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A NECESSARY AND SUFFICIENT CONDITION FOR THE OSCILLATION OF HIGHER-ORDER NEUTRAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his seventieth birthday

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Abstract. Consider the higher-order neutral delay differential equation

$$(*)\qquad \frac{d^n}{dt^n}\left(x(t)+\sum_{i=1}^{L} p_i x(t-\tau_i)-\sum_{j=1}^{M} r_j x(t-\rho_j)\right)+\sum_{k=1}^{N} q_k x(t-u_k)=0\,,$$

where the coefficients and the delays are nonnegative constants with $n\geqslant 1$ odd. Then a necessary and sufficient condition for the oscillation of $(*)$ is that the characteristic equation

$$F(\lambda):=\lambda^n+\lambda^n\sum_{i=1}^{L} p_i e^{-\lambda\tau_i}-\lambda^n\sum_{j=1}^{M} r_j e^{-\lambda\rho_j}+\sum_{k=1}^{N} q_k e^{-\lambda u_k}=0$$

has no real roots.

1. Introduction. Neutral delay differential equations are differential equations in which the highest order derivative of the unknown function appears both with and without delays. The problem of oscillations of neutral equations is of both theoretical and practical interest. For example, the equations of this type appear in networks containing lossless transmission (see [3], [10]). The oscillation theory of neutral equations has been extensively developed during the past few years (see [4]–[10]).

In this paper, we consider the oscillation of higher-order neutral delay differential equations

$$(1.1)\qquad \frac{d^n}{dt^n}\left(x(t)+\sum_{i=1}^{L} p_i x(t-\tau_i)-\sum_{j=1}^{M} r_j x(t-\rho_j)\right)+\sum_{k=1}^{N} q_k x(t-u_k)=0\,,$$

where the coefficients and the delays are nonnegative constants with $n\geqslant 1$ odd.

Let $\phi\in C([t_0-T, t_0], R)$, where $T=\max\{\tau_i, \rho_j, u_k\colon 1\leqslant i\leqslant L,\ 1\leqslant j\leqslant M,\ 1\leqslant k\leqslant N\}$. By a solution of (1.1) with initial function ϕ at t_0, we mean a function $x\in C([t_0-T, \infty), R)$ such that $x(t)=\phi(t)$ for $t_0-T\leqslant t\leqslant t_0$, $x(t)+\sum_{i=1}^{L} p_i x(t-\tau_i)-\sum_{j=1}^{M} r_j x(t-\rho_j)$ is n-times continuously differentiable, and x satisfies (1.1) for all $t\geqslant t_0$. Using the method of steps, it follows that for every continuous function ϕ, there is a unique solution of (1.1) vaild for $t\geqslant t_0$. For further questions on existence, uniqueness

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and continuous dependence, see Bellman and Cooke [1], Driver [2] and Hale [3].

As is customary, a solution is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. The characteristic equation of (1.1) is

$$F(\lambda):=\lambda^n+\lambda^n\sum_{i=1}^{L}p_ie^{-\lambda\tau_i}-\lambda^n\sum_{j=1}^{M}r_je^{-\lambda\rho_j}+\sum_{k=1}^{N}q_ke^{-\lambda u_k}=0\,. \tag{1.2}$$

Our aim is to give a necessary and sufficient condition for all solutions of (1.1) to be oscillatory. We have:

THEOREM. *All solutions of* (1.1) *oscillate if and only if the characteristic equation* (1.2) *has no real roots.*

The proof of this theorem will be given in Section 3.

For the case $n=1$, the above result was proved recently by Grammatikopoulos, Sficas and Stavroulakis [5] (see also [8], [9]). For n even, the above result can be proved by similar arguments and is omitted.

2. Lemmas. In this section we establish some useful lemmas which will be used in the proof of our main theorem.

In (1.1), without loss of generality we assume that $0<\tau_1<\tau_2<\cdots<\tau_L$, $0<\rho_1<\rho_2<\cdots<\rho_M$, $\tau_i\neq\rho_j$ $(i=1,2,\cdots,L;\,j=1,2,\cdots,M)$, and $0\leqslant u_1<u_2<\cdots<u_N$. Let $P=\sum_{i=1}^{L}P_i$, $R=\sum_{j=1}^{M}r_j$ and $Q=\sum_{k=1}^{N}q_k$.

LEMMA 1. *If $x(t)$ is a solution of* (1.1), *then each one of the following functions*

$$x(t-a),\quad \int_{t-a}^{t-b}x(u)du,\quad \dot{x}(t)\quad (\textit{if } x(t) \textit{ is continuously differentiable})$$

is also a solution of (1.1), *where a and b are real numbers.*

The proof is trivial and is omitted.

