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THE LENGTHS OF THE CLOSED GEODESICS ON A RIEMANN SURFACE WITH SELF-INTERSECTIONS

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Introduction. Let $\boldsymbol{H}=\{z=x+iy; y>0\}$ be a hyperbolic plane with the Poincaré metric $ds^2=(dx^2+dy^2)/y^2$ of constant curvature -1. The group $PSL(2, \boldsymbol{R})$ acts on $\boldsymbol{H}$ as the group of orientation-preserving isometries. A hyperbolic element γ of $PSL(2, \boldsymbol{R})$ has two fixed points in $\boldsymbol{R}\cup\{\infty\}$; the repelling fixed point α and the attracting one β. The geodesic a_γ connecting α and β is called the *axis* of γ. Let Γ be a Fuchsian group in $PSL(2, \boldsymbol{R})$ and $\phi=\phi_\Gamma: \boldsymbol{H}\to\boldsymbol{H}/\Gamma$ be the natural projection on the quotient space. Then the equivalence classes of axes $\{\eta(a_\gamma); \eta\in\Gamma\}$ of hyperbolic elements of Γ and the closed geodesics on $\boldsymbol{H}/\Gamma$ (which include a kind of geodesic segments for some cases, see § 1) are in one-to-one correspondence under the map induced by $\phi: a_\gamma\mapsto\phi(a_\gamma)$. The purpose of the present paper is to show that a closed geodesic with some self-intersections cannot be too short. To state our main theorem we first give the following condition imposed on hyperbolic elements γ of $PSL(2, \boldsymbol{R})$:

(∞) *There exists a Fuchsian group Γ containing γ and another element δ in such a way that δ does not preserve the axis a_γ of γ (that is, $a_\gamma\neq\delta(a_\gamma)$) and that a_γ and $\delta(a_\gamma)=a_{\delta\gamma\delta^{-1}}$ intersect each other.*

THEOREM. *For each hyperbolic transformation $\gamma\in PSL(2, \boldsymbol{R})$ satisfying the condition (∞), the trace of γ satisfies*

$$|\operatorname{tr}\gamma|\geqq c_0=2\cos(2\pi/7)+1=2.2469\cdots .$$

Moreover, the constant c_0 cannot be replaced by any greater value.

In the condition (∞) there are no restrictions on the Fuchsian group Γ. If, in particular, γ is contained in a Fuchsian group Γ without elliptic elments for which the condition (∞) is satisfied, then inequality $|\operatorname{tr}\gamma|\geqq 2\sqrt{2}$ holds ([5], [13]).

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1. Preliminaries. Let γ be a hyperbolic element of a Fuchsian group Γ. The projection $\phi(a_\gamma)=\phi_\Gamma(a_\gamma)$ of the axis of γ is a geodesic curve on $\boldsymbol{H}/\Gamma$. Let $l(\gamma)$ be the length of $\phi(a_\gamma)$ counting multiplicities. Then we have

$$|\operatorname{tr}\gamma|=2\cosh(l(\gamma)/2)$$

(cf. [1, p. 173]). Suppose that an elliptic element of order 2 in Γ preserves a_γ. A circular arc at a hyperbolic distance ε from a_γ is projected under ϕ onto a closed curve C_ε. We can regard $\phi(a_\gamma)$ as the degeneration of C_ε as $\varepsilon \to 0$. If C_ε are simple closed curves for small values of ε, we call also $\phi(a_\gamma)$ a *simple closed geodesic*.

A subset D of $\boldsymbol{H}$ is said to be *stable* with respect to Γ if, for all η in Γ, either

