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CERTAIN ASPECTS OF TWISTED LINEAR ACTIONS, II

Dedicated to Professor Akio Hattori on his 60th birthday

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0. Introduction. In the previous papers [1], [2], we have introduced the concept of a twisted linear action which is an analytic action of a non-compact Lie group on a sphere.

We have shown that there are uncountably many topologically distinct analytic actions of $SL(n, R)$ on an $(nk-1)$-sphere for each $n>k\geqq 2$. Furthermore, we have shown that there are uncountably many C^1-differentiably distinct but topologically equivalent analytic actions of $SL(n, R)$ on a k-sphere for each $k\geqq n\geqq 2$.

In this paper, we shall show other aspects of twisted linear actions. In particular, we shall show that there are uncountably many C^2-differentiably distinct but C^1-differentiably equivalent analytic actions of R^n on an n-sphere for each n.

1. Twisted linear actions. Here we recall the definition of twisted linear actions. Throughout this paper, a matrix means only the one with real coefficients.

1.1. Let $u=(u_i)$ and $v=(v_i)$ be column vectors in R^n. As usual, we define their inner product by $u\cdot v=\sum_i u_i v_i$ and the length of u by $\|u\|=\sqrt{u\cdot u}$. Let $M=(m_{ij})$ be a square matrix of degree n. We say that M satisfies the condition (T) if the quadratic form

$$x\cdot Mx=\sum_{i,j} m_{ij}x_i x_j$$

is positive definite. It is easy to see that M satisfies (T) if and only if

$$(\mathrm{T}') \qquad \frac{d}{dt}\|\exp(tM)x\|>0 \qquad \text{for each} \quad x\in R^n_0=R^n-\{0\}\,, \quad t\in R\,.$$

If M satisfies (T′), then

$$\lim_{t\to+\infty}\|\exp(tM)x\|=+\infty \quad \text{and} \quad \lim_{t\to-\infty}\|\exp(tM)x\|=0$$

for each $x\in R^n_0$, and hence there exists a unique real valued analytic function τ on R^n_0

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such that

$$\|\exp(\tau(x)M)x\| = 1 \qquad \text{for} \quad x \in \boldsymbol{R}_0^n .$$

Therefore, we can define an analytic mapping π^M of $\boldsymbol{R}_0^n$ onto the unit $(n-1)$-sphere S^{n-1} by

$$\pi^M(x) = \exp(\tau(x)M)x \qquad \text{for} \quad x \in \boldsymbol{R}_0^n ,$$

if M satisfies the condition (T).

1.2. Let G be a closed subgroup of $\boldsymbol{GL}(n, \boldsymbol{R})$. A square matrix M of degree n is called a G-endomorphism if $gM = Mg$ for each $g \in G$. For a G-endomorphism M satisfying the condition (T), we can define an analytic mapping

$$\xi \colon G \times S^{n-1} \to S^{n-1} \quad \text{by} \quad \xi(g, x) = \pi^M(gx) ,$$

and we see that ξ is an analytic G-action on S^{n-1}. We call $\xi = \xi^M$ a twisted linear action of G on S^{n-1} determined by the G-endomorphism M.

1.3. For a given closed subgroup G of $\boldsymbol{GL}(n, \boldsymbol{R})$, we introduce certain equivalence relations on G-endomorphisms satisfying the condition (T). Let M and N be G-endomorphisms satisfying the condition (T).

We say that M is algebraically equivalent to N, if there exist a G-automorphism A and a positive real number c satisfying

$$cN = AMA^{-1} .$$

We say that M is C^r-equivalent to N, if there exists a C^r-diffeomorphism f of S^{n-1} onto itself such that the following diagram is commutative:

$$\begin{array}{ccc} G \times S^{n-1} & \xrightarrow{1 \times f} & G \times S^{n-1} \\ \downarrow \xi^M & & \downarrow \xi^N \\ S^{n-1} & \xrightarrow{f} & S^{n-1} \end{array} .$$

We call f a G-equivariant C^r-diffeomorphism.

REMARK. It is known that (cf. [1], [2]), if M is algebraically equivalent to N, then M is C^ω-equivalent to N.

2. Certain twisted linear actions on the circle. Here we shall introduce certain twisted linear actions on the circle S^1.