LEMMA 2. *If* (1.2) *has no real roots, then we have*

$$Q>0\qquad \textit{with}\quad \tau_L<\max\{\rho_M,u_N\}\,. \tag{2.1}$$

The proof is trivial and is omitted.

LEMMA 3. *Assume that there is a nonoscillatory solution of* (1.1). *Then there is a nonoscillatory solution $w(t)$ of* (1.1) *such that either*

$$w(t)\in(\mathrm{I}):=\left\{w(t)\in C^{2n}([T^*,\infty),R)\colon(-1)^kw^{(k)}(t)>0,\ \lim_{t\to\infty}w^{(k)}(t)=0,\ k=0,1,2,\cdots,n\right\}$$

or

$$w(t)\in(\mathrm{II}):=\left\{w(t)\in C^{2n}([T^*,\infty),R)\colon w^{(k)}(t)>0,\ \lim_{t\to\infty}w^{(k)}(t)=\infty,\ k=0,1,2,\cdots,n\right\},$$

where $T^*\geqslant t_0$ *is sufficiently large.*

PROOF. As the negative of a solution of (1.1) is also a solution of the same equation, it suffices to consider that $x(t)$ is an eventually positve solution of (1.1). Set

$$z(t)=x(t)+\sum_{i=1}^{L}p_ix(t-\tau_i)-\sum_{j=1}^{M}r_jx(t-\rho_j)\,, \tag{2.2}$$

and

$$w(t)=z(t)+\sum_{i=1}^{L}p_iz(t-\tau_i)-\sum_{j=1}^{M}r_jz(t-\rho_j)\,. \tag{2.3}$$

By Lemma 1, $z(t)$ and $w(t)$ are solutions of (1.1). Then we have

$$z^{(n)}(t)=-\sum_{k=1}^{N}q_kx(t-u_k)<0\,, \tag{2.4}$$

$$w^{(n)}(t)=-\sum_{k=1}^{N}q_kz(t-u_k)\,, \tag{2.5}$$

and so $z^{(n-1)}(t)$ is eventually strictly decreasing. Also all the derivatives of z of order less than or equal to $n-1$ are monotonic functions. From (2.4) it follows that either

$$\lim_{t\to\infty}z^{(n-1)}(t)=-\infty \tag{2.6}$$

or

$$\lim_{t\to\infty}z^{(n-1)}(t)=L^* \tag{2.7}$$

is finite.

If (2.6) holds, then

$$\lim_{t\to\infty}z^{(k)}(t)=-\infty\,;\qquad k=0,1,2,\cdots,n-1\,,$$

which imply

$$\lim_{t\to\infty}w^{(n)}(t)=\infty\,,$$

and so

$$\lim_{t\to\infty}w^{(k)}(t)=\infty\,,\quad w^{(k)}(t)>0\quad \text{eventually},\ k=0,1,2,\cdots,n\,.$$

Obviously, $z(t)\in C^n([t_0+T, \infty), R)$ and $w(t)\in C^{2n}([t_0+2T, \infty), R)$. That is, $w(t)\in$ (II).

If (2.7) holds, then integrating (2.7) from t_1 to t, with t_1 sufficiently large, and letting $t\to\infty$ we find

$$L^*-z^{(n-1)}(t_1)=-\sum_{k=1}^{N} q_k \int_{t_1}^{\infty} x(s-u_k)ds$$

which implies that $x\in L^1[t_1, \infty)$. Thus, from (2.2), $z\in L^1[t_1, \infty)$ and since z is monotonic, it follows that

$$\lim_{t\to\infty} z(t)=0\,, \tag{2.8}$$

and so $L=0$. As the function $z^{(n-1)}(t)$ decreases to zero, it follows that

$$z^{(n-1)}(t)>0\,. \tag{2.9}$$

Also (2.8) implies that consecutive derivatives of z must alternate sign and tends to zero as $t\to\infty$. Thus, in view of (2.9) and the fact that n is odd, we have

$$z(t)>0\,. \tag{2.10}$$

Using the same arguments as to $w(t)$ we have

$$(-1)^k w^{(k)}(t)>0\,,\quad \lim_{t\to\infty} w^{(k)}(t)=0, \quad k=0, 1, 2, \cdots, n\,.$$

Thus, $w(t)\in$ (I) and the proof is completed.

