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A NEW CONSTRUCTION OF A COMPACTIFICATION OF C^3

Dedicated to Professor Friedrich Hirzebruch on his sixtieth birthday

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Introduction. Let (X, Y) be a smooth projective compactification of C^3, namely, X is a smooth projective 3-fold and Y is a subvariety of X such that $X-Y$ is analytically isomorphic to C^3. We will write simply as $X-Y\cong C^3$ if there is an algebraic isomorphism of $X-Y$ onto C^3. Assume that Y is normal. Then X is a Fano 3-fold of index $r(1\leq r\leq 4)$ with the second Betti number $b_2(X)=1$, and Y is a hyperplane section of X. Then, in the paper [1], we have the following results:

(i) $r=4\Rightarrow(X, Y)\cong(P^3, P^2)$

(ii) $r=3\Rightarrow(X, Y)\cong(Q^3, Q_0^2)$, where Q^3 is a smooth quadric hypersurface in P^4 and Q_0^2 is a quadric cone.

(iii) $r=2\Rightarrow(X, Y)\cong(V_5, H_5)$, where V_5 is a Fano 3-fold of degree 5 in P^6 and H_5 is a singular del Pezzo surface with exactly one rational double point of A_4-type.

(iv) $r=1\Rightarrow(X, Y)$ is not completely determined (see also [2], [3], [9]).

These 3-folds P^3, Q^3, V_5 are compactifications of C^3. In the case of $r=4$, it is clear that $P^3-\{\text{a hyperplane } P^2\}\cong C^3$. In the case of $r=3$, projecting Q^3 from the vertex of Q_0^2 to P^3, one can see that $Q^3-Q_0^2\cong C^3$. In the case of $r=2$, projecting V_5 from a line C in V_5 through the singular point x of A_4-type of H_5, one can see that $V_5-H_5\cong C^3$. Moreover, let H_5^∞ be the ruled surface swept out by lines which intersect the line C. Then H_5^∞ is a non-normal hyperplane section of V_5 such that $V_5-H_5^\infty\cong C^3$ (see [1]). In particular, H_5, H_5^∞ are members of the linear system $|H-2x|:=|\mathcal{O}_{V_5}(1)\otimes\mathcal{M}_x^2|$, where H is a member of $|\mathcal{O}_{V_5}(1)|$ and $\mathcal{M}_x$ is the maximal ideal of the local ring $\mathcal{O}_{V_5,x}$.

To see how many members of the linear system $|H-2x|$ can be normal (or non-normal) boundaries of C^3 in V_5, we will study in this paper the double projection from the singular point x of H_5. Consequently, we have a new construction of a compactification of C^3 in the case of index $r=2$.

Our main result is the following:

THEOREM. (1) *The set* $\mathfrak{A}:=\{x\in V_5$; *there is a unique line in* V_5 *through the point* $x\}$ *is not empty.*

(2) *Take a point* $x\in\mathfrak{A}$ *and a line* C *through* x. *Let* $\sigma: V_5'\to V_5$ *be the blowing up of* V_5 *at the point* x, *and put* $E:=\sigma^{-1}(x)\cong P^2$. *Then there is a* P^1*-bundle* $\pi: P(\mathcal{E})\to P^2$ *over* P^2 ($\mathcal{E}$ *is a locally free sheaf of rank* 2 *over* P^2) *and a birational map* $\rho: V_5'\to P(\mathcal{E})$,

called a flip, such that the following (i)–(iii) *hold*:

(i) *there is a smooth rational curve* f *in* $\boldsymbol{P}(\mathscr{E})$ *such that* $V'_5 - C_1$ *is isomorphic to* $\boldsymbol{P}(\mathscr{E}) - f$, *where* C_1 *is the proper transform of* C *in* V'_5,

(ii) $\Sigma := \rho(E)$ *is a rational section of* $\pi: \boldsymbol{P}(\mathscr{E}) \to \boldsymbol{P}^2$ *with a rational double point* q *of* A_2-type. In particular, $q \in f \subsetneq \Sigma$, *and*

(iii) *there is a point* $p \in \boldsymbol{P}^2$ *such that* $\pi^{-1}(p) \subsetneq \Sigma$ *and* $\Sigma - \pi^{-1}(p)$ *is isomorphic to* $\boldsymbol{P}^2 - \{p\}$.

(3) *The set* $L_\infty := \pi(f)$ *is a line in* $\boldsymbol{P}^2$ *through* p, *and* $H_5^\infty := \sigma\rho^{-1}(\pi^{-1}(L_\infty) \cup \Sigma)$ *is the ruled surface swept out by lines which intersect the line* C. *For any line* L_t $(t \neq \infty)$ *through the point* p, $H_5^t := \sigma\rho^{-1}(\pi^{-1}(L_t) \cup \Sigma)$ *is a normal surface with a rational double point of* A_4*-type. In particular,* $V_5 - H_5^\infty \cong \boldsymbol{C}^3$ *and* $V_5 - H_5^t \cong \boldsymbol{C}^3$.

COROLLARY. *For each* $x \in \mathfrak{A}$,

$$\{H_5 \in |\mathcal{O}_{V_5}(1) \otimes \mathscr{M}_x^2|;\ V_5 - H_5 \cong \boldsymbol{C}^3\} = \{H_5^t\}_{t \in \boldsymbol{C}} \cup \{H_5^\infty\}.$$

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1. Preliminaries. Let us recall some results in the paper [1]. Let (X, Y) be a projective compactification of $\boldsymbol{C}^3$ such that Y is normal. Assume that the index $r=2$. Then $(X, Y) \cong (V_5, H_5)$ (see the Introduction). Then the anti-canonical line bundle can be written as follow:

$$-K_Y \cong \mathcal{O}_Y(\Gamma),$$

where Γ is an elliptic curve not through the singularity of $Y = H_5$. Thus deg $Y = (\Gamma^2)_Y = 5$. In particular, the singular locus of Y consists of exactly one point $\{x\}$, which is of A_4-type. Let $\alpha: \tilde{Y} \to Y$ be the minimal resolution of singularity of Y and put

$$\alpha^{-1}(x) = \tilde{l}_2 \cup \tilde{f}_1 \cup \tilde{f}_2 \cup \tilde{l}_1,$$

where $\tilde{l}_i$, $\tilde{f}_i$ $(1 \le i \le 2)$ are smooth rational curves with the self-intersction number equal to -2 and the dual graph of the exceptional divisor $\alpha^{-1}(x)$ is a linear tree (see Figure 1).

On the other hand, $\tilde{Y}$ can be obtained from $\boldsymbol{P}^2$ by the blowing up of four points (infinitely near points allowed) on a smooth cubic curve Γ_0 on $\boldsymbol{P}^2$. Let $\tilde{\Gamma}$ be the proper transform of Γ_0 in $\tilde{Y}$ (see Figure 1).

In Figure 1, there exists an exceptional curve $\tilde{C}$ of the first kind with $(\tilde{C} \cdot \tilde{\Gamma})_{\tilde{Y}} = 1$. We put $C = \alpha(\tilde{C})$ and $\Gamma = \alpha(\tilde{\Gamma})$. Let H be a general hyperplane section of $X := V_5$ such that $\mathcal{O}_Y(H) = \mathcal{O}_Y(\Gamma)$. Since

$$1 = (\tilde{\Gamma} \cdot \tilde{C})_{\tilde{Y}} = (\Gamma \cdot C)_Y = (H \cdot C)_X,$$

C is a line on X. By [1, Proposition 15], C is a unique line in $\boldsymbol{P}^6$ contained in $Y \subset X$.

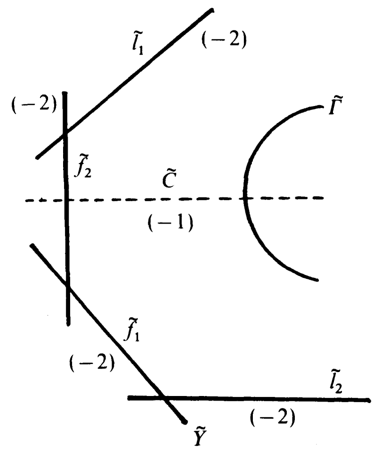


FIGURE 1

Since the multiplicity $m(\mathcal{O}_{Y,x})$ of the local ring $\mathcal{O}_{Y,x}$ is equal to two, any line through the point x must be contained in Y. Therefore C is a unique line in X through the singularity x of $Y=H_5$. Thus we have:

LEMMA 1.1. *Let $(X, Y)=(V_5, H_5)$ be a compactification of C^3 such that $Y=H_5$ is normal. Then Y has exactly one singular point x of A_4-type. Moreover, there exists a unique line C in X through the point x, which is contained in Y.*

2. Double projection from a point. We will study the double projection of $X=V_5$ from the singularity x of A_4-type of $Y=H_5$. For this purpose, let us consider the linear system

$$|H-2x|=|\mathcal{O}_X(H)\otimes\mathcal{M}_x^2|,$$

where H is a hyperplane section of X and $\mathcal{M}_x\subset\mathcal{O}_{X,x}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Let $\delta_1: X_1\to X$ be the blowing up of X at the point x and put $E_1:=\delta_1^{-1}(x)\cong P^2$. Let Y_1 and C_1 be the proper transform Y and C, respectively. Then we have:

LEMMA 2.1. $\dim|H-2x|=2$.

