

2. Preliminaries. Let $f\colon C\to P^n(C)$ be a holomorphic mapping of C into the n-dimensional complex projective space $P^n(C)$, and $(f_0, \cdots, f_n)\colon C\to C^{n+1}-\{0\}$ a reduced representation of f. Set $\|f(z)\|^2=\sum_{i=1}^n|f_i(z)|^2$. We define the characteristic function $T_f(r)$ of f by

$$T_f(r):=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta\,.$$

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This is well defined up to addition of constants. For a meromorphic function $\phi(z)$: $\boldsymbol{C}\to\boldsymbol{C}\cup\{\infty\}$, its characteristic function $T(\phi, r)$ is defined by

$$T(\phi, r):=\frac{1}{2\pi}\int_0^{2\pi}\log^+|\phi(re^{i\theta})|d\theta+N(\phi, r),$$

where $n(\phi, t)$ is the number of poles of ϕ in $|z|<t$ counting multiplicities, $N(\phi, r)=\int_0^r n(\phi, t)dt/t$ and $\log^+x=\max(\log x, 0)$. Let $\mathfrak{G}$ be a finite set of holomorphic mappings h: $\boldsymbol{C}\to\boldsymbol{P}^n(\boldsymbol{C})^*$ with $n+2\leqq q:={}^{\#}\mathfrak{G}<\infty$. Here we say that h is a moving target. Assume that

(A1) $\mathfrak{G}$ is in general position. (cf. [10, p. 7])

This means that at least one point $z_0\in\boldsymbol{C}$ exists so that $\#\mathfrak{G}(z_0)=q$ and that $\mathfrak{G}(z_0)$ is in general position. Let $(f_0, \cdots, f_n)$ and $(h_0, \cdots, h_n)$ be reduced representations of f and h, respectively. Define $N_{f,h}(r):=N(1/F, r)$ and

$$m_{f,h}(r):=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|f(re^{i\theta})\|\cdot\|h(re^{i\theta})\|}{|F(re^{i\theta})|}d\theta\geqq 0,$$

where $F(z)=\sum_{i=0}^n f_i(z)\cdot h_i(z)\not\equiv 0$. Then it is known that

$$T_f(r)+T_h(r)=N_{f,h}(r)+m_{f,h}(r)+O(1) \qquad (r\to\infty).$$

If f or h is nonconstant, then $T_f(r)+T_h(r)\to\infty$ $(r\to\infty)$ and the defect $\delta(f, h)$ for the moving target h is defined by

$$0\leqq\delta(f, h):=\liminf_{r\to\infty}\frac{m_{f,h}(r)}{T_f(r)+T_h(r)}=1-\limsup_{r\to\infty}\frac{N_{f,h}(r)}{T_f(r)+T_h(r)}\leqq 1.$$

Assume that

(A2) $T_{g^j}(r)=o(T_f(r)) \qquad (r\to\infty), \qquad$ *for all* $g^j\in\mathfrak{G}$.

Then the moving target g^j is said to grow more slowly than f, and the defect $\delta(f, g^j)$ is written as

$$\delta(f, g^j)=\liminf_{r\to\infty}\frac{m_{f,g^j}(r)}{T_f(r)}=1-\limsup_{r\to\infty}\frac{N_{f,g^j}(r)}{T_f(r)}.$$

Let $\mathfrak{K}_{\mathfrak{G}}$ be the field generated by $\mathfrak{G}$ over $\boldsymbol{C}$, that is, the field generated by elements of the form $g_{ji}=g_i^j/g_0^j$ $(i=0, \cdots, n; j=0, \cdots, q)$ over $\boldsymbol{C}$, where $(g_0^j, \cdots, g_n^j)$ is a reduced representation of g^j. By the assumption (A2), $T_\psi(r)=o(T_f(r))$ $(r\to\infty)$ for any $\Psi\in\mathfrak{K}_{\mathfrak{G}}$. Assume that

(A3) f is linearly non-degenerate over $\mathfrak{K}_{\mathfrak{G}}$,

that is, $f_0, \cdots, f_n$ are linearly independent over $\mathfrak{K}_{\mathfrak{G}}$. Then Stoll proved the following theorem.