2.1. Let G be the closed subgroup of $\boldsymbol{GL}(2, \boldsymbol{R})$ consisting of matrices in the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Then any G-endomorphism satisfying the condition (T) is written in the form

$$c\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; \qquad c>0, \quad |a|<2.$$

Denote by ξ^a the twisted linear G-action on S^1 determined by the G-endomorphism $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ satisfying $|a|<2$. Then

$$\xi^a\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = e^{\theta}\begin{pmatrix} u+(x+a\theta)v \\ v \end{pmatrix},$$

where θ is uniquely determined by the equation

$$(u+(x+a\theta)v)^2+v^2=e^{-2\theta}.$$

If $v\neq 0$, then we see that

$$\text{(1)} \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \xi^a\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}\right) \Longleftrightarrow \varepsilon = v|v|^{-1}, \quad x = uv^{-1} - a\log|v|.$$

In particular, if $a=0$, then

$$\text{(2)} \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \xi^0\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}\right) \Longleftrightarrow u = \frac{\varepsilon x}{(1+x^2)^{1/2}}, \quad v = \frac{\varepsilon}{(1+x^2)^{1/2}}.$$

Denote by E_+ (resp. E_-) the upper (resp. lower) semicircle. Then, by the above arguments, we see that the G-action ξ^a has just four orbits, two of them are fixed points and the other two of them are open orbits E_+ and E_-.

Denote by $S^1(a)$ the circle with the twisted linear G-action ξ^a. In the rest of this section, we shall show the following.

THEOREM 2.1. *Let a, b be real numbers satisfying $|a|<2$, $|b|<2$. Then, there exists an equivariant C^1-diffeomorphism from $S^1(a)$ onto $S^1(b)$. If $a\neq b$, then there is no equivariant C^2-diffeomorphism from $S^1(a)$ onto $S^1(b)$.*

2.2. Define

$$L(v)=v\log|v| \qquad \text{for} \quad v\neq 0 \quad \text{and} \quad L(0)=0.$$

Then L is a continuous function on the real line. Put

$$D(u, v; a)=((u-aL(v))^2+v^2)^{1/2}$$

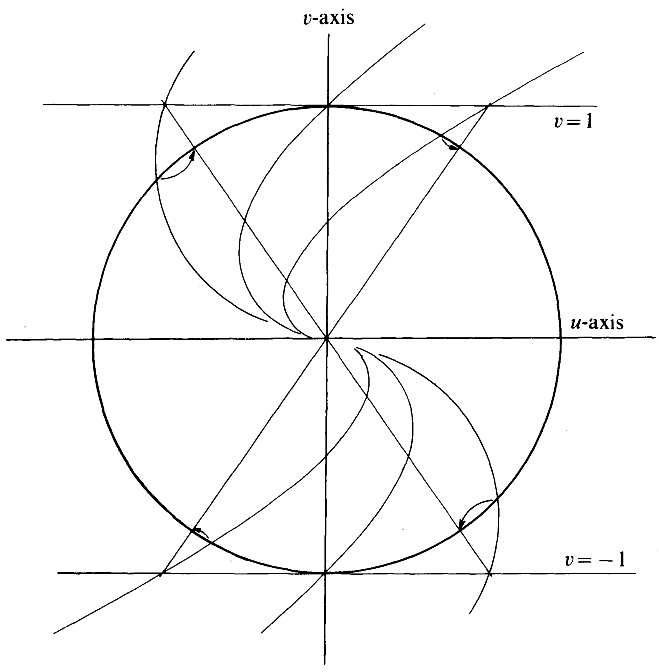
and define

(3) $$\bar{u}=(u-aL(v))D(u, v; a)^{-1}, \qquad \bar{v}=vD(u, v; a)^{-1}.$$

Then the correspondence from (u, v) to $(\bar{u}, \bar{v})$ defines a continuous mapping f_a of the circle onto itself. By 2.1(1), (2) we see that f_a is an equivariant homeomorphism from $S^1(a)$ onto $S^1(0)$.

Geometrically the above correspondence (3) is explained as follows (see Figure). Consider integral curves of the linear system

$$\dot{u}=u+av, \qquad \dot{v}=v.$$



FIGURE

If $v\neq 0$, then there is just one point $(\varepsilon x, \varepsilon)$ on the integral curve through (u, v), where $\varepsilon=v|v|^{-1}$, and we can define $(\bar{u}, \bar{v})$ as the intersection point of the circle and the line segment joining the origin and $(\varepsilon x, \varepsilon)$.

By (3), we obtain

$$\frac{\partial \bar{u}}{\partial u}(u, v)=v^2D^{-3}, \qquad \frac{\partial \bar{u}}{\partial v}(u, v)=-v(u+av)D^{-3},$$

$$\frac{\partial \bar{v}}{\partial u}(u, v)=-v(u-aL(v))D^{-3}, \qquad \frac{\partial \bar{v}}{\partial v}(u, v)=(u+av)(u-aL(v))D^{-3}$$

for $v \neq 0$, where $D = D(u, v; a)$, and we obtain directly

$$\frac{\partial \bar{u}}{\partial u}(u, 0) = \frac{\partial \bar{u}}{\partial v}(u, 0) = \frac{\partial \bar{v}}{\partial u}(u, 0) = 0\,, \qquad \frac{\partial \bar{v}}{\partial v}(u, 0) = |u|^{-1}\,.$$

Let us show $(\partial \bar{u}/\partial v)(u, 0) = 0$, for completeness.