PROOF. Let us consider the exact sequences:

$$0\longrightarrow\mathcal{O}_{X_1}(\delta_1^*H-E_1)\longrightarrow\mathcal{O}_{X_1}(\delta_1^*H)\longrightarrow\mathcal{O}_{E_1}\longrightarrow 0$$

$$0\longrightarrow\mathcal{O}_{X_1}(\delta_1^*H-2E_1)\longrightarrow\mathcal{O}_{X_1}(\delta_1^*H-E_1)\longrightarrow\mathcal{O}_{E_1}(1)\longrightarrow 0$$

Since $\dim|H-x|=\dim H-1$, we have

$$H^0(X_1, \mathcal{O}_{X_1}(\delta_1^*H-E_1))\cong C^6, \quad\text{and}\quad H^1(X_1, \mathcal{O}_{X_1}(\delta_1^*H-E_1))\cong 0$$

Let $\mathscr{L} := \mathrm{Tr}_{E_1}|\delta_1^* H - E_1| \subseteq \mathcal{O}_{E_1}(1)|$ be the trace of the linear system $|\delta_1^* H - E_1|$ on E_1. Since $|\delta_1^* H - E_1|$ has no fixed component and no base point on X_1, neither does $\mathscr{L}$ on E_1. Therefore $\mathscr{L} = |\mathcal{O}_{E_1}(1)|$. Thus, we have a surjection

$$H^0(X_1, \mathcal{O}_{X_1}(\delta_1^* H - E_1)) \twoheadrightarrow H^0(E_1, \mathcal{O}_{E_1}(1)) \cong \boldsymbol{C}^3 .$$

This means that

$$H^0(X_1, \mathcal{O}_{X_1}(\delta_1^* H - 2E_1)) \cong \boldsymbol{C}^3 , \quad \text{and} \quad H^1(X_1, \mathcal{O}_{X_1}(\delta_1^* H - 2E_1)) \cong 0 . \qquad \text{q.e.d.}$$

By Lemma 2.1, we have rational maps

$$\Phi := \Phi_{|H-2x|} : X \dashrightarrow \boldsymbol{P}^2 , \quad \text{and} \quad \Phi^{(1)} := \Phi_{|\delta_1^* H - 2E_1|} : X_1 \dashrightarrow \boldsymbol{P}^2 .$$

Since $(\delta_1^* H - 2E_1)\cdot C_1 = -1 < 0$, C_1 is a base curve of the linear system $|\delta_1^* H - 2E_1|$.

Next, we will study the singularities of Y_1. Let Δ be a small neighborhood of x in X with a local coordinate system (z_1, z_2, z_3). Since the singularity $x \in Y = H_5$ is of A_4-type and C intersects the component $\tilde{f}_2$ of $\alpha^{-1}(x)$ in $\tilde{Y}$ (see Figure 1), we may assume that

$$(2.1) \qquad \begin{aligned} \Delta \cap Y &= \{z_1 \cdot z_2 = z_3^5\} \subsetneq \Delta \quad \text{with} \quad x = (0, 0, 0) , \\ \Delta \cap C &= \{z_1 = z_3^2, z_2 = z_3^3\} \subsetneq \Delta . \end{aligned}$$

By an easy calculation, we find that Y_1 has exactly one singular point x_1 of A_2-type. Then there exists a birational morphism $\mu_1 : \tilde{Y} \to Y_1$ such that

$$\mu_1^{-1}(x_1) = \tilde{f}_1 \cup \tilde{f}_2 , \quad \text{and} \quad \tilde{Y} - (\tilde{f}_1 \cup \tilde{f}_2) \overset{\mu_1}{\cong} Y_1 - \{x_1\} \text{ (isomorphic)} .$$

We put $l_i^{(1)} := \mu_1(\tilde{l}_i)$ $(1 \le i \le 2)$ and $C_1 = \mu_1(\tilde{C})$. Then we have

$$(2.2) \qquad E_1 \cdot Y_1 = l_1^{(1)} + l_2^{(1)} .$$

In particular, $l_1^{(1)}$, $l_2^{(1)}$ are two distinct lines on $E_1 \cong \boldsymbol{P}^2$ and C_1 is the proper transform of C in X_1.

Since $Y_1 \in |\delta_1^* H - 2E_1|$, by (2.2), we have

$$\mathcal{O}_{Y_1}(Y_1) = \mathcal{O}_{Y_1}(\delta_1^* H - 2E_1) = \mathcal{O}_{Y_1}(\Gamma^{(1)} - 2l_1^{(1)} - 2l_2^{(1)}) ,$$

where $\Gamma^{(1)} = \delta_1^*(Y|_H) = \mu_1(\tilde{\Gamma})$. We have

$$(2.3) \qquad \mu_1^* \mathcal{O}_{Y_1}(\Gamma^{(1)} - 2l_1^{(1)} - 2l_2^{(1)}) \cong \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma} - 2\tilde{f}_1 - 2\tilde{f}_2 - 2\tilde{l}_1 - 2\tilde{l}_2) \cong \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma} - 2Z) ,$$

where $Z = \tilde{f}_1 + \tilde{f}_2 + \tilde{l}_1 + \tilde{l}_2$ is the fundamental cycle of the singularity x associated with the resolution $(\tilde{Y}, \alpha)$. From the exact sequence

$$(2.4) \qquad 0 \longrightarrow \mathcal{O}_{X_1} \longrightarrow \mathcal{O}_{X_1}(Y_1) \longrightarrow \mathcal{O}_{Y_1}(Y_1) \longrightarrow 0 ,$$

we have

$$H^0(Y_1, \mathcal{O}_{Y_1}(Y_1)) \cong H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma} - 2Z)) \cong \boldsymbol{C}^2 ,$$

since $H^0(X_1, \mathcal{O}_{X_1}(Y_1)) \cong C^3$ by Lemma 2.1. Let $\{\psi_0, \psi_1\}$ be a basis of $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{\Gamma}-2Z))$ such that

$$(2.5)\qquad \begin{aligned}(\psi_0) &= 3\tilde{C}+2\tilde{f}_2+\tilde{f}_1+\tilde{f}_0 \\ (\psi_1) &= 5\tilde{C}+4\tilde{f}_2+2\tilde{f}_1+\tilde{l}_1\,,\end{aligned}$$

where $\tilde{f}_0$ is a smooth rational curve in $\tilde{Y}$ such that $(\tilde{f}_0{}^2)_{\tilde{Y}}=0$ and $(\tilde{f}_0 \cdot \tilde{l}_2)_{\tilde{Y}}=1$ (in fact, $\tilde{Y}$ can be regarded as a ruled surface over a smooth rational curve, which has $\tilde{f}_0$ as a fiber $\tilde{l}_2$ as a section). Since

$$(\psi_0) \cap (\psi_1) = \tilde{C} \cup \tilde{f}_1 \cup \tilde{f}_2\,,$$

we have the base locus

$$\mathrm{Bs}\,|\mathcal{O}_{Y_1}(Y_1)| = C_1 \ni x_1\,.$$

By (2.4), since $H^1(X_1, \mathcal{O}_{X_1})=0$, we have the base locus

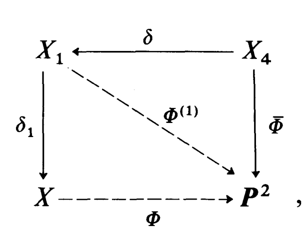
$$\mathrm{Bs}\,|\mathcal{O}_{X_1}(Y_1)| = C_1 \ni x_1\,.$$

Since $\mathrm{Pic}\, X \cong Z\mathcal{O}_X(H)$, $|H-2x|$ has no fixed component, hence neither does $|\delta_1^* H - 2E_1|$. Thus we have the following:

LEMMA 2.2. *The linear system $|\delta_1^* H - 2E_1|$ on X_1 has no fixed component, but has the base locus*

$$\mathrm{Bs}\,|\delta_1^* H - 2E_1| = C_1 \ni x_1\,.$$

3. Resolution of indeterminancy. The indeterminancy of the rational map $\Phi^{(1)}: X_1 \dashrightarrow P^2$ can be resolved as follows: First, let us consider the blowing up $\delta_2: X_2 \to X_1$ of X_1 along $C_1 \cong P^1$. Then $C_1' = \delta_2^{-1}(C_1) \cong F_2$. Next, let us consider the blowing up $\delta_3: X_3 \to X_2$ of X_2 along the negative section C_2 of $C_1' \cong F_2$. Then $C_2' := \delta_3^{-1}(C_2) \cong F_2$. Finally, let us consider the blowing up $\delta_4: X_4 \to X_3$ of X_3 along the negative section C_3 of $C_2' \cong F_2$. Then, we have a morphism $\bar{\Phi}: X_4 \to P^2$ and the following diagram:



where $\delta := \delta_2 \circ \delta_3 \circ \delta_4$. This is a desired resolution of the indeterminancy of the rational map $\Phi^{(1)}: X_1 \dashrightarrow P^2$.