LEMMA 4. *Assume that* (2.1) *holds, and that there is a nonoscillatory solution* $x(t)$ *of* (1.1). *Then*

(i) *if* $x(t)\in$ (I), *then there is a solution* $z(t)$ *of* (1.1) *which belongs to Class* (I), *such that*

$$\Lambda^+(z):=\{\lambda>0;\ z^{(n)}(t)+\lambda^n z(t)\leqslant 0\}\neq\varnothing\,;$$

(ii) *if* $x(t)\in$ (II), *then there is a solution* $z(t)$ *of* (1.1) *which belongs to Class* (II), *such that the set*

$$\Lambda^-(z):=\{\lambda>0;\ -z^{(n)}(t)+\lambda^n(t)\leqslant 0\}\neq\varnothing\,.$$

PROOF. (i) Let $x(t)\in$ (I). Set

$$z(t)=x(t)+\sum_{i=1}^{L} p_i x(t-\tau_i)-\sum_{j=1}^{M} r_j x(t-\rho_j)\,. \tag{2.11}$$

It is easy to see that $z(t)\in$ (I).

We consider the following two cases:

Case 1. $u_N\geqslant\tau_L$. As $x(t)$ is positive and decreasing, it follows from (2.11) that

(2.12) $$z(t) < x(t) + \sum_{i=1}^{L} p_i x(t-\tau_i) < (1+P)x(t-u_N).$$

On the other hand,

(2.13) $$z^{(n)}(t) = -\sum_{k=1}^{N} q_k x(t-u_k) \leqslant -q_N x(t-u_N).$$

Combining (2.12) and (2.13) we obtain $z^{(n)}(t) + (q_N/(1+P))z(t) \leqslant 0$. Thus $\lambda = (q_n/(1+P))^{1/n} \in \Lambda^+(z)$.

Case 2. $\rho_M > \tau_L > u_N$. From (2.11) we obtain

(2.14) $$(1+P)x(t-\tau_L) > z(t)$$

and, as $z(t) > 0$, it follows that $(1+P)x(t-\tau_L) > r_M x(t-\rho_M)$, that is,

(2.15) $$x(t+(\rho_M-\tau_L)) > (r_M/(1+P))x(t).$$

Let k be the first positive integer such that $\tau_L - u_N \leqslant k(\rho_M - \tau_L)$. Then by (2.13) we have

$$\begin{aligned} 0 &\geqslant z^{(n)}(t) + q_N x(t-u_N) = z^{(n)}(t) + q_N x(t-\tau_L+(\tau_L-u_N)) \\ &\geqslant z^{(n)}(t) + q_N x(t-\tau_L+k(\rho_M-\tau_L)) \\ &\geqslant z^{(n)}(t) + q_N (r_M/(1+P))^k x(t-\tau_L) \qquad \text{(by (2.15))} \\ &\geqslant z^{(n)}(t) + (q_N r_M^k/(1+P)^{k+1}) z(t). \qquad \text{(by (2.14))} \end{aligned}$$

Thus $\lambda = (q_N r_M^k/(1+P)^{k+1})^{1/n} \in \Lambda^+(z)$. The proof of Part (i) is completed.

(ii) Let $x(t) \in \text{(II)}$. Set

$$z(t) = -x(t) - \sum_{i=1}^{L} p_i x(t-\tau_i) + \sum_{j=1}^{M} r_j x(t-\rho_j).$$

It is easy to see that $z(t) \in \text{(II)}$. As $x(t)$ and $z(t)$ are positive and increasing, it follows that

(2.16) $$z(t) < \sum_{j=1}^{M} r_j x(t-\rho_j) < Rx(t-\rho_1)$$

and

(2.17) $$x(t) < \sum_{j=1}^{M} r_j x(t-\rho_j) < Rx(t-\rho_1).$$

Let k be the first positive integer such that $u_1 \leqslant k\rho_1$. Then

$$\begin{aligned}0&=-z^{(n)}(t)+\sum_{k=1}^{N}q_kx(t-u_k)\\&\geqslant -z^{(n)}(t)+q_1x(t-u_1)\geqslant -z^{(n)}(t)+q_1x(t-k\rho_1)\\&\geqslant -z^{(n)}(t)+(q_1/R^{k-1})x(t-\rho_1)\qquad\text{(by (2.17))}\\&\geqslant -z^{(n)}(t)+(q_1/R^k)z(t).\qquad\text{(by (2.16))}\end{aligned}$$

Thus, $\lambda=(q_1/R^k)^{1/n}\in\Lambda^-(z)$, and so $\Lambda^-(z)\neq\varnothing$. The proof of the lemma is completed.

LEMMA 5. *Assume that* (2.1) *holds, and that there is a nonoscillatory solution* $x(t)$ *of* (1.1). *Then*

(i) *if* $x(t)\in(\mathrm{I})$ *with* $\Lambda^+(x)\neq\varnothing$, *then the set* $\Lambda^+(x)$ *has an upper bound which is independent of* x;

(ii) *if* $x(t)\in(\mathrm{II})$ *with* $\Lambda^-(x)\neq\varnothing$, *then the set* $\Lambda^-(x)$ *has an upper bound which is independent of* x.