$$\eta(D)=D \quad \text{or} \quad \eta(D)\cap D=\varnothing$$

([1, 6.3]); $\Gamma_D=\{\eta\in\Gamma;\ \eta(D)=D\}$ is called the *stabilizer* of D. Then $\phi(a_\gamma)$ is simple closed if and only if a_γ is stable with respect to Γ. Assume that $\phi(a_\gamma)$ is simple closed and γ generates the maximal cyclic subgroup $\langle\gamma\rangle$ of the stabilizer Γ_{a_γ} of a_γ. Let $C(\omega, a_\gamma)$ denote the hyperbolic ω-neighborhood of a_γ. If $C(\omega, a_\gamma)$ is stable with respect to Γ, we call its projection $\phi(C(\omega, a_\gamma))$ a *collar* of width ω about $\phi(a_\gamma)$. For a positive number l, let $\omega(l)$ be the value determined by $2\sinh\omega(l)=(\sinh l/2)^{-1}$. Then the collar lemma ([4]) says that, if $\Gamma_{a_\gamma}=\langle\gamma\rangle$, $C(\omega(l(\gamma)), a_\gamma)$ is stable with respect to Γ. If Γ_{a_γ} contains an elliptic element δ of order 2, we can find a subgroup G of Γ such that $\Gamma=G\cup G\delta$ and $G_{a_\gamma}=\langle\gamma\rangle$. By the collar lemma $C(\omega(l(\gamma)), a_\gamma)$ is stable with respect to G. Then $C(\omega(l(\gamma)), a_\gamma)$ is preserved by δ and hence stable with respect to Γ.

2. Two-generator Fuchsian groups of the first kind.

2.1. Let us observe that in order to prove the inequality in the theorem it suffices to consider hyperbolic transformations in two-generator Fuchsian groups of the first kind.

Let γ satisfy the condition (∞) with respect to a Fuchsian group Γ. An element δ of Γ does not preserve a_γ, and a_γ and $a_{\delta\gamma\delta^{-1}}$ intersect each other. Let Γ' be the group generated by γ and δ. Then obviously γ satisfies (∞) with respect to Γ'.

We replace Γ by the above Γ' and proceed with the two-generator group Γ. Note that Γ is non-elementary, because the endpoints of a_γ and $a_{\delta\gamma\delta^{-1}}$ are limit points of Γ. Assume that Γ is of the second kind. Following the method described in Bers's paper [2], we shall construct the Nielsen extension of $\boldsymbol{H}/\Gamma$. For a greater detail, see [2]. Let Ω be the region of discontinuity for the action of Γ on the extended complex plane $\boldsymbol{C}\cup\{\infty\}$. The $J=\Omega\cap(\boldsymbol{R}\cup\{\infty\})$ is a union of open intervals, and $\Omega=\boldsymbol{H}\cup\boldsymbol{H}^*\cup J$, where $\boldsymbol{H}^*$ is the lower half plane. Let $\psi:\Omega\to\Omega/\Gamma$ be the natural projection and $\chi:\boldsymbol{H}\to\Omega$ be a universal covering mapping. If K is the Fuchsian group leaving $\psi\circ\chi$ invariant, then Ω/Γ is represented by $\boldsymbol{H}/K$ (Maclachlan [8]). Let f be the identity on $\boldsymbol{H}/\Gamma=\psi(\boldsymbol{H})$. Then there exists a conformal mapping $f_1:\boldsymbol{H}\to f_1(\boldsymbol{H})\subset\boldsymbol{H}$ which makes the following diagram commute:

$$\begin{array}{ccc}
\boldsymbol{H} & \xrightarrow{\ f_1\ } & \boldsymbol{H} \\
\phi_\Gamma \downarrow & & \downarrow \phi_K=\psi\circ\chi \\
\boldsymbol{H}/\Gamma & \xrightarrow{\ f\ } & \boldsymbol{H}/K=\Omega/\Gamma
\end{array}$$