NOTATION.

$\bar{C}'_j$: the proper transform of C'_j in X_4 $(1 \leq j \leq 2)$.
$f_j^{(j+1)}$: a fiber of the ruled surface C'_j.
C_{j+1}: a section of C'_j.
K_{X_j}: a canonical divisor on X_j.
$N_{C_j|X_j}$: the normal bundle of C_j in X_j.
Y_{j+1}: the proper transform of Y_j in X_{j+1}.
E_{j+1}: the proper transform of E_j in X_{j+1}.
$l_i^{(j+1)}$ $(i=1, 2)$: the proper transform of $l_i^{(j)}$ in X_{j+1}.
x_j: the singular point of $Y_j (1 \leqq j \leqq 2)$.
Δ_j: a neighborhood of x_j in X_j with a local coordinate system $(z_1, z_2, z_3) = (z_1^j, z_2^j, z_3^j)$.

For the proof, we need the following:

LEMMA 3.1 (Morrison [7]). *Let S be a surface with only one singularity x of A_n-type in a smooth projective 3-fold X. Let $E \subset S \subset X$ be a smooth rational curve in X. Let $\mu: \tilde{S} \to S$ be the minimal resolution of the singularity of S and put*

$$\mu^{-1}(x) = \bigcup_{j=1}^{n+1} C_j ,$$

where C_j's $(1 \leq j \leq n+1)$ are smooth rational curve with

$$(C_j^2)_{\tilde{S}} = -2 \qquad (1 \leq j \leq n+1) ,$$

$$(C_j \cdot C_{j+1})_{\tilde{S}} = 1 \qquad (1 \leq j \leq n) ,$$

$$(C_i \cdot C_j)_{\tilde{S}} = 0 \qquad \text{if } |i-j| \geq 2 .$$

Let $\tilde{E}$ be the proper transform of E in $\tilde{S}$. Assume that

(i) *$N_{\tilde{E}|\tilde{S}} \cong \mathcal{O}_{\tilde{E}}(-1)$, where $N_{\tilde{E}|\tilde{S}}$ is the normal bundle of $\tilde{E}$ in $\tilde{S}$, and*

(ii) *$\deg N_{E|X} = -2$, where $N_{E|X}$ is the normal bundle of E in X.*

Then we have

(1) *$N_{E|X} \cong \mathcal{O}_E \oplus \mathcal{O}_E(-2)$ if $x \in E$ and $(C_j \cdot \tilde{E})_{\tilde{S}} = 1$ for $j=1$ or $n+1$, or*

(2) *$N_{E|X} \cong \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1)$ if $x \notin E$.*

PROOF. In the proof of Theorem 3.2 in Morrison [7], we have only to replace the conormal bundle $\tilde{N}^*_{\tilde{E}|\tilde{S}} = \mathcal{O}_{\tilde{E}}(2)$ by $N^*_{\tilde{E}|\tilde{S}} = \mathcal{O}_{\tilde{E}}(1)$. q.e.d.

(Step I). Since $(K_{X_1} \cdot C_1) = 0$, we have $\deg N_{C|X_1} = -2$. Since $x_1 \in C_1$ and the normal bundle $N_{\tilde{C}|\tilde{Y}} \cong \mathcal{O}_{\tilde{C}}(-1)$ (see §2), by Lemma 3.1, we have

$$N_{C_1|X_1} \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_1}(-2) . \tag{3.1}$$

Since the singularity x_1 of Y_1 is of A_2-type and $(\tilde{C} \cdot \tilde{f}_2)_{\tilde{Y}} = 1$, we may assume that

$$\Delta_1 \cap Y_1 = \{z_1 z_2 = z_3^2\} \subsetneq \Delta_1 \tag{3.2}$$
$$\Delta_1 \cap C_1 = \{z_1 = z_3,\, z_2 = z_3^2\} \subsetneq \Delta_1 .$$

(Step II). Let $\delta_2 : X_2 \to X_1$ be the blowing up of X_1 along $C_1 \cong \boldsymbol{P}^1$. By (3.1), we have $\delta_2^{-1}(C_1) =: C_1' \cong \boldsymbol{F}_2$. By (3.2), we find that Y_2 has exactly one singularity x_2 of A_1-type. Then there exists a birational morphism $\mu_2 : \tilde{Y} \to Y_2$ such that $\mu_2^{-1}(x_2) = \tilde{f}_2$ and $\tilde{Y} - \tilde{f}_2 \cong Y_2 - \{x_2\}$. Furthermore, we have

(i) $C_2 = \mu_2(\tilde{C})$ is the negative section of $C_1' \cong \boldsymbol{F}_2$,
(ii) $Y_2 \cdot C_1' = f_1^{(2)} + C_2$,
(iii) $f_1^{(2)} = \mu_2(f_1) \subseteq Y_2 \cap E_2 \cap C_1'$ and $l_i^{(2)} = \mu_2(\tilde{l}_i) \subseteq Y_2 \cap E_2 (1 \le i \le 2)$,
(iv) $(l_i^{(2)} \cdot l_i^{(2)})_{E_2} = 0 (1 \le i \le 2)$ and $(f_1^{(2)} \cdot f_1^{(2)})_{E_2} = -1$.

Since $K_{X_2} = \delta_2^* K_{X_1} + C_1'$, we have $(K_{X_2} \cdot C_2) = 0$. Hence $\deg N_{C_2 | X_2} = -2$. Since $x_2 \in C_2$, by Lemma 3.1, we have

$$N_{C_2 | X_2} \cong \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(-2) . \tag{3.3}$$

Furthermore, we may assume that

$$\Delta_2 \cap Y_2 = \{z_1 z_2 = z_3^2\} \subsetneq \Delta_2 , \tag{3.4}$$
$$\Delta_2 \cap C_2 = \{z_1 = z_2 = z_3\} \subsetneq \Delta_2 .$$

(Step III). Let $\delta_3 : X_3 \to X_2$ be the blowing up of X_2 along C_2. By (3.3), we have $\delta_3^{-1}(C_2) =: C_2' \cong \boldsymbol{F}_2$. By (3.4), we find that Y_3 is a smooth surface. Then there exists an isomorphism $\mu_3 : \tilde{Y} \xrightarrow{\sim} Y_3$. Furthermore, we have:

(i) $C_3 = \mu_3(\tilde{C})$ is the negative section of $C_2' \cong \boldsymbol{F}_2$,
(ii) $Y_3 \cdot C_2' = f_2^{(3)} + C_3$,
(iii) $f_1^{(3)} = \mu_3(\tilde{f}_1) \subseteq Y_3 \cap \bar{C}_1' \cap E_3$, $f_2^{(3)} = \mu_3(\tilde{f}_2) \subseteq Y_3 \cap C_2' \cap E_3$, and $l_i^{(3)} = \mu_3(\tilde{l}_i) \subseteq Y_3 \cap E_3 (1 \le i \le 2)$,
(iv) $(l_1^{(3)} \cdot l_1^{(3)})_{E_3} = (f_2^{(3)} \cdot f_2^{(3)})_{E_3} = -1$, $(l_2^{(3)} \cdot l_2^{(3)})_{E_3} = 0$, $(C_3 \cdot l_1^{(3)})_{Y_3} = 0$, $(C_3 \cdot f_2^{(3)})_{Y_3} = 1$.

Since $(K_{X_3} \cdot C_3) = 0$, we have $\deg N_{C_3 | X_3} = -2$. Since Y_3 is smooth, by Lemma 3.1, we have

$$N_{C_3 | X_3} \cong \mathcal{O}_{C_3}(-1) \oplus \mathcal{O}_{C_3}(-1) . \tag{3.5}$$

(Step IV). Let $\delta_4 : X_4 \to X_3$ be the blowing up of X_3 along $C_3 \cong \boldsymbol{P}^1$. By (3.5), we have $\delta_4^{-1}(C_3) =: C_3' \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Since Y_3 is smooth, we also have an isomorphism $\mu_4 : \tilde{Y} \xrightarrow{\sim} Y_4$. We identify $\tilde{Y}$ and Y_4 via the isomorphism μ_4, and put, for simplicity, $\tilde{f}_i := \mu_4(\tilde{f}_i)$, $\tilde{l}_i := \mu_4(\tilde{l}_i) (1 \le i \le 2)$, $\tilde{\Gamma} := \mu_4(\tilde{\Gamma})$ and $\tilde{C} := \mu_4(\tilde{C})$. Then we have

(i) $\tilde{f}_i \subseteq Y_4 \cap E_4$, $\tilde{l}_i \subseteq Y_4 \cap E_4 (1 \le i \le 2)$, $\tilde{f} := f_3^{(4)} \subseteq C_3' \cap E_4$,
(ii) $\tilde{C} := C_4$ is a section of $C_3' \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ with $(\tilde{C} \cdot \tilde{C})_{C_3'} = 0$,

(iii) $Y_4 \cdot C'_3 = \tilde{C}$,

(iv) $(\tilde{l}_1 \cdot \tilde{l}_1)_{E_4} = -1$, $(\tilde{l}_2 \cdot \tilde{l}_2)_{E_4} = 0$, $(\tilde{f}_1 \cdot \tilde{f}_1)_{E_4} = (\tilde{f}_2 \cdot \tilde{f}_2)_{E_4} = -2$, $(\tilde{f} \cdot \tilde{f})_{E_4} = -1$.