PROOF. (i) Let $x(t)\in(\mathrm{I})$ with $\Lambda^+(x)\neq\varnothing$. Set

$$z(t)=x(t)+\sum_{i=1}^{L}p_ix(t-\tau_i)-\sum_{j=1}^{M}r_j(t-\rho_j).$$

It is easy to see that $z(t)\in(\mathrm{I})$, and so

(2.18) $\qquad (-1)^kz^{(k)}(t)$ is positive and decreasing for $k=0,1,2,\cdots,n$.

We consider the following cases:

Case 1. $\rho_M>\tau_L$. From (2.18) we have

$$(-1)^kx^{(k)}(t)+\sum_{i=1}^{L}p_i(-1)^kx^{(k)}(t-\tau_i)-\sum_{j=1}^{M}r_j(-1)^kx^{(k)}(t-\rho_j)>0,$$

which implies

$$(-1)^kx^{(k)}(t)>(r_M/(1+P))(-1)^kx^{(k)}(t-(\rho_M-\tau_L))$$

for $k=0,1,2,\cdots,n$. Let $M^*=r_M/(1+P)$, $w=\rho_M-\tau_L>0$. Then

(2.19) $$(-1)^kx^{(k)}(t)>M^*(-1)^kx^{(k)}(t-w),\quad k=0,1,2,\cdots,n.$$

Now, we want to prove that $\lambda_0=-(1/w)\ln M^*\notin\Lambda^+(x)$. Otherwise, $\lambda_0\in\Lambda^+(x)$ which means that

$$x^{(n)}(t)+\lambda_0^nx(t)\leqslant 0.$$

Set

$$y(t)=x^{(n-1)}(t)-\lambda_0x^{(n-2)}(t)+\cdots+\lambda_0^{n-1}x(t).$$

Then

$$\dot{y}(t)+\lambda_0 y(t)=x^{(n)}(t)+\lambda_0^n x(t)\leqslant 0\,,$$

and by (2.19) we have

$$y(t)>M^*y(t-w)\,. \tag{2.20}$$

Let $\phi(t)=e^{\lambda_0 t}y(t)$. Then

$$\dot{\phi}(t)=e^{\lambda_0 t}(\dot{y}(t)+\lambda_0 y(t))\leqslant 0$$

and so $\phi(t)$ is non-increasing. Thus, $\phi(t)\leqslant\phi(t-w)$ which implies

$$y(t)\leqslant e^{-\lambda_0 w}y(t-w)=M^*y(t-w)\,.$$

It contradicts (2.20).

Case 2. $u_N>\tau_L$. For $k=0, 1, 2, \cdots, n$,

$$(-1)^k z^{(k)}(t)=(-1)^k x^{(k)}(t)+\sum_{i=1}^{L} p_i(-1)^k x^{(k)}(t-\tau_i)-\sum_{j=1}^{M} r_j(-1)^k x^{(k)}(t-\rho_j)$$

which implies

$$(-1)^k z^{(k)}(t)<(1+P)(-1)^k x^{(k)}(t-\tau_L)\,. \tag{2.21}$$

On the other hand, for $k=0, 1, 2, \cdots, n$,

$$(-1)^{n+k-1}z^{(n+k)}(t)=-\sum_{k=1}^{N} q_k(-1)^k x^{(k)}(t-u_k)$$

which implies

$$(-1)^{n+k-1}z^{(n+k)}(t)\leqslant -q_N(-1)^k x^{(k)}(t-u_N)\,. \tag{2.22}$$

For $u_n>\tau_L$, there is a $b>0$ such that $u_N>\tau_L+nb$. Integrating (2.22) from t to $t+b$, we obtain

$$(-1)^{n+k-1}(z^{(n+k-1)}(t+b)-z^{(n+k-1)}(t))\leqslant -q_N b(-1)^k x^{(k)}(t-(u_N-b))\,.$$

Noting that $(-1)^{n+k-1}z^{(n+k-1)}(t+b)>0$, We have

$$(-1)^{n+k-2}z^{(n+k-1)}(t)<-q_N b(-1)^k x^{(k)}(t-(u_N-b))\,,$$

and then after n steps we obtain

$$(-1)^{k-1}z^{(k)}(t)<-q_N b^n(-1)^k x^{(k)}(t-(u_N-nb))\,,$$

that is,

$$(-1)^k z^{(k)}(t)\geqslant q_N b^n(-1)^k x^{(k)}(t-(u_N-nb))\,. \tag{2.23}$$

Combining (2.21) and (2.23), we obtain

$$(1+P)(-1)^k x^{(k)}(t-\tau_L) > q_N b^n(-1)^k x^{(k)}(t-(u_N-nb)),$$

that is,

$$(-1)^k x^{(k)}(t) > M^*(-1)^k x^{(k)}(t-w^*),$$

where $M^*=q_N b^n/(1+P)$, $w^*=u_N-\tau_L-nb>0$. Then, as in Case 1, we can show that $\lambda_0=-(1/w^*)\ln M^*$ is an upper bound of $\Lambda^+(x)$.