The set $\psi(J)$ consists of a finite number of simple closed curves $C_1, \cdots, C_t$. Each C_j is a geodesic with respect to the hyperbolic metric on H/K induced by ϕ_K. Thus $f_1(H)$, a lift of $\psi(H)$, is a convex region bounded by axes of hyperbolic elements corresponding to $C_1, \cdots, C_t$. Let K_1 be the stabilizer of $f_1(H)$ in K. For any hyperbolic half plane D of $H \setminus f_1(H)$, a hyperbolic element which has ∂D as the axis generates the stabilizer $(K_1)_D$ of D in K_1. Hence H/K_1 is obtained from $\psi(H)$ by attaching the ring domains of the form $D/(K_1)_D$ to $C_1, \cdots, C_t$. We call $N(H/\Gamma)=H/K_1$ the *Nielsen extension* of H/Γ. By replacing K by a conjugation of K in $PSL(2, R)$, we can normalize f_1 so that f_1 fixes $i=\sqrt{-1}$. Since K_1 is the group of covering transformations leaving $\phi_K|_{f_1(H)}$ invariant, f_1 induces an isomorphism $\theta_1: \Gamma \to K_1$ defined by $\theta_1(\eta) \circ f_1 = f_1 \circ \eta$ for $\eta \in \Gamma$. We define inductively $N_s(H/\Gamma)=H/K_s$ $(s=2, 3, \cdots)$ to be $N(H/K_{s-1})$ by using a similar conformal mapping $f_s: H \to H$ to the f_1 as above, such that $f_s(i)=i$. Let $\theta_s: K_{s-1} \to K_s$ be the isomorphism defined by $\theta_s(\eta) \circ f_s = f_s \circ \eta$ for $\eta \in K_{s-1}$.

We set $\Theta_s = \theta_s \circ \cdots \circ \theta_1$ and $F_s = f_s \circ \cdots \circ f_1$. Then $\Theta_s: \Gamma \to K_s$ is an isomorphism and F_s is conformal and fixes i. Moreover we set $\gamma_s = \Theta_s(\gamma)$ and $\delta_s = \Theta_s(\delta)$ for the generators γ and δ of Γ. Let $d(,)$ denote the hyperbolic distance in H. The Ahlfors-Schwarz lemma (a holomorphic mapping is distance decreasing between Riemann surfaces with hyperbolic metrics) yields that for $s=1, 2, \cdots$,

$$d(F_s(z), i) < d(z, i) \qquad \text{for} \quad z \in H$$

and, in particular, that

$$d(\gamma_s(i), i) < d(\gamma(i), i) \quad \text{and} \quad d(\delta_s(i), i) < d(\delta(i), i)\,.$$

These inequalities imply that $\{F_s\}$ is locally uniformly bounded in H and that $\{\gamma_s\}$ and $\{\delta_s\}$ contain subsequences converging in $PSL(2, R)$. By replacing them by suitable subsequences, we may assume that

$$F_s \to F \qquad \gamma_s \to \gamma_0 \quad \text{and} \quad \delta_s \to \delta_0\,.$$

By a theorem of Jørgensen ([6, Theorem 1]), the group K_0 generated by γ_0 and δ_0 is a non-elementary Fuchsian group and there is an isomorphism $\Theta: \Gamma \to K_0$ such that $\Theta(\gamma)=\gamma_0$ and $\Theta(\delta)=\delta_0$. By the limiting process, we have that

$$\Theta(\eta) \circ F = F \circ \eta \qquad \text{for} \quad \eta \in \Gamma\,. \tag{2.1}$$

From this it follows that F is not constant, since $F(\gamma(i)) = \gamma_0(i) \neq i = i = F(i)$. Hence F is conformal. By proceeding precisely as in [2], we can see that K_0 is of the first kind. We call H/K_0 the *infinite Nielsen extension* of H/Γ.

To verify that γ_0 is hyperbolic and satisfies the condition (∞), note first either one of the following cases occurs:

(1) The endopints of $F(a_\gamma)$ separate those of $F(a_{\delta\gamma\delta^{-1}})$, or

(2) $F(a_\gamma)$ and $F(a_{\delta\gamma\delta^{-1}})$ have a common fixed point.

The case (2) occurs in particular if γ_0 is parabolic. By (2.1) γ_0 fixes the endpoints of

$F(a_\gamma)$ and $\delta_0\gamma_0\delta_0^{-1}$ fixes those of $F(a_{\delta\gamma\delta^{-1}})$. Then the case (2) is impossible, since K_0 is a non-elementary Fuchsian group. Hence only the case (1) occurs, and γ_0 satisfies the condition (∞) with respect to K_0. Finally by the Ahlfors-Schwarz lemma, if z is a point of a_γ,

$$|\operatorname{tr}\gamma| = 2\cosh(d(z, \gamma(z))/2) > 2\cosh(d(F(z), \gamma_0 F(z))/2) \geqq |\operatorname{tr}\gamma_0|\,.$$

Thus for our purpose, it suffices to consider two-generator Fuchsian groups of the first kind.