$$\frac{\partial \bar{u}}{\partial v}(u, 0) = \lim_{v \to 0} \frac{\bar{u}(u, v) - \bar{u}(u, 0)}{v} = \lim_{v \to 0} \frac{u - aL(v) - |u|^{-1} uD}{vD}$$

$$= \lim_{v \to 0} \frac{(u - aL(v))^2 - D^2}{(u - aL(v) + |u|^{-1} uD) vD} = \lim_{v \to 0} \frac{-v}{(u - aL(v) + |u|^{-1} uD) D} = 0\,.$$

Hence we see that f_a is C^1-differentiable. Moreover, we obtain

$$\frac{d}{du} \bar{u}(u, (1 - u^2)^{1/2}) = \frac{1 + au(1 - u^2)^{1/2}}{D^3} > 0 \qquad \text{for} \quad -1 < u < 1$$

and

$$\frac{d}{dv} \bar{v}((1 - v^2)^{1/2}, v) = \frac{((1 - v^2)^{1/2} - aL(v))(1 + av(1 - v^2)^{1/2})}{(1 - v^2)^{1/2} D^3} > 0$$

for $|v| \ll 1$. Hence we see that f_a is a C^1-diffeomorphism by the inverse function theorem.

Consequently, we see that a composite mapping $f_b^{-1} f_a$ is an equivariant C^1-diffeomorphism from $S^1(a)$ onto $S^1(b)$. This proves the first half of Theorem 2.1.

2.3. Next, we shall show that the composite mapping $f_b^{-1} f_a$ is not C^2-differentiable at a point $(1, 0)$ if $a \neq b$.

$$f_b^{-1} f_a \quad \text{maps} \quad ((1 - v^2)^{1/2}, v) \quad \text{to} \quad ((1 - w^2)^{1/2}, w)\,,$$

where $w = w(v)$ is a C^1-diffeomorphism of an open interval $(-1, 1)$ onto itself satisfying $w(0) = 0$. By 2.1(1), we obtain

$$v^{-1}((1 - v^2)^{1/2} - aL(v)) = w^{-1}((1 - w^2)^{1/2} - bL(w))\,. \tag{4}$$

Differentiating both sides of (4) as functions of the variable v, we obtain

$$\frac{av + (1 - v^2)^{-1/2}}{-v^2} = \frac{bw + (1 - w^2)^{-1/2}}{-w^2} \cdot \frac{dw}{dv}$$

Therefore

$$\frac{dw}{dv} = \frac{av + (1 - v^2)^{-1/2}}{bw + (1 - w^2)^{-1/2}} (wv^{-1})^2\,.$$

Moreover, we obtain

$$\frac{d^2w}{dv^2}=(wv^{-1})^2\frac{d}{dv}\left(\frac{av+(1-v^2)^{-1/2}}{bw+(1-w^2)^{-1/2}}\right)+\frac{av+(1-v^2)^{-1/2}}{bw+(1-w^2)^{-1/2}}(2wv^{-1})\frac{d}{dv}(wv^{-1})$$

and

$$\frac{d}{dv}(wv^{-1})=\frac{(a-b)w^2}{v^2(bw+(1-w^2)^{-1/2})}+\frac{w(w(1-w^2)^{1/2}-v(1-v^2)^{1/2})}{v^3(bw+(1-w^2)^{-1/2})(1-v^2)^{1/2}(1-w^2)^{1/2}}.$$

By (4), we obtain

$$\begin{aligned}w(1-w^2)^{1/2}-v(1-v^2)^{1/2}&=(v+w)((1-w^2)^{1/2}-(1-v^2)^{1/2})+awL(v)-bvL(w)\\&=\frac{(v+w)(v^2-w^2)}{(1-v^2)^{1/2}+(1-w^2)^{1/2}}+vw(a\log|v|-b\log|w|).\end{aligned}$$

Moreover, we obtain

$$\lim_{v\to 0}(wv^{-1})=\lim_{v\to 0}\frac{(1-w^2)^{1/2}-bL(w)}{(1-v^2)^{1/2}-aL(v)}=1\,,\qquad \lim_{v\to 0}\frac{d}{dv}\left(\frac{av+(1-v^2)^{-1/2}}{bw+(1-w^2)^{-1/2}}\right)=a-b\,.$$

Hence we obtain

$$\lim_{v\to 0}\frac{d^2w}{dv^2}=\lim_{v\to 0}(a-b)(3+2\log|v|)\,.$$

Therefore, we see that $w=w(v)$ is not C^2-differentiable at $v=0$ if $a\neq b$. Consequently, we see that the composite mapping $f_b^{-1}f_a$ is not C^2-differentiable at the point $(1, 0)$ if $a\neq b$.