Thus we have Figure 2 (see also Pagoda (5.8) in Reid [10]).

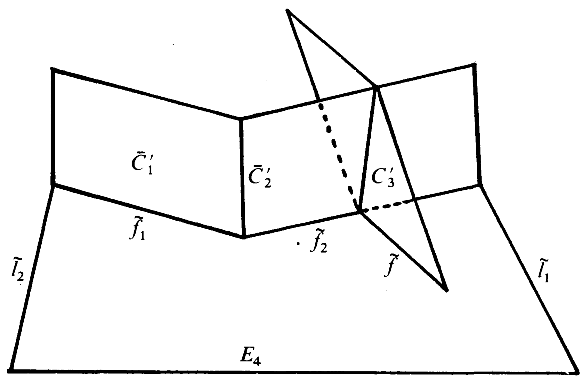


FIGURE 2

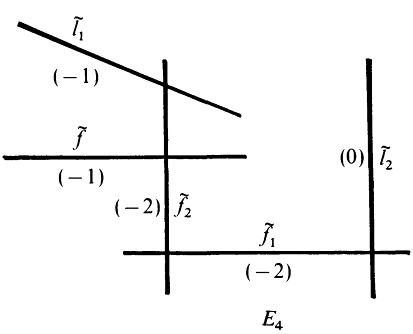


FIGURE 3

Now, since $Y_{j+1} = \delta^*_{j+1} Y_j - C'_j (1 \le j \le 3)$, we have

$$Y_4 = \delta^*_4 \delta^*_3 \delta^*_2 \delta^*_1 H - 2\delta^*_4 \delta^*_3 \delta^*_2 E - 3C'_3 - 2\bar{C}'_2 - \bar{C}'_1 .$$

Therefore we have

$$\mathcal{O}_{Y_4}(Y_4) = \mathcal{O}_{Y_4}(\tilde{\Gamma} - 2Z - \tilde{f}_1 - 2\tilde{f}_2 - 3\tilde{C}) = \mathcal{O}_{Y_4}(\tilde{f}_0)(\cong \mathcal{O}_{\tilde{Y}}(\tilde{f}_0)) ,$$

where $Z = \tilde{l}_1 + \tilde{l}_2 + \tilde{f}_1 + \tilde{f}_2$ (see (2.3)). Since $\tilde{f}_0$ is a general fiber of the rational ruled surface $\tilde{Y} = Y_4$, $|\mathcal{O}_{Y_4}(\tilde{f}_0)|$ has no fixed component and no base point. Thus, it defines a morphism $\varphi := \varphi_{|\mathcal{O}_{Y_4}(\tilde{f}_0)|} : Y_4 \to \boldsymbol{P}^1$. Then $Y_4 \xrightarrow{\varphi} \boldsymbol{P}^1$ is a ruled surface over $\boldsymbol{P}^1$ with exactly one singular fiber $2\tilde{C} + 2\tilde{f}_2 + 2\tilde{f}_1 + \tilde{l}_1$. In particular, $\tilde{l}_2$ is a section. Let us consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{X_4} \longrightarrow \mathcal{O}_{X_4} \longrightarrow \mathcal{O}_{Y_4}(Y_4) \longrightarrow 0 .$$

Since $H^1(X_4, \mathcal{O}_{X_4})=0$ and the linear system $|\mathcal{O}_{Y_4}(Y_4)|$ has no fixed component and no base point, neigher does $|Y_4|:=|\mathcal{O}_{X_4}(Y_4)|$. Therefore, it defines a morphism $\bar{\Phi}:=\bar{\Phi}_{|Y_4|}: X_4 \to \boldsymbol{P}^2$ of X_4 onto $\boldsymbol{P}^2$ such that $\bar{\Phi}^*\mathcal{O}_{\boldsymbol{P}^2}(1)=\mathcal{O}_{X_4}(Y_4)$. Thus, we have the following:

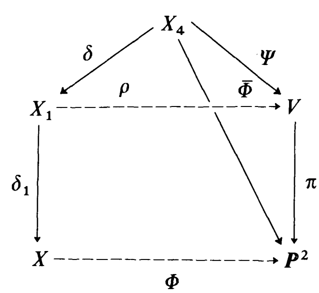
PROPOSITION 3.2. *There exists a morphism* $\bar{\Phi}: X_4 \to \boldsymbol{P}^2$ *of* X_4 *onto* $\boldsymbol{P}^2$ *with* $\bar{\Phi}^*\mathcal{O}_{\boldsymbol{P}^2}(1)=\mathcal{O}_{X_4}(Y_4)$, *which is a resolution of the indeterminancy of the rational map* $\Phi^{(1)}: X_1 \dashrightarrow \boldsymbol{P}^2$.

4. Structure of V_5. Let X_4, Y_4, and $C'_3 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be as in §3. Since

$$N_{C_3|X_3} \cong \mathcal{O}_{C_3}(-1) \oplus \mathcal{O}_{C_3}(-1),$$

by Corollary 5.6 in [10], there exists a birational morphism $\phi: X_4 \to V$ of X_4 onto a smooth 3-fold V with the second Betti number $b_2(V)=2$, and a morphism $\pi: V \to \boldsymbol{P}^2$ of V onto $\boldsymbol{P}^2$, and a birational map $\rho: X_1 \dashrightarrow V$ which is called a flip such that $\rho=\phi\circ\delta^{-1}$ and $\bar{\Phi}=\pi\circ\phi$. Thus we have the diagram $(*)$:

(*)



In particular, $f:=\phi(\bar{C}'_1 \cup \bar{C}'_2 \cup C'_3)$ is a smooth rational curve in V, and

$$X_4-(\bar{C}'_1 \cup \bar{C}'_2 \cup C'_3) \xrightarrow[\sim]{\phi} V-f \xleftarrow[\sim]{\rho} X_1 - C_1 . \tag{4.1}$$

We put $A:=\phi(Y_4)$ and $\Sigma:=\phi(E_4)$. Then,

$$-K_V=2A+2\Sigma . \tag{4.2}$$

$$\mathcal{O}_V(A)=\pi^*\mathcal{O}_{\boldsymbol{P}^2}(1) . \tag{4.3}$$

Indeed, since $-K_{X_1}=2\delta_1^*H-2E_1=2Y_1+2E_1$ and $\mathcal{O}_{X_4}(Y_4)=\bar{\Phi}^*\mathcal{O}_{\boldsymbol{P}^2}(1)$, by (4.1), we have (4.2), (4.3). We put $l_i:=\phi(\bar{l}_i)(1\le i\le 2)$ and $L_0:=\pi(l_2)\subsetneq \boldsymbol{P}^2$. Then l_i's are smooth rational curves in V and L_0 is a line in $\boldsymbol{P}^2$. In particular, $\pi|_A: A \to L_0$ has a structure of the $\boldsymbol{P}^1$-boundle $\boldsymbol{F}_1$ with l_1 a fiber and l_2 the negative section. Moreover, Σ has only one singularity q of A_2-type. The rational curves l_1, l_2, f, which are also contained in Σ, intersect only at the point q (see Figure 4).

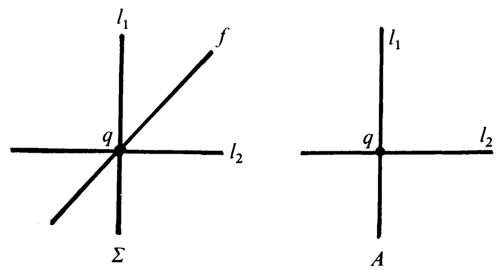


FIGURE 4

By construction, $\sigma := \phi|_{E_4}: E_4 \to \Sigma$ is the minimal resolution of the singularity of Σ with $\sigma^{-1}(q) = \tilde{f}_1 \cup \tilde{f}_2$, and $l_i = \sigma(\tilde{l}_i)(1 \le i \le 2)$, $f = \sigma(\tilde{f})$ (see (i)–(iv) of Step IV and Figure 4). We put $\lambda := \pi|_\Sigma: \Sigma \to \boldsymbol{P}^2$. Then

$$(\lambda \circ \sigma)(\tilde{f}_1 \cup \tilde{f}_2 \cup \tilde{l}_1) = L_0 \cdot L_\infty = \{\mathrm{p}\} \text{ (a point)}, \tag{4.4}$$

where $L_\infty := \pi(f)$ is a line in $\boldsymbol{P}^2$.