The classification of all two-generator Fuchsian groups has been already completed (see [10], [11]). We write the signature as $(g; m_1, \cdots, m_r, \infty, \cdots, \infty)$ with ∞ repeated s times, instead of $(g; m_1, \cdots, m_r; s; 0)$, which is employed in [1]. The signatures of two-generator Fuchsian groups of the first kind are: (a) $(1; p)$, $2 \geqq p$, (b) $(0; 2, 2, 2, p)$, p odd $\geqq 3$ and (c) $(0; p, q, r)$, $2 \leqq p, q, r$ and $1/p + 1/q + 1/r < 1$ (the signatures of triangle groups).

2.2. We consider a Fuchsian group Γ with signature $(1; p)$, $p \geqq 2$. The surface $\boldsymbol{H}/\Gamma$ is either a torus with $\phi_\Gamma: \boldsymbol{H} \to \boldsymbol{H}/\Gamma$ branched over a single point if $p < \infty$, or a once-punctured torus if $p = \infty$.

Let $\gamma \in \Gamma$ be a hyperbolic element satisfying the condition (∞) with respect to Γ. Assume first that $\phi(a_\gamma)$ does not intersect a simple closed geodesic g on $\boldsymbol{H}/\Gamma$. Let D be a lift of $\boldsymbol{H}/\Gamma \setminus g$ to $\boldsymbol{H}$ containing a_γ. Then γ satisfies the condition (∞) with respect to the stabilizer Γ_D of D. Now Γ_D is of the second kind. By considering the infinite Nielsen extension of $\boldsymbol{H}/\Gamma_D$ as in 2.1, we can find a Fuchsian group G and an isomorphism $\theta: \Gamma_D \to G$ such that $|\operatorname{tr}\theta(\gamma)| < |\operatorname{tr}\gamma|$ and $\theta(\gamma)$ satisfies (∞) with respect to G. The signature of G is $(0; p, \infty, \infty)$, which we shall treat later.

Assume next that $\phi(a_\gamma)$ intersects every simple closed geodesic on $\boldsymbol{H}/\Gamma$. If a simple closed geodesic has a collar of width ω for which $2\cosh\omega \geqq c_0 = 2\cos(2\pi/7) + 1$, the length of $\phi(a_\gamma)$ is greater than 2ω, and hence $|\operatorname{tr}\gamma| \geqq c_0$. Thus we may assume that for every simple closed geodesic the maximal width of collars satisfies $2\cosh\omega < c_0$. By the collar lemma our assumption means that every simple closed geodesic has length greater than l_0 with $(2\sinh l_0/2)^{-1} = \sinh(\cosh^{-1}(c_0/2))$. Here note that $2\cosh l_0/2 > 2.7 > c_0$. The curve $\phi(a_\gamma)$ can be divided into some simple closed curves $C_1, \cdots, C_n$. At least one of them, say C_1, is not contractible to the projection of the elliptic fixed point (or the puncture) of the torus. Then the length of $\phi(a_\gamma)$ is greater than l_0, since it is greater than the length of the simple closed geodesic freely homotopic to C_1. Thus we have $|\operatorname{tr}\gamma| > c_0$.

2.3. Next we consider a hyperbolic element γ satisfying the condition (∞) with respect to a Fuchsian group Γ with signature $(0; 2, 2, 2, p)$, p odd $\geqq 3$. $\boldsymbol{H}/\Gamma$ is a sphere and ϕ_Γ is branched over four points with branching orders 2, 2, 2 and p. As in 2.2 we may assume that $\phi(a_\gamma)$ intersects every simple closed geodesic on $\boldsymbol{H}/\Gamma$, all of which have length greater than l_0 as above.