2.4. Finally, we shall show that there is no equivariant C^2-diffeomorphism from $S^1(a)$ onto $S^1(b)$ if $a\neq b$.

Suppose that there is an equivariant C^2-diffeomorphism f from $S^1(a)$ onto $S^1(b)$. Then, we can assume that $f(E_+)=E_+$, because the correspondence from (u, v) to $(-u, -v)$ is an equivariant C^ω-diffeomorphism of $S^1(a)$ onto itself. Moreover, we can assume $f((0, 1))=(0, 1)$, because the abelian group G acts transitively on E_+ via ξ^a.

Consequently, we can assume $f=f_b^{-1}f_a$ on the closure of E_+. Hence we obtain $a=b$ by the arguments in 2.3. Therefore, we see that there is no equivariant C^2-diffeomorphism from $S^1(a)$ onto $S^1(b)$ if $a\neq b$. This proves the second half of Theorem 2.1.

3. First generalization.

3.1. Let G_n be the closed subgroup of $GL(n+1, \boldsymbol{R})$ consisting of matrices in the form

$$(*)\qquad \begin{pmatrix} 1 & x_1 & \cdots & x_n \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

Denote by $[x_1, \cdots, x_n]$ the above matrix. Then G_n is an abelian Lie group isomorphic to $\boldsymbol{R}^n$. Moreover, any G_n-endomorphism satisfying the condition (T) is written in the form

$$c[a_1, \cdots, a_n]; \qquad c>0, \quad a_1^2+\cdots+a_n^2<4.$$

For $\boldsymbol{a}=(a_1, \cdots, a_n)$ satisfying $a_1^2+\cdots+a_n^2<4$, denote by $\xi^{[\boldsymbol{a}]}$ the twisted linear G_n-action on S^n determined by the G_n-endomorphism $[a_1, \cdots, a_n]$. Then

$$\xi^{[\boldsymbol{a}]}([x_1, \cdots, x_n], (u_0, u_1, \cdots, u_n))=e^{\theta}(u_0+(x_1+a_1\theta)u_1+\cdots+(x_n+a_n\theta)u_n, u_1, \cdots, u_n),$$

where θ is uniquely determined by the equation

$$(u_0+(x_1+a_1\theta)u_1+\cdots+(x_n+a_n\theta)u_n)^2+u_1^2+\cdots+u_n^2=e^{-2\theta}.$$

If $(u_1, \cdots, u_n)\neq(0, \cdots, 0)$, then we see that

$$(u_0, u_1, \cdots, u_n)=\xi^{[\boldsymbol{a}]}([x_1, \cdots, x_n], (0, v_1, \cdots, v_n))$$

if and only if

$$(1)\qquad \begin{aligned} &v_j=u_j(1-u_0^2)^{-1/2} \qquad \text{for} \quad 1\leqq j\leqq n, \\ &u_0=x_1u_1+\cdots+x_nu_n+(a_1u_1+\cdots+a_nu_n)\log(u_1^2+\cdots+u_n^2)^{1/2}. \end{aligned}$$

In particular, if $(a_1, \cdots, a_n)=(0, \cdots, 0)$, then

$$(u_0, u_1, \cdots, u_n)=\xi^{[0,\cdots,0]}([x_1, \cdots, x_n], (0, v_1, \cdots, v_n))$$

if and only if

$$(2)\qquad \begin{aligned} &u_j=\frac{v_j}{(1+(x_1v_1+\cdots+x_nv_n)^2)^{1/2}} \qquad \text{for} \quad 1\leqq j\leqq n, \\ &u_0=\frac{x_1v_1+\cdots+x_nv_n}{(1+(x_1v_1+\cdots+x_nv_n)^2)^{1/2}}. \end{aligned}$$

By the above arguments, we see that the G_n-action $\xi^{[\boldsymbol{a}]}$ has just two fixed points

$$(1, 0, \cdots, 0), \qquad (-1, 0, \cdots, 0)$$

and each of the other orbits is diffeomorphic to an open interval.

Denote by $S^n(\boldsymbol{a})$ the n-sphere with the twisted linear G_n-action $\xi^{[\boldsymbol{a}]}$. In the rest of this section, we shall show the following.

THEOREM 3.1. *Let* $\boldsymbol{a}=(a_1, \cdots, a_n)$ *and* $\boldsymbol{b}=(b_1, \cdots, b_n)$, *for* $n\geqq 1$. *Suppose*

$$a_1^2+\cdots+a_n^2<4\,,\qquad b_1^2+\cdots+b_n^2<4\,.$$

Then, there exists a G_n*-equivariant* C^1*-diffeomorphism from* $S^n(\boldsymbol{a})$ *onto* $S^n(\boldsymbol{b})$. *If* $\boldsymbol{a}\neq\boldsymbol{b}$, *then there is no* G_n*-equivariant* C^2*-diffeomorphism from* $S^n(\boldsymbol{a})$ *onto* $S^n(\boldsymbol{b})$.