(ii) Let $x(t)\in(\text{II})$ with $\Lambda^-(x)\neq\varnothing$. Set

$$z(t)=-x(t)-\sum_{i=1}^{L} p_i x(t-\tau_i)+\sum_{j=1}^{M} r_j x(t-\rho_j). \tag{2.24}$$

It is easy to see that $z(t)\in(\text{II})$ and, for $k=0, 1, 2, \cdots, n$, $x^{(k)}(t)$ and $z^{(k)}(t)$ are positive and increasing. It follows that

$$x^{(k)}(t)<\sum_{j=1}^{M} r_j x^{(k)}(t-\rho_j)\leqslant Rx^{(k)}(t-\rho_1) \tag{2.25}$$

for $k=0, 1, 2, \cdots, n$. We now want to show that $\lambda_0=(1/\rho_1)\ln R$ is an upper bound of $\Lambda^-(x)$. Otherwise, $\lambda_0\in\Lambda^-(x)$ which means that

$$-x^{(n)}(t)+\lambda_0^n x(t)\leqslant 0. \tag{2.26}$$

Set $y(t)=x^{(n-1)}(t)+\lambda_0 x^{(n-2)}(t)+\cdots+\lambda_0^{n-1}x(t)$. Then, from (2.25),

$$y(t)<Ry(t-\rho_1), \tag{2.27}$$

and, from (2.26),

$$\dot{y}(t)-\lambda_0 y(t)=x^{(n)}(t)-\lambda_0^n x(t)\geqslant 0.$$

Let $\phi(t)=e^{-\lambda_0 t}y(t)$. Then

$$\dot{\phi}(t)=e^{-\lambda_0 t}(\dot{y}(t)-\lambda_0 y(t))\geqslant 0,$$

and so $\phi(t)$ is non-decreasing. Thus, $\phi(t)\geqslant\phi(t-\rho_1)$ which implies

$$y(t)\geqslant e^{\lambda_0\rho_1}y(t-\rho_1)=Ry(t-\rho_1).$$

This contradicts (2.27) and the proof is completed.

3. **Main result.** Our main result is the following:

THEOREM. *All solutions of* (1.1) *oscillate if and only if the characteristic equation* (1.2) *has no real roots.*

PROOF. Assume first that (1.2) has a real root λ. Then, obviously, (1.1) has a nonoscillatory solution $x(t)=e^{\lambda t}$.

Assume, conversely, that there is a nonoscillatory solution of (1.1). Then we want to prove that (1.2) has a real root. Otherwise, assume that (1.2) has no real roots. Then, by Lemma 2, (2.1) holds. Thus, $F(\infty)=F(-\infty)=\infty$, and $\alpha:=\min_{\lambda\in R}F(\lambda)>0$. This implies

$$-\lambda^n-\lambda^n\sum_{i=1}^{L}p_ie^{-\lambda\tau_i}+\lambda^n\sum_{j=1}^{M}r_je^{-\lambda\rho_j}-\sum_{k=1}^{N}q_ke^{-\lambda u_k}\leqslant-\alpha \tag{3.1}$$

or

$$\lambda^n+\lambda^n\sum_{i=1}^{L}p_ie^{\lambda\tau_i}-\lambda^n\sum_{j=1}^{M}r_je^{\lambda\rho_j}-\sum_{k=1}^{N}q_ke^{\lambda u_k}\leqslant-\alpha\,. \tag{3.2}$$

By Lemmas 3 and 4, if (1.1) has a nonoscillatory solution, then (1.1) has a nonoscillatory solution $x(t)$ which belongs to Class (I) with $\Lambda^+(x)\neq\varnothing$ or to Class (II) with $\Lambda^-(x)\neq\varnothing$.

We consider the following two cases:

Case 1. $x(t)\in(\mathrm{I})$ with $\Lambda^+(x)\neq\varnothing$. Let $\lambda\in\Lambda^+(x)$. By Lemma 5(i) there is a $\lambda_0>0$ such that λ_0 is an upper bound of $\Lambda^+(x)$ which is independent of x. Let $T=\max\{\rho_M, u_N\}$, and set

$$y(t)=T_1x(t):=x(t)+\sum_{i=1}^{L}p_ix(t-\tau_i)-\sum_{j=1}^{M}r_jx(t-\rho_j)\,. \tag{3.3}$$