For a general fiber F of the morphism $\pi: V \to \boldsymbol{P}^2$, we have, by (4.2),

$$\deg K_F = (K_V \cdot F) = -2(\Sigma \cdot F) \le -2 .$$

Hence, $F \cong \boldsymbol{P}^1$ and $(\Sigma \cdot F)_V = 1$, where K_F is a canonical divisor on F. Therefore Σ is a meromorphic section of $\pi: V \to \boldsymbol{P}^2$.

PROPOSITION 4.1. *$\pi: V \to \boldsymbol{P}^2$ is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$ and Σ is a holomorphic section on $\boldsymbol{P}^2 - \{p\}$.*

PROOF. By construction,

$$\boldsymbol{C}^3 \cong X - Y \xleftarrow[\sim]{\delta_1} X_1 - (Y_1 \cup E_1) \xrightarrow[\sim]{\rho} V - (A \cup \Sigma) .$$

In particular, $\pi: V - (A \cup \Sigma) \to \boldsymbol{P}^2 - L_0$ is an affine morphism. Assume that there exists an irreducible divisor D on V such that $\pi(D) = \{\text{one point}\}$. Then the one-dimensional scheme $D \cap \Sigma$ is contracted to one point, hence, $\mathrm{Supp}(D \cap \Sigma) = l_1$. Since $l_1 \subseteq A = \pi^{-1}(L_0)$ and $\pi|_A: A \to L_0$ is a $\boldsymbol{P}^1$-bundle, this is a contradiction. Thus π is equi-dimensional, hence, π is a proper flat morphism. Let G be an arbitrary scheme-theoric fiber. Then $(\Sigma \cdot G)_V = 1$. Since $V - (A \cup \Sigma) \cong \boldsymbol{C}^3$ contains no compact analytic curve, G must be irreducible. Since $(K_V \cdot G) = -2(\Sigma \cdot G) = -2$, we see that G is a smooth rational curve. Therefore $\pi: V \to \boldsymbol{P}^2$ is a smooth proper morphism. By the upper semicontinuity theorem, we have that $R^1\pi_*\mathcal{O}_V(\Sigma) = 0$ and $\pi_*\mathcal{O}_V(\Sigma)$ is a vector bundle of rank 2 over $\boldsymbol{P}^2$. Moreover, for every point $x \in \boldsymbol{P}^2$,

$$\pi_*\mathcal{O}_V(\Sigma) \otimes \boldsymbol{C}(x) \cong H^0(\pi^{-1}(x), \mathcal{O}_V(\Sigma) \otimes \mathcal{O}_{\pi^{-1}(x)}) \cong H^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(1)) \cong \boldsymbol{C}^2 .$$

Thus the natural homomorphism $\pi^*\pi_*\mathcal{O}_V(\Sigma) \twoheadrightarrow \mathcal{O}_V(\Sigma)$ is surjective and induces an isomorphism $V \cong \boldsymbol{P}(\pi_* \mathcal{O}_V(\Sigma))$ over $\boldsymbol{P}^2$. The rest is clear. q.e.d.

REMARK. π is the contraction of an extremal ray of the smooth projective 3-fold V.

Finally, we will study the vector bundle $\pi_* \mathcal{O}_V(\Sigma)$ of rank 2 over $\boldsymbol{P}^2$.

LEMMA 4.2. $\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3l_1) \otimes \mathcal{O}_V(A)$.

PROOF. Since the singularity of Σ is a rational double point, we have $\sigma^* K_\Sigma = K_{E_4} = -2\tilde{f}_1 - \tilde{f}_2 - 3\tilde{l}_2$, hence, $K_\Sigma = -3l_2$. On the other hand, since $K_\Sigma = (K_V + \Sigma)|_\Sigma = -2A|_\Sigma - \Sigma|_\Sigma$, we have $\Sigma|_\Sigma = -2A|_\Sigma + 3l_2$. Since $A|_\Sigma = l_1 + l_2$, we have $\Sigma|_\Sigma = -3l_1 + A|_\Sigma$, namely, $\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3l_1) \otimes \mathcal{O}_V(A)$. q.e.d.

Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(\Sigma) \longrightarrow \mathcal{O}_\Sigma(\Sigma) \longrightarrow 0 .$$

Taking π_*, we have

$$(4.5) \qquad 0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^2} \longrightarrow \pi_* \mathcal{O}_V(\Sigma) \longrightarrow \pi_* \mathcal{O}_\Sigma(\Sigma) \longrightarrow 0 .$$

Taking π^* in (4.5), we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_V & \longrightarrow & \pi^*\pi_*\mathcal{O}_V(\Sigma) & \longrightarrow & \pi^*\pi_*\mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 \\ & & \wr\| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_V & \longrightarrow & \mathcal{O}_V(\Sigma) & \longrightarrow & \mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 . \end{array}$$

In particular, we have a surjection

$$\pi^*\pi_*\mathcal{O}_\Sigma(\Sigma) \twoheadrightarrow \mathcal{O}_\Sigma(\Sigma) .$$

We put $\lambda := \pi|_\Sigma : \Sigma \to \boldsymbol{P}^2$. Taking λ^* in (4.5), we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_\Sigma & \longrightarrow & \lambda^*\pi_*\mathcal{O}_V(\Sigma) & \longrightarrow & \lambda^*\pi_*\mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 \\ & & \downarrow & & \wr\| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \lambda^*\pi_*\mathcal{O}_V(\Sigma) & \xrightarrow{\ \tau\ } & \mathcal{O}_\Sigma(\Sigma) & \longrightarrow & 0 , \end{array}$$

where $\mathcal{K} := \ker \tau$ is a line bundle, and the image of the global section 1 of $\mathcal{O}_\Sigma$ via map $\mathcal{O}_\Sigma \to \mathcal{K}$ defines an effective Cartier divisor D with Supp $D = l_1$.

PROPOSITION 4.3. *$\lambda^*\pi_*\mathcal{O}_V(\Sigma)$ is an extension of $\mathcal{O}_\Sigma(\Sigma)$ by $\mathcal{O}_\Sigma(3l_1)$.*

PROOF. We have only to prove that $D = 3l_1$. Since $\lambda^*(\det(\pi_*\mathcal{O}_V(\Sigma))) = \mathcal{O}_\Sigma(\Sigma) \otimes \mathcal{O}_\Sigma(3l_1)$, we have $(\Sigma \cdot l_1)_\Sigma + (D \cdot l_1)_\Sigma = 0$. Since $\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3l_1) \otimes \mathcal{O}_V(A)$ by Lemma 4.2, we must have $D = 3l_1$, and also, by (4.3), we have $\det(\pi_*\mathcal{O}_V(\Sigma)) = \mathcal{O}_{\boldsymbol{P}^2}(1)$. q.e.d.

REMARK. We put $\mathscr{J} := \lambda_* \mathcal{O}_\Sigma(-3l_1)$. Then $\mathscr{J}$ is an ideal locally generated by two polynomials xy and $y - x^2$ over $\boldsymbol{C}[x, y]$. We put $\mathscr{E} := \pi_* \mathcal{O}_V(\Sigma)$. Since $\mathcal{O}_\Sigma(\Sigma) = \mathcal{O}_\Sigma(-3l_1) \otimes \lambda^* \mathcal{O}_{\boldsymbol{P}^2}(1)$, by (4.5), we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^2} \longrightarrow \mathscr{E} \longrightarrow \mathscr{J} \cdot \mathcal{O}_{\boldsymbol{P}^2}(1) \longrightarrow 0 .$$

By Lemma 1.3.4 [8, p. 186–p. 187], $\mathscr{E}$ is a stable vector bundle of rank 2 over $\boldsymbol{P}^2$.

Thus we have finally the following:

PROPOSITION 4.4. *Let (X_1, Y_1), $E_1 \cong \boldsymbol{P}^2$, C_1 be as in §1. Then one can construct a birational map $\rho: X_1 \to \boldsymbol{P}(\mathscr{E})$ of X_1 to a $\boldsymbol{P}^1$-bundle $\pi: \boldsymbol{P}(\mathscr{E}) \to \boldsymbol{P}^2$ ($\mathscr{E}$ is a stable vector bundle of rank two over $\boldsymbol{P}^2$) with the following properties:*

(1) *There is a smooth rational curve f contained in $\Sigma := \rho(E_1)$ such that*
$X_1 - C_1 \overset{\rho}{\cong} \boldsymbol{P}(\mathscr{E}) - f$ *(isomorphic).*

(2) *There is a point $p \in \boldsymbol{P}^2$ such that $\pi^{-1}(p) \subsetneq \Sigma$ and*
$\Sigma - \pi^{-1}(p) \cong \boldsymbol{P}^2 - \{p\}$.