3.2. Define

$$L=L(u_1, \cdots, u_n; \boldsymbol{a})=(a_1u_1+\cdots+a_nu_n)\log(u_1^2+\cdots+u_n^2)^{1/2}$$

for $(u_1, \cdots, u_n)\neq(0, \cdots, 0)$ and $L(0, \cdots, 0; \boldsymbol{a})=0$. Then L is a continuous function on the n-plane. Put

$$D=D(u_0, u_1, \cdots, u_n; \boldsymbol{a})=((u_0-L)^2+u_1^2+\cdots+u_n^2)^{1/2}$$

and define

$$\text{(3)}\qquad \bar{u}_0=(u_0-L)D^{-1}\,,\qquad \bar{u}_j=u_jD^{-1}\quad (1\leqq j\leqq n)\,.$$

Then the correspondence from $(u_0, u_1, \cdots, u_n)$ to $(\bar{u}_0, \bar{u}_1, \cdots, \bar{u}_n)$ defines a continuous mapping f of the n-sphere onto itself. We see that f induces the identity mapping on the $(n-1)$-sphere determined by the equation $u_0=0$. By 3.1(1), (2) we see that f is a G_n-equivariant homeomorphism from $S^n(\boldsymbol{a})$ onto $S^n(\boldsymbol{0})$, where $\boldsymbol{0}=(0, \cdots, 0)$.

By (3), we obtain

$$\frac{\partial\bar{u}_0}{\partial u_0}=(u_1^2+\cdots+u_n^2)D^{-3}\,,\qquad \frac{\partial\bar{u}_j}{\partial u_0}=-u_j(u_0-L)D^{-3}\quad(1\leqq j\leqq n)\,,$$

$$\frac{\partial\bar{u}_0}{\partial u_j}=-((u_1^2+\cdots+u_n^2)\frac{\partial L}{\partial u_j}+u_j(u_0-L))D^{-3}\quad(1\leqq j\leqq n)\,,$$

$$\frac{\partial\bar{u}_i}{\partial u_j}=\left(\delta_{ij}D^2-u_iu_j+(u_0-L)u_i\frac{\partial L}{\partial u_j}\right)D^{-3}\quad(1\leqq i, j\leqq n)\,,$$

for $(u_1, \cdots, u_n)\neq(0, \cdots, 0)$, where

$$\frac{\partial L}{\partial u_j}=\frac{u_j(a_1u_1+\cdots+a_nu_n)}{u_1^2+\cdots+u_n^2}+a_j\log(u_1^2+\cdots+u_n^2)^{1/2}\quad(1\leqq j\leqq n)\,,$$

and we obtain directly

$$\frac{\partial\bar{u}_0}{\partial u_j}=\frac{\partial\bar{u}_j}{\partial u_0}=0\quad(0\leqq j\leqq n)\,,\qquad \frac{\partial\bar{u}_i}{\partial u_j}=\frac{\delta_{ij}}{|u_0|}\quad(1\leqq i, j\leqq n)$$

for $(u_1, \cdots, u_n)=(0, \cdots 0)$. Hence we see that f is C^1-differentiable.

By the geometric meaning of the construction (3), we see that f induces a C^ω-diffeomorphism from $S^n(\boldsymbol{a})-\{(\varepsilon, 0, \cdots, 0)\}$ onto $S^n(\boldsymbol{0})-\{(\varepsilon, 0, \cdots, 0)\}$.

Moreover, we obtain

$$\frac{\partial}{\partial u_j}\bar{u}_i(\varepsilon(1-u_1^2-\cdots-u_n^2)^{1/2}\,,\quad u_1,\cdots,u_n)=\delta_{ij}\quad(1\leqq i,j\leqq n)$$

at the point $(\varepsilon, 0, \cdots, 0)$. Hence we see that $f=f_{\boldsymbol{a}}$ is a C^1-diffeomorphism from $S^n(\boldsymbol{a})$ onto $S^n(\boldsymbol{0})$ by the inverse function theorem.

Consequently, we see that a composite mapping $f_{\boldsymbol{b}}^{-1}f_{\boldsymbol{a}}$ is a G_n-equivariant C^1-diffeomorphism from $S^n(\boldsymbol{a})$ onto $S^n(\boldsymbol{b})$. This proves the first half of Theorem 3.1.

3.3. Next, we shall show that there is no G_n-equivariant C^2-diffeomorphism from $S^n(\boldsymbol{a})$ onto $S^n(\boldsymbol{b})$ if $\boldsymbol{a}\neq\boldsymbol{b}$.