Obviously, $y(t)$ is a solution of (1.1) and $y(t)\in(\mathrm{I})$. Then

$$y^{(n)}(t)=-\sum_{k=1}^{N}q_kx(t-u_k)\,. \tag{3.4}$$

Set

$$z(t)=T_2y(t):=-y^{(n)}(t)+\lambda y^{(n-1)}(t)-\lambda^2y^{(n-2)}(t)+\cdots-\lambda^{n-1}\dot{y}(t)\,. \tag{3.5}$$

It is easy to see that $z(t)$ is also a solution of (1.1) with $z(t)\in(\mathrm{I})$, and so $z(t)$ can be expressed as

$$z(t)=\int_t^\infty dt_1\int_{t_1}^\infty dt_2\cdots\int_{t_{n-2}}^\infty z^{(n-1)}(s)ds\,. \tag{3.6}$$

From (3.4) and (3.5), we have

$$z^{(n)}(t)+\lambda z^{(n-1)}(t)=-y^{(2n)}(t)-\lambda^ny^{(n)}(t)=\sum_{k=1}^{N}q_k(x^{(n)}(t-u_k)+\lambda^nx(t-u_k))\leqslant0$$

for t sufficiently large since $\lambda\in\Lambda^+(x)$. Let $\phi(t)=e^{\lambda t}z^{(n-1)}(t)$. Then

$$\dot{\phi}(t)=e^{\lambda t}(z^{(n)}(t)+\lambda z^{(n-1)}(t))\leqslant0\,,$$

and so $\phi(t)$ is decreasing. Set

$$w(t)=T_3z(t):=z^{(n-1)}(t)+\sum_{i=1}^{L}p_iz^{(n-1)}(t-\tau_i)-\sum_{j=1}^{M}r_jz^{(n-1)}(t-\rho_j)\tag{3.7}$$

$$+\sum_{k=1}^{N}q_k\int_{t-T}^{t-u_k}z(u)du+\lambda^n\sum_{j=1}^{M}r_j\int_{t-T}^{t-\rho_j}z(u)du\,.$$

By Lemma 1, $w(t)$ is a solution of (1.1). Then

$$\dot{w}(t)=-\sum_{k=1}^{N}q_kz(t-T)-\lambda^n\sum_{j=1}^{M}r_j(z(t-T)-z(t-\rho_j))\,,$$

$$w^{(k)}(t)=-\sum_{k=1}^{N}q_kz^{(k-1)}(t-T)-\lambda^n\sum_{j=1}^{M}r_j(z^{(k-1)}(t-T)-z^{(k-1)}(t-\rho_j))\,,\quad k=2,\cdots,n\,.$$

As $z(t)\in(\mathrm{I})$, it follows that $w(t)\in(\mathrm{I})$. Let

$$\mu=\alpha/(1+P+Q/\lambda^n+R)e^{\lambda T}\,.\tag{3.8}$$

We now show that

$$w^{(n)}(t)+(\lambda^n+\mu)w(t)\leqslant 0\,.\tag{3.9}$$

Indeed,

$$\begin{aligned}
&w^{(n)}(t)+(\lambda^n+\mu)w(t)\\
&=-\sum_{k=1}^{N}q_kz^{(n-1)}(t-T)-\lambda^n\sum_{j=1}^{M}r_j(z^{(n-1)}(t-T)-z^{(n-1)}(t-\rho_j))\\
&\quad+(\lambda^n+\mu)\left(z^{(n-1)}(t)+\sum_{i=1}^{L}p_iz^{(n-1)}(t-\tau_i)-\sum_{j=1}^{M}r_jz^{(n-1)}(t-\rho_j)\right.\\
&\quad\left.+\sum_{k=1}^{N}q_k\int_{t-T}^{t-u_k}z(u)du+\lambda^n\sum_{j=1}^{M}r_j\int_{t-T}^{t-\rho_j}z(u)du\right)\\
&=\lambda^nz^{(n-1)}(t)+\lambda^n\sum_{i=1}^{L}p_iz^{(n-1)}(t-\tau_i)+\lambda^n\sum_{j=1}^{M}r_j\left(-z^{(n-1)}(t-T)+\lambda^n\int_{t-T}^{t-\rho_j}z(u)du\right)\\
&\quad+\sum_{k=1}^{N}q_k\left(-z^{(n-1)}(t-T)+\lambda^n\int_{t-T}^{t-u_k}z(u)du\right)+\left(z^{(n-1)}(t)+\sum_{i=1}^{L}p_iz^{(n-1)}(t-\tau_i)\right.\\
&\quad\left.-\sum_{j=1}^{M}r_jz^{(n-1)}(t-\rho_j)+\sum_{k=1}^{N}q_k\int_{t-T}^{t-u_k}z(u)du+\lambda^n\sum_{j=1}^{M}r_j\int_{t-T}^{t-\rho_j}z(u)du\right).
\end{aligned}$$