THEOREM ([10, Theorem 6.19]). *Assume that* (A1)~(A3) *hold. Then*

$$\sum_{g^j \in \mathfrak{G}} \delta(f, g^j) \leqq n(n+1) .$$

REMARK. The proof should be easily extend to meromorphic mappings $f: \boldsymbol{C}^m \to \boldsymbol{P}^n(\boldsymbol{C})$ by means of a result of Biancofiore-Stoll [1] or Vitter [11].

3. Proof of the theorem. We now give a short proof of this theorem.

We may assume that $g_0^j(z) \not\equiv 0$ $(j=1, \cdots, q)$ by a linear change L of $\boldsymbol{P}^n(\boldsymbol{C})^* \cong \boldsymbol{P}^n(\boldsymbol{C})$. Set $\tilde{f} = L \circ f$ and $\tilde{g}^j = \tilde{L} \circ g^j$ $(j=1, \cdots, q)$. Then it follows that $T_f(r) = T_{\tilde{f}}(r) + O(1)$, $T_{g^j}(r) = T_{\tilde{g}^j}(r) + O(1)$ and $N_{f,g^j}(r) = N_{\tilde{f},\tilde{g}^j}(r)$, so $\delta(f, g^j) = \delta(\tilde{f}, \tilde{g}^j)$ $(j=1, \cdots, q)$. It is known that

$$T(g_{ji}, r) - O(1) \leqq T_{g^j}(r) \leqq \sum_{i=0}^{n} T(g_{ji}, r) + O(1) \qquad (r \to \infty) ,$$

$(i=0, \cdots, n;\ j=1, \cdots, q)$. This yields $T(g_{ji}, r) = o(T_f(r))$ $(r \to \infty)$, $(i=0, \cdots, n;\ j=1, \cdots, q)$. Let $F_j(z) = \sum_{i=0}^{n} g_{ji}(z) \cdot f_i(z)$ $(j=1, \cdots, q)$. Then the assumption (A3) yields $F_j(z) \not\equiv 0$. Let $\mathscr{L}(s)$ be the vector space over $\boldsymbol{C}$ spamed by the set

$$\{\textstyle\prod_{\substack{1 \leqq j \leqq q \\ 0 \leqq i \leqq n}} g_{ji}^{s_{ji}} \mid s_{ji} \ \text{ are non-negative integers with } \ \sum_{\substack{1 \leqq j \leqq q \\ 0 \leqq i \leqq n}} s_{ji} = s\} ,$$

$\{b_1, \cdots, b_k\}$ be a basis of $\mathscr{L}(s)$ and $\{c_1, \cdots, c_l\}$ a basis of $\mathscr{L}(s+1)$. Then it is evident that $\mathscr{L}(s) \subset \mathscr{L}(s+1)$ and $k \leqq l$.

Set

$$J := W(b_1 f_0, \cdots, b_k f_0, c_1 f_1, \cdots, c_l f_1, \cdots, c_1 f_n, \cdots, c_l f_n)$$

and

$$J_j := W(b_1 F_j, \cdots, b_k F_j, c_1 f_1, \cdots, c_l f_1, \cdots, c_1 f_n, \cdots, c_l f_n) ,$$

where $W(h_1, \cdots, h_m)$ denotes the Wronskian determinant of $h_1, \cdots, h_m$. Then it is easy to see that $J = J_j$ $(j=1, \cdots, q)$ and $J \not\equiv 0$ by the assumption (A3). At any $z \in \boldsymbol{C}$, the F_j's may be ordered as

$$|F_1(z)| \leqq |F_2(z)| \leqq \cdots \leqq |F_q(z)| \leqq +\infty .$$

Since $\mathfrak{G}$ is in general position, we can find a function $C(z)$ independent of the arrangement of $F_1, \cdots, F_q$ so that

$$\log \|f(z)\| \leqq \log |F_j(z)| + \log |C(z)| \leqq +\infty \qquad (j=n+1, \cdots, q)$$

and

$$\int_0^{2\pi} \log^+ |C(re^{i\theta})| d\theta = o(T_f(r)) \quad (r \to \infty) .$$