Denote by $G_n(i)$ the closed subgroup of G_n consisting of matrices in the form

$$[x_1, \cdots, x_n]\,;\qquad x_i=0\,,$$

and by $F_i(\boldsymbol{a})$ the fixed point set of the restricted $G_n(i)$-action on $S^n(\boldsymbol{a})$. Then we see that

$$F_i(\boldsymbol{a})=\{(u_0, \cdots, u_n)\in S^n\,|\,u_j=0 \text{ for } j\neq 0, i\}\,.$$

Define a C^ω-diffeomorphism h_i from S^1 onto $F_i(\boldsymbol{a})$ by the correspondence from (u, v) to $(u, 0, \cdots, 0, v, 0, \cdots, 0)$. Then, we obtain

$$\xi^{[\boldsymbol{a}]}([x_1, \cdots, x_n], h_i(u, v))=h_i(\xi^{a_i}([x_i], (u, v)))\,. \tag{4}$$

Now, we suppose that there is a G_n-equivariant C^2-diffeomorphism f from $S^n(\boldsymbol{a})$ onto $S^n(\boldsymbol{b})$. Then, f induces naturally a G_n-equivariant C^2-diffeomorphism from $F_i(\boldsymbol{a})$ onto $F_i(\boldsymbol{b})$. Then, by (4), we obtain an equivariant C^2-diffeomorphism from $S^1(a_i)$ onto $S^1(b_i)$ for each $i=1, \cdots, n$. Then we obtain $\boldsymbol{a}=\boldsymbol{b}$ by Theorem 2.1. This proves the second half of Theorem 3.1.

4. Second generalization.

4.1. Let G_n^* be the closed subgroup of $\boldsymbol{GL}(n+1, \boldsymbol{R})$ consisting of matrices in the form

$$\begin{pmatrix} 1 & & 0 & x_1 \\ & \ddots & & \vdots \\ & & 1 & x_n \\ 0 & & & 1 \end{pmatrix}. \tag{**}$$

Denote by $[x_1, \cdots, x_n]^*$ the above matrix. Then G_n^* is an abelian Lie group isomorphic to $\boldsymbol{R}^n$. Moreover, any G_n^*-endomorphism satisfying the condition (T) is written in the form

$$c[a_1, \cdots, a_n]^*\,;\qquad c>0\,,\quad a_1^2+\cdots+a_n^2<4\,.$$

For $\boldsymbol{a}=(a_1, \cdots, a_n)$ satisfying $a_1^2+\cdots+a_n^2<4$, denote by $\xi^{[\boldsymbol{a}]*}$ the twisted linear G_n^*-action on S^n determined by the G_n^*-endomorphism $[a_1, \cdots, a_n]^*$. Then

$$\xi^{[\boldsymbol{a}]*}([x_1, \cdots, x_n]^*, (u_1, \cdots, u_{n+1}))$$
$$=e^{\theta}(u_1+(x_1+a_1\theta)u_{n+1}, \cdots, u_n+(x_n+a_n\theta)u_{n+1}, u_{n+1}),$$

where θ is uniquely determined by the equation

$$(u_1+(x_1+a_1\theta)u_{n+1})^2+\cdots+(u_n+(x_n+a_n\theta)u_{n+1})^2+u_{n+1}^2=e^{-2\theta}.$$

If $u_{n+1}\neq 0$, then we see that

$$(u_1, \cdots, u_{n+1})=\xi^{[\boldsymbol{a}]*}([x_1, \cdots, x_n]^*, (0, \cdots, 0, \varepsilon))$$

if and only if

$$\varepsilon=\frac{u_{n+1}}{|u_{n+1}|}, \quad x_j=\frac{u_j}{u_{n+1}}-a_j\log|u_{n+1}| \qquad (1\leqq j\leqq n). \tag{1}$$

In particular, if $(a_1, \cdots, a_n)=(0, \cdots, 0)$, then

$$(u_1, \cdots, u_{n+1})=\xi^{[0,\cdots,0]*}([x_1, \cdots, x_n]^*, (0, \cdots, 0, \varepsilon))$$

if and only if

$$\begin{aligned} u_j&=\frac{\varepsilon x_j}{(1+x_1^2+\cdots+x_n^2)^{1/2}} \qquad \text{for} \quad 1\leqq j\leqq n, \\ u_{n+1}&=\frac{\varepsilon}{(1+x_1^2+\cdots+x_n^2)^{1/2}}. \end{aligned} \tag{2}$$

Denote by E_+ (resp. E_-) the upper (resp. lower) hemisphere determined by the inequality $u_{n+1}>0$ (resp. $u_{n+1}<0$). Then, by the above arguments, we see that E_+ and E_- are open orbits of the G_n^*-action $\xi^{[\boldsymbol{a}]*}$ and the other points are fixed points.

Denote by $S^n(\boldsymbol{a})^*$ the n-sphere with the twisted linear G_n^*-action $\xi^{[\boldsymbol{a}]*}$. In the rest of this section, we shall show the following.