(3) *$L_0 := \pi(A)$ is a line in $\boldsymbol{P}^2$ through p, where $A := \rho(Y_1)$. In particular, $\pi|_A: A \to L_0$ is a $\boldsymbol{P}^1$-bundle over L_0.*

(4) $X - Y \overset{\delta_1}{\cong} X_1 - (Y_1 \cup E_1) \overset{\rho}{\cong} \boldsymbol{P}(\mathscr{E}) - (A \cup \Sigma)$.

5. A construction and the proof of Theorem. Take any fixed line L_∞ in $\boldsymbol{P}^2$ and a point $p \in L_\infty$. Let $L_t (t \in \boldsymbol{C}, t \neq \infty)$ be a line in $\boldsymbol{P}^2$ through the point p. Let E_4 be a rational surface obtained from $\boldsymbol{P}^2$ by succession of three blowing ups at p (infinitely near points allowed). Let $\mu: E_4 \to \boldsymbol{P}^2$ be the projection with $\mu^{-1}(p) = \tilde{f}_1 \cup \tilde{f}_2 \cup \tilde{l}_1$, where $(\tilde{f}_i \cdot \tilde{f}_i)_{E_4} = -2 (1 \leq i \leq 2)$, $(\tilde{l}_1 \cdot \tilde{l}_1)_{E_4} = -1$, $(\tilde{f}_1 \cdot \tilde{f}_2)_{E_4} = 1$, $(\tilde{f}_1 \cdot \tilde{l}_1)_{E_4} = 0$, and $(\tilde{f}_2 \cdot \tilde{l}_1)_{E_4} = 1$. Let $\tilde{f}$ (resp. $\tilde{l}_2$) be the proper transform of L_∞ (resp. L_t) in E_4. Let $\sigma: E_4 \to \Sigma$ be the contraction of the exceptional set $\tilde{f}_1 \cup \tilde{f}_2$, and put $f := \sigma(\tilde{f})$, $l_i := \sigma(\tilde{l}_i)$ $(i = 1, 2)$. Then there is a birational morphism $\lambda: \Sigma \to \boldsymbol{P}^2$ such that $\lambda(l_1) = p$, $\lambda(l_2) = L_t$, $\lambda(f) = L_\infty$. Thus we have the following diagram:

$$\begin{array}{ccc} E_4 & & \\ {\scriptstyle\sigma}\downarrow & \searrow{\scriptstyle\mu} & \\ \Sigma & \xrightarrow{\ \lambda\ } & \boldsymbol{P}^2 \end{array}$$

(see Figure 5).

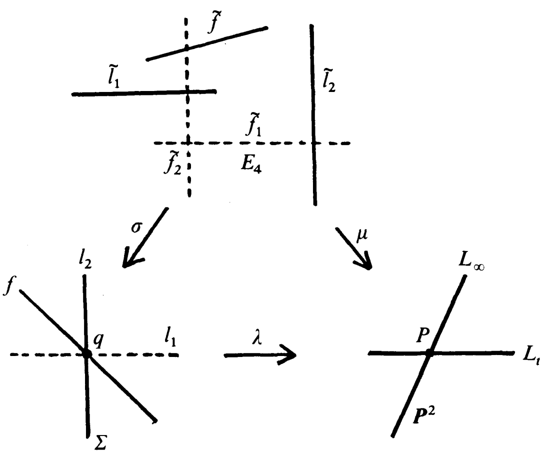


FIGURE 5

LEMMA 5.1. *As Q-divisors, we have*

$$\sigma^* l_1 \sim_{Q} \tilde{l}_1 + \frac{1}{3}\tilde{f}_1 + \frac{2}{3}\tilde{f}_2$$

(5.1)
$$\sigma^* l_2 \sim_{Q} \tilde{l}_2 + \frac{2}{3}\tilde{f}_1 + \frac{1}{3}\tilde{f}_2$$

$$\sigma^* f \sim_{Q} \tilde{f} + \frac{1}{3}\tilde{f}_1 + \frac{2}{3}\tilde{f}_2 ,$$

and the linear equivalences

$$\tilde{l}_1 + \tilde{f}_2 + \tilde{f}_3 \sim \tilde{l}_2$$

(5.2)
$$l \sim l_2 + l_1 \sim f + 2l_1$$

$$K_{E_4} = \sigma^* K_\Sigma \sim \sigma^*(-3l) + \tilde{f}_1 + 2\tilde{f}_2 + 3\tilde{l}_1 ,$$

where K_{E_4} is a canonical divisor on E_4, and $l := \lambda^* \mathcal{O}_{P^2}(1)$.

PROOF. Since $(\sigma^* l_1 \cdot \tilde{f}_i) = (\sigma^* l_2 \cdot \tilde{f}_i) = (\sigma^* f \cdot \tilde{f}_i) = 0$ for $i = 1, 2$, we have (5.1). By a similar calculation, we have (5.2). q.e.d.

Now, we will prove the existence of a vector bundle of rank 2 over P^2 which is an extension of $\mathcal{O}_\Sigma(-3l_1 + l)$ by $\mathcal{O}_\Sigma(3l_1)$.

LEMMA 5.2. (1) $\mathrm{Ext}^1_\Sigma(\mathcal{O}_\Sigma(-3l_1+l), \mathcal{O}_\Sigma(3l_1)) \cong \mathrm{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3l_1+l), \sigma^*\mathcal{O}_\Sigma(3l_1))$.

(2) $\mathrm{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3l_1+l), \sigma^*\mathcal{O}_\Sigma(3l_1)) \twoheadrightarrow \mathrm{Ext}^1_{\tilde{l}_1}(\sigma^*\mathcal{O}_\Sigma(-3l_1+l)\otimes\mathcal{O}_{\tilde{l}_i}, \sigma^*\mathcal{O}_\Sigma(3l_1)\otimes \mathcal{O}_{\tilde{l}_i})$ *is surjective.*

(3) $\dim \mathrm{Ext}^1_\Sigma(\mathcal{O}_\Sigma(-3l_1+l), \mathcal{O}_\Sigma(3l_1))=3$ *and*

$$\dim \mathrm{Ext}^1_{\tilde{l}_1}(\sigma^*\mathcal{O}_\Sigma(-3l_1+l)\otimes\mathcal{O}_{\tilde{l}_1}, \sigma^*\mathcal{O}_\Sigma(3l_1)\otimes\mathcal{O}_{\tilde{l}_1})=1.$$

PROOF. (1) $\mathrm{Ext}^1_\Sigma(\mathcal{O}_\Sigma(-3l_1+l), \mathcal{O}_\Sigma(3l_1))\cong H^1(\Sigma, \mathcal{O}_\Sigma(6l_1-l))$ and $\mathrm{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3l_1+l), \sigma^*\mathcal{O}_\Sigma(3l_1))\cong H^1(E_4, \sigma^*\mathcal{O}_\Sigma(6l_1-l))$, we have only to prove $H^1(\Sigma, \mathcal{O}_\Sigma(6l_1-l))\xrightarrow{\sim} H^1(E_4, \sigma^*\mathcal{O}_\Sigma(6l_1-l))$, which is clear, since $R^1\sigma_*\mathcal{O}_{E_4}=0$.

(2) We have only to prove that the morphism

$$H^1(E_4, \sigma^*\mathcal{O}_\Sigma(6l_1-l)) \twoheadrightarrow H^1(\tilde{l}_1, \sigma^*\mathcal{O}_\Sigma(6l_1-l)\otimes\mathcal{O}_{\tilde{l}_1})$$

is surjective. For this purpose, let us consider the exact sequence:

$$0\longrightarrow \sigma^*\mathcal{O}_\Sigma(6l_1-l)\otimes\mathcal{O}_{E_4}(-\tilde{l}_1)\longrightarrow \sigma^*\mathcal{O}_\Sigma(6l_1-l)\longrightarrow \sigma^*\mathcal{O}_\Sigma(6l_1-l)\otimes\mathcal{O}_{\tilde{l}_1}\longrightarrow 0\,.$$

By Lemma 5.1, we have

$$\sigma^*\mathcal{O}_\Sigma(6l_1-l)\cong\mathcal{O}_{E_4}(6\tilde{l}_1+2\tilde{f}_1+4\tilde{f}_2-\sigma^*l)\cong\mathcal{O}_{E_4}(2K_{E_4}+5\sigma^*l)$$

hence,

$$H^2(E_4, \mathcal{O}_{E_4}(2K_{E_4}+5\sigma^*l-\tilde{l}_1))\cong H^0(E_4, \mathcal{O}_{E_4}(-K_{E_4}-5\sigma^*l)$$
$$\cong H^0(E_4, \mathcal{O}_{E_4}(-2\sigma^*l-\tilde{f}_1-2\tilde{f}_2-2\tilde{l}_1))\cong 0\,.$$

Therefore, we have a surjection

$$H^1(E_4, \sigma^*\mathcal{O}_{E_4}(6l_1-l))\twoheadrightarrow H^1(\tilde{l}_1, \sigma^*\mathcal{O}_{E_4}(6l_1-l)\otimes\mathcal{O}_{\tilde{l}_1})\,.$$

(3) Since $(\sigma^*(-3l_1+l)\cdot\tilde{l}_1)_{E_4}=1$, $(\sigma^*(3l_1)\cdot\tilde{l}_1)_{E_4}=-1$, we have

$\mathrm{Ext}^1_{\tilde{l}_1}(\sigma^*\mathcal{O}_\Sigma(-3l_1+l)\otimes\mathcal{O}_{\tilde{l}_1}, \sigma^*\mathcal{O}_\Sigma(3l_1)\otimes\mathcal{O}_{\tilde{l}_1})\cong\mathrm{Ext}^1_{\boldsymbol{P}^1}(\mathcal{O}(1), \mathcal{O}(-1))\cong H^1(\boldsymbol{P}^1, \mathcal{O}(-2))\cong\boldsymbol{C}$.