Noting that $z^{(n-1)}(t)=\phi(t)e^{-\lambda t}$ with $\phi(t)$ decreasing, we obtain from (3.6),

$$\lambda^n \sum_{k=1}^{N} q_k \int_{t-T}^{t-u_k} z(u)du = \sum_{k=1}^{N} q_k \lambda^n \int_{t-T}^{t-u_k} du \int_u^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{n-2}}^{\infty} z^{(n-1)}(v)dv$$

$$\leqslant \sum_{k=1}^{N} q_k \phi(t-T)\lambda^n \int_{t-T}^{t-u_k} du \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{n-2}}^{\infty} e^{-\lambda v}dv \leqslant \sum_{k=1}^{N} q_k \phi(t-T)(e^{-\lambda(t-T)} - e^{-\lambda(t-u_k)}) .$$

Similarly,

$$\sum_{j=1}^{M} r_j \lambda^n \int_{t-T}^{t-\rho_j} z(u)du \leqslant \sum_{j=1}^{M} r_j \phi(t-T)(e^{-\lambda(t-T)} - e^{-\lambda(t-\rho_j)}) .$$

Then we have

$$w^{(n)}(t) + (\lambda^n + \mu)w(t) \leqslant e^{-\lambda t}\phi(t-T)\left(\left(\lambda^n \sum_{i=1}^{L} p_i e^{\lambda \tau_i} - \lambda^n \sum_{j=1}^{M} r_j e^{\lambda \rho_j} - \sum_{k=1}^{N} q_k e^{\lambda u_k} + \lambda^n\right)\right.$$

$$\left. + \mu(1 + P + Q/\lambda^n + R)e^{\lambda T}\right) \leqslant e^{-\lambda t}\phi(t-T)(-\alpha + \alpha) \qquad \text{(by (3.2) and (3.8))}$$

$$= 0 .$$

Thus, (3.9) holds which means that $(\lambda^n + \mu)^{1/n} \in \Lambda^+(w)$.

Set

$$w(t) = Ux(t) := T_3(T_2(T_1 x(t))) ,$$

and set $x_0 = x$, $x_1 = Ux_0$, and in general,

$$x_k = Ux_{k-1} , \qquad k = 1, 2, \cdots .$$

We observe tht $x_k(t) \in \text{(I)}$ with $\Lambda^+(x_k) \neq \varnothing$, and that $\lambda \in \Lambda^+(x) = \Lambda^+(x_0)$ implies $(\lambda^n + \mu)^{1/n} \in \Lambda^+(w) = \Lambda^+(x_1)$ and after k steps we obtain

$$(\lambda^n + k\mu)^{1/n} \in \Lambda^+(x_k) , \qquad k = 1, 2, \cdots$$

which is a contradiction since λ_0 is a common bound for all $\Lambda^+(x_k)$.

Case 2. $x(t) \in \text{(II)}$ with $\Lambda^-(x) \neq \varnothing$. Let $\lambda \in \Lambda^-(x)$. By Lemma 5 (ii), there is an upper bound λ_0 of $\Lambda^-(x)$ which is independent of x. Let $b = \min\{\tau_1, \rho_1, u_1\}$, and set

$$y(t) = T_1 x(t) := -x(t) - \sum_{i=1}^{L} p_i x(t-\tau_i) + \sum_{j=1}^{M} r_j x(t-\rho_j) , \tag{3.10}$$

$$z(t) = T_2 y(t) := y^{(n)}(t) + \lambda y^{(n-1)}(t) + \cdots + \lambda^{n-1}\dot{y}(t) . \tag{3.11}$$

It is easy to see that $y(t)$ and $z(t)$ are solutions of (1.1), and that $y(t) \in \text{(II)}$, $z(t) \in \text{(II)}$. From (3.10), (3.11),

$$z^{(n)}(t) - \lambda z^{(n-1)}(t) = y^{(2n)}(t) - \lambda^n y^{(n)}(t) = \sum_{k=1}^{N} q_k(x^{(n)}(t-u_k) - \lambda^n x(t-u_k)) \geqslant 0$$

since $\lambda \in \Lambda^-(x)$. Let $\phi(t)=e^{-\lambda t}z^{(n-1)}(t)$. Then

$$\dot{\phi}(t)=e^{-\lambda t}(z^{(n)}(t)-\lambda z^{(n-1)}(t))\geqslant 0 ,$$

and so $\phi(t)$ is increasing. Set

$$w(t)=T_3 z(t):=-z^{(n-1)}(t)-\sum_{i=1}^{L} p_i z^{(n-1)}(t-\tau_i)+\sum_{j=1}^{M} r_j z^{(n-1)}(t-\rho_j)$$
$$+\sum_{k=1}^{N} q_k \int_{t-u_k}^{t-b} z(u)du+\sum_{i=1}^{L} p_i \lambda^n \int_{t-\tau_i}^{t-b} z(u)du . \tag{3.12}$$