Indeed, by (A1) we can write

$$f_i(z)=\sum_{j=1}^{n+1} A_{ij}(z)\cdot F_j(z)\,,$$

where $A_{ij}\in\mathfrak{R}_{\mathfrak{G}}$ are rational functions of g_{ji}'s ($i=0,\cdots,n;\ j=1,\cdots,n+1$). Hence we have

$$\|f(z)\|\leqq\sqrt{n+1}\max_{0\leqq i\leqq n}|f_i(z)|\leqq\sqrt{n+1}\left\{\sum_{i=0}^{n}\sum_{j=1}^{n+1}|A_{ij}|\right\}\cdot|F_{n+1}(z)|\,.$$

So we set

$$C(z)=\sqrt{n+1}\left\{\sum_{i,j}|A_{ij}(z)|\right\},$$

where the summation is taken over all A_{ij} corresponding to all combinations $F_{j_1},\cdots,F_{j_{n+1}}$ of $F_1,\cdots,F_q$. Then we see that

$$\int_0^{2\pi}\log^+|C(re^{i\theta})|d\theta=o(T_f(r))\quad(r\to\infty)\,.$$

Therefore we have

$$\log\frac{|F_1\cdots F_q|^k}{|J|^n}=\log|F_{n+1}\cdots F_q|^k-\log\frac{|J_1\cdots J_n|}{|F_1\cdots F_n|^k}$$

$$=\log|F_{n+1}\cdots F_q|^k-\sum_{j=1}^{n}\log\frac{|J_j|}{|F_j|^k\cdot\left|\prod_{i=1}^{n}f_i\right|^l}-nl\cdot\log\left|\prod_{i=1}^{n}f_i\right|$$

$$=\log|F_{n+1}\cdots F_q|^k-\sum_{j=1}^{n}\log|D_j|-nl\log\left|\prod_{i=1}^{n}f_i\right|,$$

where

$$|D_j|=\frac{|J_j|}{\left(|F_j|^k\left|\prod_{i=1}^{n}f_i\right|^l\right)}$$

$$=\begin{vmatrix} b_1, & \cdots, & b_k, & c_1, & \cdots, & c_l \\ \dfrac{(b_1F_j)'}{F_j}, & \cdots, & \dfrac{(b_kF_j)'}{F_j}, & \dfrac{(c_1f_1)'}{f_1}, & \cdots, & \dfrac{(c_lf_n)'}{f_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \dfrac{(b_1F_j)^{(n+k-1)}}{F_j}, & \cdots, & \dfrac{(b_kF_j)^{(n+k-1)}}{F_j}, & \dfrac{(c_1f_1)^{(n+k-1)}}{f_1}, & \cdots, & \dfrac{(c_lf_n)^{(n+k-1)}}{f_n} \end{vmatrix}$$

and

$$\int_0^{2\pi} \log^+ |D_j| d\theta = o(T_f(r)) \quad (r \to \infty) \quad //, \quad (j=1, \cdots, n).$$

The notation "//" means that the stated inequality holds outside an exceptional intervals of finite total length. Hence we have

$$\log |F_{n+1} \cdots F_q|^k = \log \frac{|F_1 \cdots F_q|^k}{|J|^n} + \sum_{j=1}^{n} \log |D_j| + nl \log \left| \prod_{i=1}^{n} f_i \right| .$$

Integrating both sides along the circle $\{z \in \boldsymbol{C} \mid |z|=r\}$, we have

$$k(q-n)T_f(r) \leqq k \sum_{j=1}^{q} N_{f,g^j}(r) + \frac{n}{2\pi} \int_0^{2\pi} \log \frac{1}{|J|} d\theta$$

$$+ \sum_{j=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |D_j| d\theta + \frac{nl}{2\pi} \int_0^{2\pi} \log \left| \prod_{i=1}^{n} f_i \right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |C| d\theta$$

$$\leqq k \sum_{j=1}^{q} N_{f,g^j}(r) + n^2 l T_f(r) + o(T_f(r)) \quad (r \to \infty) \quad // .$$

Thus we have

$$\sum_{j=1}^{q} \delta(f, g^j) \leqq n + \frac{n^2 l}{k} .$$

By Steinmentz' lemma (cf. [8, p. 138] or [10, Lemma 3.12]), lim $\inf_{s \to \infty} l/k = 1$. Therefore we obtain

$$\sum_{j=1}^{q} \delta(f, g^j) \leqq n(n+1) . \qquad \text{q.e.d.}$$

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