If $\phi(a_\gamma)$ is a closed curve in the usual sense, then a closed curve in $\phi(a_\gamma)$ bounds either a disc containing two projections of elliptic fixed points of order 2, or two discs each of which contains one projection of such a point. Then the curve has length $>l_0$. If an elliptic element of order 2 preserves a_γ, then we consider the closed curve C_ε as in § 1. By applying the same argument to C_ε, we know the length of C_ε is greater than l_0. By letting $\varepsilon \to 0$, the length of $\phi(a_\gamma)$ is not less than l_0. Therefore we can conclude that $|\operatorname{tr} \gamma| > c_0$.

3. Triangle groups. In this section we shall only be concerned with triangle groups. Hence we abbreviate the notation of a signature $(0; p, q, r)$ to (p, q, r). We write $(p, q, r) \geqq (p', q', r')$ if the inequalities $p \geqq p'$, $q \geqq q'$ and $r \geqq r'$ hold simultaneously.

3.1. We assume first that p, q and r are all finite. The triangle group with signature (p, q, r) has a group presentation $\{A, B;\ A^p = B^q = (B^{-1}A^{-1})^r = 1\}$. A triangle group $\Gamma = \Gamma(p, q, r)$ is generated by the following two matrices:

$$A = E(p) = \begin{bmatrix} \cos(\pi/p) & -\sin(\pi/p) \\ \sin(\pi/p) & \cos(\pi/p) \end{bmatrix} \quad \text{and}$$

$$(3.1) \qquad B = \begin{bmatrix} \cos(\pi/q) & -\lambda^{-1}\sin(\pi/q) \\ \lambda \sin(\pi/q) & \cos(\pi/q) \end{bmatrix},$$

where the constant $\lambda = \lambda(p, q, r) > 1$ is to be determined. Since $C = B^{-1}A^{-1}$ is elliptic of order r and $\operatorname{tr} A > 0$ and $\operatorname{tr} B > 0$, we have ([9, p. 489, Corollary])

$$\operatorname{tr} C = 2\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right) - (\lambda + \lambda^{-1})\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right) = -2\cos\left(\frac{\pi}{r}\right).$$

Now we obtain

$$(3.2) \qquad \lambda = \lambda(p, q, r) = \frac{E + \left(E^2 - \sin^2\left(\frac{\pi}{p}\right)\sin^2\left(\frac{\pi}{q}\right)\right)^{1/2}}{\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right)},$$

where $E = \cos(\pi/r) + \cos(\pi/p)\cos(\pi/q)$. Denote by p_X the fixed point in $\boldsymbol{H}$ of an elliptic transformation X of $PSL(2, \boldsymbol{R})$. Then we have $p_A = i$, $p_B = \lambda^{-1} i$,

$$p_C = \frac{-(\lambda - \lambda^{-1})\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right) + 2\sin\left(\frac{\pi}{r}\right)i}{2\left(\lambda\cos\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right) + \sin\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\right)} \quad \text{and}$$

(3.3)

$$p_D = \frac{-(\lambda^2-1)\sin\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{p}\right)+\lambda i}{\lambda^2\cos^2\left(\frac{\pi}{p}\right)+\sin^2\left(\frac{\pi}{p}\right)},$$

where $D=ABA^{-1}$. If $p<\infty$, we can define $\Gamma(p, q, \infty)$ and $\Gamma(p, \infty, \infty)$ to be the limit of $\Gamma(p, q, r)$ as $r\to\infty$ and $q, r\to\infty$, respectively. We define $\Gamma(\infty, \infty, \infty)$ to be the group

$$\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \boldsymbol{Z}); \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod 2\right\}.$$

The groups $\Gamma(p, q, r)$ and $\Gamma(p', q', r')$ are conjugate to each other in $PSL(2, \boldsymbol{R})$ if and only if (p', q', r') is a permutation of (p, q, r). We classify the signatures into four types:

Type I	(i)	(p, q, r)	with $4\leqq p\leqq q\leqq r$ and $p<\infty$
	(ii)	$(p, 3, r)$	with $5\leqq p\leqq r$
	(iii)	$(4, 3, r)$	with $5\leqq r$
Type II	(i)	$(2, q, r)$	with $5\leqq q\leqq r$
	(ii)	$(2, 4, r)$	with $7\leqq r$
Type III	(i)	$(2, 4, r)$	with $r=5$ and 6
	(ii)	$(2, 3, r)$	with $7\leqq r$
Type IV		$(3, 4, 4)$ and $(3, 3, r)$	with $4\leqq r$.