Then

$$\dot{w}(t)=\sum_{k=1}^{N} q_k z(t-b)+\lambda^n \sum_{i=1}^{L} p_i(z(t-b)-z(t-\tau_i)) .$$

As $z(t)\in$(II), it follows that

$$w^{(k)}(t)=\sum_{k=1}^{N} q_k z^{(k-1)}(t-b)+\lambda^n \sum_{i=1}^{L} p_i(z^{(k-1)}(t-b)-z^{(k-1)}(t-\tau_i))\rightarrow\infty$$

as $t\rightarrow\infty$ for $k=1, 2, \cdots, n$. Obviously, $w(t)\rightarrow\infty$ as $t\rightarrow\infty$. Thus, $w(t)\in$(II). Let

$$\mu=\alpha/(P+Q/\lambda^n+R) . \tag{3.13}$$

We now show that

$$-w^{(n)}(t)+(\lambda^n+\mu)w(t)\leqslant 0 . \tag{3.14}$$

Indeed,

$$-w^{(n)}(t)+(\lambda^n+\mu)w(t)=-\lambda^n z^{(n-1)}(t)+\sum_{i=1}^{L}\lambda^n p_i\left(-z^{(n-1)}(t-b)+\lambda^n\int_{t-\tau_i}^{t-b} z(u)du\right)$$
$$+\sum_{j=1}^{M}\lambda^n r_j z^{(n-1)}(t-\rho_j)+\sum_{k=1}^{N} q_k\left(-z^{(n-1)}(t-b)+\lambda^n\int_{t-u_k}^{t-b} z(u)du\right)$$
$$+\mu\left(-z^{(n-1)}(t)-\sum_{i=1}^{L} p_i z^{(n-1)}(t-\tau_i)+\sum_{j=1}^{M} r_j z^{(n-1)}(t-\rho_j)\right.$$
$$\left.+\sum_{k=1}^{N} q_k\int_{t-u_k}^{t-b} z(u)du+\sum_{i=1}^{L}\lambda^n p_i\int_{t-\tau_i}^{t-b} z(u)du\right).$$

For $z(t)\in$(II), there is a t^* such that for $t\geqslant t^*$

$$z^{(k)}(t)>0 , \qquad k=0, 1, 2, \cdots, n .$$

Then it is possible to extend the definition of $z^{(k)}(t)$ to $t<t^*$ such that for $k=0, 1, 2, \cdots, n$, $z^{(k)}(t)$ is continuous and increasing on $(-\infty, \infty)$ and $z^{(k)}(t)\rightarrow 0$ as

$t \to -\infty$. Then $z(t)$ can be expressed as

$$z(t) = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-2}} z^{(n-1)}(s)ds . \tag{3.15}$$

Noting that $z^{(n-1)}(t) = \phi(t)e^{\lambda t}$ with $\phi(t)$ increasing, we have by (3.15)

$$\begin{aligned}\int_{t-u_k}^{t-b} z(u)du &\leqslant \phi(t-b) \int_{t-u_k}^{t-b} du \int_{-\infty}^{u} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-2}} e^{\lambda s} ds \\ &= \phi(t-b)(e^{\lambda(t-b)} - e^{\lambda(t-u_k)})/\lambda^n .\end{aligned}$$

Similarly,

$$\int_{t-\tau_i}^{t-b} z(u)du \leqslant \phi(t-b)(e^{\lambda(t-b)} - e^{\lambda(t-\tau_i)})/\lambda^n .$$

Then

$$\begin{aligned}&-w^{(n)}(t) + (\lambda^n + \mu)w(t) \\ &\leqslant \phi(t-b)e^{\lambda t}\left(\left(-\lambda^n - \sum_{i=1}^{L} p_i \lambda^n e^{-\lambda\tau_i} + \sum_{j=1}^{M} r_j \lambda^n e^{-\lambda\tau_i} - \sum_{k=1}^{N} q_k e^{-\lambda u_k}\right) + \mu(P + Q/\lambda^n + R)\right) \\ &\leqslant \phi(t-b)e^{\lambda t}(-\alpha + \alpha) = 0 . \qquad \text{(by (3.1) and (3.13))}\end{aligned}$$

It follows that (3.14) holds which means that $(\lambda^n + \mu)^{1/n} \in \Lambda^-(w)$.

Set $w(t) = Ux(t) := T_3(T_2(T_1 x(t)))$, and let $x_0 = x$, $x_1 = Ux_0$, and in general,

$$x_k = Ux_{k-1} , \qquad k = 1, 2, \cdots$$

and, as in Case 1, we are led to a contradiction. This proves the theorem.

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