THEOREM 4.1. *Let $\boldsymbol{a}=(a_1, \cdots, a_n)$ and $\boldsymbol{b}=(b_1, \cdots, b_n)$, for $n\geqq 2$.* Suppose

$$a_1^2+\cdots+a_n^2<4, \qquad b_1^2+\cdots+b_n^2<4.$$

*Then, there exists a G_n^*-*equivariant homeomorphism from *$S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{b})^*$. If $\boldsymbol{a}\neq\boldsymbol{b}$, then there is no G_n^*-equivariant C^1-diffeomorphism from $S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{b})^*$.*

4.2. Define

$$L(v)=v\log|v| \qquad \text{for} \quad v\neq 0 \quad \text{and} \quad L(0)=0.$$

Then L is a continuous function on the real line. Put

$$D=((u_1-a_1L(u_{n+1}))^2+\cdots+(u_n-a_nL(u_{n+1}))^2+u_{n+1}^2)^{1/2}$$

and define

(3) $$\bar{u}_j=(u_j-a_jL(u_{n+1}))D^{-1}\quad (1\leqq j\leqq n)\,,\quad \bar{u}_{n+1}=u_{n+1}D^{-1}\,.$$

Then the correspondence from $(u_1, \cdots, u_{n+1})$ to $(\bar{u}_1, \cdots, \bar{u}_{n+1})$ defines a continuous mapping $f=f_{\boldsymbol{a}}$ of the n-sphere onto itself. By 4.1(1), (2) we see that f is a G_n^*-equivariant homeomorphism from $S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{0})^*$.

Consequently, we see that the composite mapping $f_{\boldsymbol{b}}^{-1}f_{\boldsymbol{a}}$ is a G_n^*-equivariant homeomorphism from $S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{b})^*$. This proves the first half of Theorem 4.1.

4.3. Next, we shall show that the composite mapping $F=f_{\boldsymbol{b}}^{-1}f_{\boldsymbol{a}}$ is not C^1-differentiable at a point $(0, \cdots, 0, 1, 0, \cdots, 0)$, if $n\geqq 2$ and $\boldsymbol{a}\neq\boldsymbol{b}$. F maps $(u_1, \cdots, u_{n+1})$ to $(w_1, \cdots, w_{n+1})$, where

$$w_j=w_j(u_1, \cdots, u_{n+1})\qquad (1\leqq j\leqq n+1)$$

are continuous mappings. Then, by 4.1(1), we see that

(4) $$(u_j-a_jL(u_{n+1}))w_{n+1}=(w_j-b_jL(w_{n+1}))u_{n+1}$$

for $1\leqq j\leqq n$.

For each k $(1\leqq k\leqq n)$, define a C^ω-differentiable mapping

$$c_k(s)=(u_1^k(s), \cdots, u_{n+1}^k(s))$$

from an open interval $(-1, 1)$ to the n-sphere by

$$u_j^k(s)=\delta_{kj}(1-s^2)^{1/2}\qquad \text{for}\quad 1\leqq j\leqq n\,,\qquad u_{n+1}^k(s)=s\,,$$

and put

$$F(c_k(s))=(w_1^k(s), \cdots, w_{n+1}^k(s))\,.$$

By (4), we obtain

$$\frac{w_{n+1}^k(s)}{s}=\frac{w_{n+1}^k(s)}{u_{n+1}^k(s)}=\frac{w_k^k(s)-b_kL(w_{n+1}^k(s))}{u_k^k(s)-a_kL(u_{n+1}^k(s))}\,,$$

and hence

$$\lim_{s\to 0}\frac{w_{n+1}^k(s)}{s}=1\,,$$

because $w_k^k(0)=u_k^k(0)=1$ and $w_{n+1}^k(0)=u_{n+1}^k(0)=0$. Moreover, by (4), we obtain

$$\frac{w_j^k(s)}{w_{n+1}^k(s)}=(b_j-a_j)\log|s|+b_j\log|w_{n+1}^k(s)s^{-1}|$$

for each j $(\neq k, n+1)$. Hence we obtain

$$\frac{dw_j^k}{ds}(0)=\lim_{s\to 0}\frac{w_j^k(s)}{s}=\lim_{s\to 0}\frac{w_j^k(s)}{w_{n+1}^k(s)}=\lim_{s\to 0}(b_j-a_j)\log|s|$$

for each j ($\neq k, n+1$).

Therefore, if the mapping F is C^1-differentiable at $c_k(0)$, then we obtain $a_j=b_j$ for each j ($\neq k, n+1$). Consequently, we see that if $n\geqq 2$ and F is C^1-differentiable at each point $c_k(0)$ $(1\leqq k\leqq n)$, then $\boldsymbol{a}=\boldsymbol{b}$.

4.4. Finally, we shall show that there is no G_n^*-equivariant C^1-diffeomorphism from $S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{b})^*$, if $n\geqq 2$ and $\boldsymbol{a}\neq\boldsymbol{b}$.