Note that except for $\Gamma(\infty, \infty, \infty)$ any triangle group $\Gamma(p, q, r)$ is conjugate to a group with a signature listed above. As we have seen in §2, it suffices to show the following for the proof of the inequality in the theorem:

PROPOSITION 3.1. *For any hyperbolic transformation γ contained in a triangle group it holds that* $|\operatorname{tr}\gamma|\geqq c_0=2\cos(2\pi/7)+1$.

Any hyperbolic element γ of $\Gamma(\infty, \infty, \infty)$ satisfies $|\operatorname{tr}\gamma|\geqq 3>c_0$. The groups $\Gamma(3, 4, 4)$ and $\Gamma(3, 3, r)$ $(r\geqq 4)$ are conjugate to a subgroup of $\Gamma(2, 4, 6)$ and $\Gamma(2, 3, 2r)$, respectively, in $PSL(2, \boldsymbol{R})$ (see [3], [12]). Therefore we need only to consider the triangle groups with signatures of type I, II and III. For more details about triangle groups, see [7].

3.2. We consider then the triangle group $\Gamma(p, q, r)$ with $p<\infty$. Let $Q=Q(p, q, r)$ be the hyperbolic quadrilateral with vertices p_A, p_B, p_C and p_D (see Figure 3.1). Poincaré's theorem ([1, 9.8]) implies that Q is a fundamental domain for $\Gamma=\Gamma(p, q, r)$. Define $R=R(p, q, r)$ by

$$R=\bar{Q} \quad \text{for} \quad p\geqq 3 \quad \text{and} \quad R=\bar{Q}\cup A(\bar{Q}) \quad \text{for} \quad p=2$$

(here $\bar{Q}$ is the closure of Q). If $p \geqq 3$, label the sides $p_A p_D$, $p_A p_B$, $p_B p_C$ and $p_C p_D$ of R by the letters A, A^{-1}, C and C^{-1}, respectively. If $p=2$, label the sides $p_D p_C$, $p_D A(p_C)$, $p_B A(p_C)$ and $p_B p_C$ by A, A^{-1}, C and C^{-1}, respectively. We call the side labelled by the letter A the *A-side* of R and so on. By abuse of notation we denote by $d(A, C)$ the hyperbolic distance between the A-side and the C-side. Since R is symmetric with repect to the hyperbolic line through p_A and p_C if $p \geqq 3$, or through p_B and p_D if $p=2$, $d(A, C)$

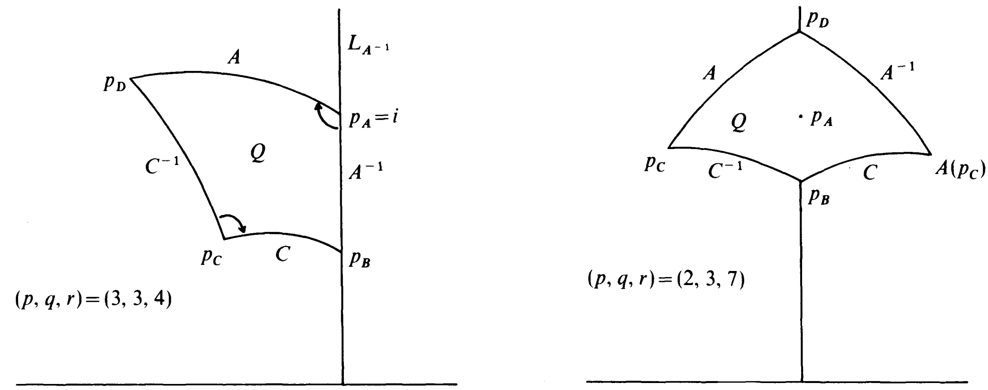

FIGURE 3.1

is also the hyperbolic distance between the A^{-1}-side and the C^{-1}-side. We shall estimate the value of $d(A, C)$.