Finally, we prove that $H^1(E_4, \mathcal{O}_{E_4}(2K_{E_4}+5\sigma^*l))\cong\boldsymbol{C}^3$. By Lemma 5.1, we have

$$2K_{E_4}+5\sigma^*l=-\sigma^*l+2\tilde{f}_1+4\tilde{f}_2+6\tilde{l}_1\,.$$

Since $\tilde{f}_1\cup\tilde{f}_2\cup\tilde{l}_1$ can be contracted to a smooth point, we have

$$H^0(E_4, \mathcal{O}_{E_4}(-\sigma^*l+2\tilde{f}_1+4\tilde{f}_2+6\tilde{l}_1))=0\,,$$

$$H^2(E_4, \mathcal{O}_{E_4}(-\sigma^*l+2\tilde{f}_1+4\tilde{f}_2+6\tilde{l}_1))\cong H^0(E_4, \mathcal{O}_{E_4}(-2\sigma^*l-\tilde{f}_1-2\tilde{f}_2-3\tilde{l}_1))=0\,.$$

By the Riemann-Roch theorem, we have easily

$$\dim H^1(E_4, \mathcal{O}_{E_4}(-\sigma^*l+2\tilde{f}_1+4\tilde{f}_2+6\tilde{l}_1))=3\,,$$

hence, $H^1(E_4, \mathcal{O}_{E_4}(2K_{E_4}+5\sigma^*l))\cong\boldsymbol{C}^3$. q.e.d.

The following is well-known (cf. [8]):

LEMMA 5.3. *Let $v: S\to T$ be the blowing up at the point p on a smooth surface T, and put $v^{-1}(p)=C$. Then a vector bundle $\mathcal{E}$ on S is the pull back of a vector bundle* *on T if and only if*

$$\mathscr{E}\,|_C \cong \mathcal{O}_C^{\otimes r},$$

where $r = \operatorname{rank} \mathscr{E}$.

Let $\mathscr{E} := \mathscr{E}_\xi$ be the vector bundle on E_4 determined by an element $\xi \in \operatorname{Ext}^1_{E_4}(\sigma^*\mathcal{O}_\Sigma(-3l_1+l), \sigma^*\mathcal{O}_\Sigma(3l_1))$, where the image of ξ by the surjection in Lemma 5.2, (2) is not zero. Then $\mathscr{E} \otimes \mathcal{O}_{\bar{l}_1}$ induces a non-split exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{l}_1}(-1) \longrightarrow \mathscr{E} \otimes \mathcal{O}_{\bar{l}_1} \longrightarrow \mathcal{O}_{\bar{l}_1}(1) \longrightarrow 0,$$

hence, $\mathscr{E} \otimes \mathcal{O}_{\bar{l}_1} \cong \mathcal{O}_{\bar{l}_1} \oplus \mathcal{O}_{\bar{l}_1}$.

On the other hand, we have

$$\sigma^*\mathcal{O}_\Sigma(-3l_1+l) \otimes \mathcal{O}_{\bar{f}_i} \cong \mathcal{O}_{\bar{f}_i},\ \sigma^*\mathcal{O}_\Sigma(3l_1) \otimes \mathcal{O}_{\bar{f}_i} \cong \mathcal{O}_{\bar{f}_i}$$

for $i=1, 2$. Thus $\mathscr{E} \otimes \mathcal{O}_{\bar{f}_i} \cong \mathcal{O}_{\bar{f}_i}^{\oplus 2}$ for $i=1, 2$.

By Lemma 5.3, there exists a vector bundle $\mathscr{E}$ on $\boldsymbol{P}^2$ such that $\mathscr{E} = \mu^*\mathscr{E}$, and then we have an exact sequence

$$0 \longrightarrow \sigma^*\mathcal{O}_\Sigma(3l_1) \longrightarrow \mu^*\mathscr{E} \longrightarrow \sigma^*\mathcal{O}_\Sigma(-3l_1+l) \longrightarrow 0. \tag{5.3}$$

Taking σ_*, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma(3l_1) \longrightarrow \lambda^*\mathscr{E} \longrightarrow \mathcal{O}_\Sigma(-3l_1+l) \longrightarrow 0. \tag{5.4}$$

Further, taking λ_*, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^2} \longrightarrow \mathscr{E} \longrightarrow \lambda_*\mathcal{O}_\Sigma(-3l_1) \otimes \mathcal{O}_{\boldsymbol{P}^2}(1) \longrightarrow 0, \tag{5.5}$$

since $R^1\lambda_*\mathcal{O}_\Sigma(3l_1) = 0$ by the Grauert-Riemenschneider vanishing theorem.

We remark that $\lambda: \Sigma \to \boldsymbol{P}^2$ is the blowing up of $\boldsymbol{P}^2$ along the ideal $\mathscr{J} := \lambda_*\mathcal{O}_\Sigma(-3l_1)$. By (5.4), we have a $\boldsymbol{P}^1$-bundle $V := \boldsymbol{P}(\mathscr{E}) \xrightarrow{\pi} \boldsymbol{P}^2$ and a rational section $\Sigma \subsetneq V$.

LEMMA 5.4. $\mathscr{E} \otimes \mathcal{O}_{L_t} \cong \mathcal{O}_{L_t}(1) \oplus \mathcal{O}_{L_t}$.

PROOF. Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma(3l_1) \otimes \mathcal{O}_{l_2} \longrightarrow \lambda^*\mathscr{E} \otimes \mathcal{O}_{l_2} \longrightarrow \mathcal{O}_\Sigma(-3l_1+l) \otimes \mathcal{O}_{l_2} \longrightarrow 0.$$

Since $(3l_1 \cdot l_2)_\Sigma = (l \cdot l_2)_\Sigma = 1$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^1}(1) \longrightarrow \lambda^*\mathscr{E} \otimes \mathcal{O}_{\boldsymbol{P}^1} \longrightarrow \mathcal{O}_{\boldsymbol{P}^1} \longrightarrow 0.$$

Therefore, $\lambda^*\mathscr{E} \otimes \mathcal{O}_{l_2} \cong \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}$. q.e.d.

COROLLARY 5.5. $\pi^{-1}(L_t) =: A$ *is the* $\boldsymbol{P}^1$*-bundle* $\boldsymbol{F}_1$ *over* $L_t \cong \boldsymbol{P}^1$.

LEMMA 5.6. $N_{f|V} \cong \mathcal{O}_{\boldsymbol{P}^1}(-2) \oplus \mathcal{O}$, *where* $N_{f|V}$ *is the normal bundle of* $f\,(\subsetneq \Sigma)$ *in* V.

PROOF. Let K_V be a canonical divisor on V. Then we have

$$K_V=\pi^*(K_{P^2}+\det \mathscr{E})-2\Sigma=-2A-2\Sigma\ .$$

Since $\mathcal{O}_\Sigma(\Sigma)=\mathcal{O}_\Sigma(-3l_1+l)$, we have $(K_V\cdot f)=(-4l+6l_1\cdot f)_\Sigma=-4+4=0$. Thus, by Lemma 3.1, we have the claim. q.e.d.

LEMMA 5.7. *$V-(\Sigma\cup A)$ is algebraically isomorphic to C^3.*

PROOF. Since $\Sigma-\pi^{-1}(p)\longrightarrow P^2-\{p\}$ and $p\in L_t$, the morphism $\pi|_{P(\mathscr{E})-(\Sigma\cup A)}$: $P(\mathscr{E})-(\Sigma\cup A)\to P^2-L_t$ gives an algebraic C-bundle structure on $P^2-L_t\cong C^2$. Therefore, by Quillen [10], we have $P(\mathscr{E})-(\Sigma\cup A)\cong C^3$. q.e.d.