Suppose that there is a G_n^*-equivariant C^1-diffeomorphism f from $S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{b})^*$. Then, we can assume that

$$f(0, \cdots, 0, 1)=(0, \cdots, 0, 1),$$

for the same reason as in 2.4. Hence we can assume $f=f_{\boldsymbol{b}}^{-1}f_{\boldsymbol{a}}$ on the closure of the upper hemisphere E_+. Hence we obtain $\boldsymbol{a}=\boldsymbol{b}$ by the arguments in 4.3. This proves the second half of Theorem 4.1.

5. Concluding remark.

5.1. Let G be a closed subgroup of $GL(n, \boldsymbol{R})$ and let M and N be G-endomorphisms satisfying the condition (T). We say that M is weakly C^r-equivalent to N, if there exist an automorphism α of G and a C^r-diffeomorphism f of S^{n-1} onto itself such that the following diagram is commutative:

$$\begin{array}{ccc} G\times S^{n-1} & \xrightarrow{\alpha\times f} & G\times S^{n-1} \\ \Big\downarrow \xi^M & & \Big\downarrow \xi^N \\ S^{n-1} & \xrightarrow{f} & S^{n-1}. \end{array}$$

We call f a weakly G-equivariant C^r-diffeomorphism.

5.2. For $\boldsymbol{x}=(x_1, \cdots, x_n)$, denote by $[\boldsymbol{x}]$ and $[\boldsymbol{x}]^*$ the matrices in the form 3.1(*) and 4.1(**), respectively. We shall show the following result due to a colleague, Shin-ichi Watanabe.

THEOREM 5.2. *Let* $\boldsymbol{a}=(a_1, \cdots, a_n)$ *and* $\boldsymbol{b}=(b_1, \cdots, b_n)$, *for* $n\geqq 1$. *Suppose*

$$0<a_1^2+\cdots+a_n^2<4, \qquad 0<b_1^2+\cdots+b_n^2<4.$$

Then, (i) *there exists a weakly* G_n*-equivariant analytic diffeomorphism from* $S^n(\boldsymbol{a})$ *onto* $S^n(\boldsymbol{b})$, *and* (ii) *there exists a weakly* G_n^**-equivariant analytic diffeomorphism from* $S^n(\boldsymbol{a})^*$ *onto* $S^n(\boldsymbol{b})^*$.

PROOF. We see that there exist P, Q in $GL(n, \boldsymbol{R})$ satisfying $\boldsymbol{a}=\boldsymbol{b}P$ and $\boldsymbol{b}=\boldsymbol{a}^tQ$. Denote

$$P^{(1)}=\begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix}, \qquad Q_{(1)}=\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. Define automorphisms α_P of G_n and α_Q^* of G_n^* by

$$\alpha_P([x])=[xP^{-1}], \qquad \alpha_Q^*([x]^*)=[x^tQ]^*,$$

respectively. Define an analytic diffeomorphism f_P from $S^n(\boldsymbol{a})$ onto $S^n(\boldsymbol{b})$ by

$$f_P(\boldsymbol{u})=\pi^{[\boldsymbol{b}]}(P^{(1)}\boldsymbol{u}) \qquad \text{for} \quad \boldsymbol{u}=(u_0, \cdots, u_n),$$

and an analytic diffeomorphism f_Q^* from $S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{b})^*$ by

$$f_Q^*(\boldsymbol{u})=\pi^{[\boldsymbol{b}]^*}(Q_{(1)}\boldsymbol{u}) \qquad \text{for} \quad \boldsymbol{u}=(u_1, \cdots, u_{n+1}).$$

Then, we see that the following diagrams are commutative:

$$\begin{array}{ccc} G_n \times S^n(\boldsymbol{a}) & \xrightarrow{\alpha_P \times f_P} & G_n \times S^n(\boldsymbol{b}) \\ \downarrow \xi^{[\boldsymbol{a}]} & & \downarrow \xi^{[\boldsymbol{b}]} \\ S^n(\boldsymbol{a}) & \xrightarrow{f_P} & S^n(\boldsymbol{b}), \end{array}$$

$$\begin{array}{ccc} G_n^* \times S^n(\boldsymbol{a})^* & \xrightarrow{\alpha_Q^* \times f_Q^*} & G_n^* \times S^n(\boldsymbol{b})^* \\ \downarrow \xi^{[\boldsymbol{a}]^*} & & \downarrow \xi^{[\boldsymbol{b}]^*} \\ S^n(\boldsymbol{a})^* & \xrightarrow{f_Q^*} & S^n(\boldsymbol{b})^*. \end{array}$$

Therefore, f_P is a weakly G_n-equivariant analytic diffeomorphism from $S^n(\boldsymbol{a})$ onto $S^n(\boldsymbol{b})$, and f_Q^* is a weakly G_n^*-equivariant analytic diffeomorphism from $S^n(\boldsymbol{a})^*$ onto $S^n(\boldsymbol{b})^*$. q.e.d.

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