Let $\phi_1: V_1\to V:=P(\mathscr{E})$ be the blowing up along f and put $C_1'=\phi_1^{-1}(f)$. Then $C_1'\cong F_2$ by Lemma 5.6. Let Σ_1 be the proper transform of Σ in V_1. Then Σ_1 has the singularity q_1 of A_1-type, and there exists a birational morphism $v_1: E_4\to\Sigma_1$ such that $v_1^{-1}(q_1)=\tilde{f}_2$ and $E_4-\tilde{f}_2\overset{v_1}{\cong}\Sigma_1-\{q_1\}$. We put $f_1^{(1)}:=v_1(\tilde{f}_1)$ and $f^{(1)}:=v_1(\tilde{f})$. Then $\Sigma_1\cdot C_1'=f_1^{(1)}+f^{(1)}$. In particular, $f_1^{(1)}$ is a fiber and $f^{(1)}$ is the negative section of $C_1'\cong F_2$. Since $q_1\in f^{(1)}$ and $(K_{V_1}\cdot f^{(1)})=(K_V\cdot f)=0$, by Lemma 3.1, we have

$$N_{f^{(1)}|V_1}\cong\mathcal{O}\oplus\mathcal{O}(-2)\ .$$

Let $\phi_2: V_2\to V_1$ be the blowing up along the curve $f^{(1)}$ and put $C_2'=\phi_2^{-1}(f^{(1)})\cong F_2$. Let Σ_2 be the proper transform of Σ_1 in V_2. Then Σ_2 is a smooth surface and there is an isomorphism $v_2: E_4\xrightarrow{\sim}\Sigma_2$. We put $f_i^{(2)}:=v_2(\tilde{f}_i)$ $(i=1, 2)$ and $f^{(2)}=v_2(\tilde{f})$. Then we have $\Sigma_2\cdot C_2'=f_2^{(2)}+f^{(2)}$. In particular, $f_2^{(2)}$ is a fiber and $f^{(2)}$ is the negative section of $C_2'=F_2$. Since $(K_{V_2}\cdot f^{(2)})=(K_{V_1}\cdot f^{(1)})=0$ and Σ_2 is smooth, by Lemma 3.1, we have

$$N_{f^{(2)}|V_2}\cong\mathcal{O}(-1)\oplus\mathcal{O}(-1)\ .$$

Let $\phi_3: V_3\to V_2$ be the blowing up along $f^{(2)}$ and put $C_3'=\phi_3^{-1}(f^{(2)})\cong P^1\times P^1$. Let $\tilde{C}$ be a fiber of the ruled surface $\phi_3|_{C_3'}: C_3'\to f^{(2)}$, and Σ_3 be the proper transform of Σ_2 in V_3. Then Σ_3 is a smooth surface and there exists an isomorphism $v_3: E_4\xrightarrow{\sim}\Sigma_3$. We put $\tilde{f}_i:=v_3(\tilde{f}_i)$ $(i=1, 2)$, $\tilde{f}=v_3(\tilde{f})$, $\tilde{l}_i:=v_3(\tilde{l}_i)$ $(i=1, 2)$. Then, $\Sigma_3\cdot C_3'=\tilde{f}$. In particular, $(\tilde{f}\cdot\tilde{f})_{C_3'}=0$ and $(\tilde{f}\cdot\tilde{C})_{C_3'}=1$ (see Step IV and Figure 2 in §4).

Since $C_3'\cong P(\mathcal{O}-1)\oplus\mathcal{O}(-1))$, by Corollary 5.6 in [10], C_3' can be blown down along the fiber $\tilde{f}$. After step by step blowing down, we finally have a smooth 3-fold X_1 with $b_2(X_1)=2$ and the contraction morphism $\delta: V_3\to X_1$. We put $C_1:=\delta(C_3'\cup\bar{C}_2'\cup\bar{C}_1')$, $E_1:=\delta(\Sigma_3)$, and $Y_1:=\delta(A_3)$, where $\bar{C}_j'$ $(j=1, 2)$, A_3 are the proper transforms of C_j' $(j=1, 2)$, $A=\pi^{-1}(L_t)$ in V_3, respectively. Then, by construction, one can easily see that C_1 is a smooth rational curve in X_1 with $C_1\subset Y_1$, $E_1\cong P^2$, and Y_1 is a singular del Pezzo surface with a singularity of A_2-type. We put $\rho':=(\phi_1\circ\phi_2\circ\phi_3)^{-1}\circ\delta$. Then ρ' is a birational map of V onto X_1 such that $\rho': V-f\cong X_1-C$ (isomorphic). Since $K_V=-2A-2\Sigma$, we have $K_{X_1}=-2Y_1-2E_1$. Since $E_1\cdot Y_1=l_1^{(1)}+l_2^{(1)}$, by the adjunction formula, $\mathcal{O}_{E_1}(E_1)=\mathcal{O}_{E_1}(-l_j^{(1)})$ for $j=1, 2$, where $l_j^{(1)}:=\delta(\tilde{l}_j)$ is a line in $E_1\cong P^2$. Thus E_1 can be blown down to a point x of a smooth projective 3-fold X.

Let $\delta_1 : X_1 \to X$ be the contraction morphism. Then $Y := \delta_1(Y_1)$ has a singularity of A_4-type at $x = \delta_1(E_1)$. Since all the transformations above are performed on the divisor $\Sigma \subsetneq V$, we have $X - Y \simeq V - (\Sigma \cup A) \cong C^3$ (by Lemma 5.7). Thus, (X, Y) is a smooth projective compactification of C^3 such that Y is a singular del Pezzo surface with a singularity of A_4-type. This implies that X is a Fano 3-fold of index 2 with $\operatorname{Pic} X \cong \boldsymbol{Z}\mathcal{O}_X(Y)$. Since Y has a singularity of A_4-type, we have $\deg N_Y = \deg(-K_Y) = 5$, where $N_Y := [Y]|_Y$ (resp. K_Y) is the normal bundle of Y in X (resp. a canonical divisor on Y). Thus, X is a Fano 3-fold V_5 of degree 5 in $\boldsymbol{P}^6$ by the anti-canonical embedding. In particular, $C := \delta_1(C_1)$ is a unique line in X through the point $x = \delta_1(E_1)$ on X. Thus we have the following:

PROPOSITION 5.8. (1) $\delta_1(E_1) =: x \in \mathfrak{A} \neq \emptyset$.

(2) *There is a birational map* $\rho' : \boldsymbol{P}(\mathscr{E}) \dashrightarrow V'_5 =: X_1$ *such that*

$$\boldsymbol{P}(\mathscr{E}) - f \xrightarrow[\sim]{\rho'} X_1 - C_1 \ (isomorphic),$$

where V'_5 *is the blowing up of* V_5 *at the point* $\delta_1(E_1) = x \in V_5$.

(3) $H^t_5 := \delta_1(\rho'(\Sigma \cup \pi^{-1}(L_t))$ *is a singular del Pezzo surface with singularity of* A_4-*type. In particular,* $V_5 - H^t_5 \cong C^3$.

By Propositions 4.4 and 5.8, we have the proof of the assertions (1), (2) and a half part of (3) in our main theorem. The rest can be proved as follows:

For any fiber $\pi^{-1}(p')$ $(p \neq p' \in L_\infty)$, let $l_{p'}$ be the proper transform of $\pi^{-1}(p') \subsetneq \boldsymbol{P}(\mathscr{E})$ in $V'_5 = X_1$. By construction, $l_{p'} \cap C_1 \neq \emptyset$, $(l_{p'} \cdot Y_1) = 1$, and $(l_{p'} \cdot E_1) = 0$. Thus $H^\infty_5 := \delta_1(\rho'(\Sigma \cup \pi^{-1}(L_\infty)))$ is a ruled variety swept out by lines which intersect the line C.

We also have $V_5 - H^\infty_5 \cong C^3$. By Lemma 1.1, H^∞_5 cannot be normal. This completes the proof of the theorem.

Finally, we will prove the corollary. Let L be any line in $\boldsymbol{P}^2$ which does not pass through the point $p \in \boldsymbol{P}^2$. We put $H_5 := \delta_1(\rho'(\Sigma \cup \pi^{-1}(L)))$. Then, H_5 is a member of the linear system $|\mathcal{O}_{V_5}(1) \otimes \mathcal{M}^2_x|$. Thus, H_5 contains a unique line C through the point x. We can see that

$$V_5 - H_5 \overset{\delta_1}{\cong} V'_5 - \delta_1^{-1}(H_5) \overset{\rho}{\cong} \boldsymbol{P}(\mathscr{E}) - (\Sigma \cup \pi^{-1}(L)).$$

Since $\boldsymbol{P}(\mathscr{E}) - (\Sigma \cup \pi^{-1}(L))$ is a C-bundle over $C^2 - \{0\}$, $V_5 - H_5 \ncong C^3$. Therefore we have the corollary.

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