LEMMA 3.2. *For the triangle groups $\Gamma(p, q, r)$ the following inequalities hold: If (p, q, r) is a signature of type* I *or type* II,

$$2\cosh(d(A, C)/2) > c_0 = 2\cos(2\pi/7) + 1 ,$$

and if (p, q, r) is of type III,

$$2\cosh d(A, C) \geqq c_0 .$$

PROOF. (i) *Case of type* I. For a while we treat arbitrary signatures (p, q, r) with $3 \leqq p, q, r$. Define L_A to be the hyperbolic line which is the extension of the A-side of R. Define L_C, $L_{A^{-1}}$ and $L_{C^{-1}}$ similarly. The hyperbolic distance $d(L_A, L_C)$ between L_A and L_C satisfies $d(L_A, L_C) \leqq d(A, C)$. We write $D(p, q, r) = d(L_A, L_C)$ when we are concerened with the signature (p, q, r).

We shall show that L_A and L_C are disjoint. Suppose that L_A and L_C meet in a point $\tilde{p}_0$. Let $\tilde{p}_1$ and $\tilde{p}_2$ be the vertices on the side of R closest to $\tilde{p}_0$. Hence $\{\tilde{p}_1, \tilde{p}_2\} = \{p_A, p_B\}$ or $\{p_D, p_C\}$. Consider the hyperbolic triangle $\tilde{\Delta}$ with vertices $\tilde{p}_0$, $\tilde{p}_1$ and $\tilde{p}_2$. Since the angle at each vertex of R is not greater than $\pi/3$, the angle sum of $\tilde{\Delta}$ exceeds π. This is a contradiction ([1, 7.13 Corollary]).

We prove then that:

(3.4) $D(p, q, r) = D(r, q, p)$, $D(p, q, r) < D(p, q, r+1)$ and $D(p, q, q) < D(p, q, +1, q)$.

The first equality holds, since $R(p, q, r)$ is congruent to $R(r, q, p)$ in hyperbolic geometric sense. The line L_A is the same for the signatures (p, q, r) and $(p, q, r+1)$. On the other hand L_C for (p, q, r) and that for $(p, q, r+1)$ are different. We distinguish them by writing $L_C(r)$ and $L_C(r+1)$, respectively. Two lines $L_C(r)$ and $L_C(r+1)$ meet $L_{A^{-1}}$ in the points $\lambda(p, q, r)^{-1}i$ and $\lambda(p, q, r+1)^{-1}i$ with the same angle π/q. From (3.2) follows $\lambda(p, q, r) < \lambda(p, q, r+1)$. Hence $L_C(r)$ separates $L_C(r+1)$ from L_A. Thus $D(p, q, r) < D(p, q, r+1)$. Let S be the hyperbolic quadrilateral with vertices p_A, p_B, p_C and $A(p_C)$. Then $S = E(2p)(Q(p, r, q))$, where $E(2p)$ is the transformation given in (3.1). It follows from this that $d(p_A, p_C) = \log \lambda(p, r, q)$. As before we write $L_C(q)$, $L_C(q+1)$, $p_C(q)$ and $p_C(q+1)$ to distinguish L_C's and p_C's for the signatures (p, q, q) and $(p, q+1, q)$. Let M be the hyperbolic bisector of the angle which L_A and $L_{A^{-1}}$ make at p_A. Then $L_C(q)$ and $L_C(q+1)$ meet M in $p_C(q)$ and $p_C(q+1)$, respectively, with the same angle π/r. Since $d(p_A, p_C(q)) = \log \lambda(p, q, q) < \log \lambda(p, q+1, q) < d(p_A, p_C(q+1))$, $L_C(q)$ separates $L_C(q+1)$ from L_A. Thus $D(p, q, q) < D(p, q+1, q)$.

By combinations of relations (3.4), we can obtain

$$D(p, q, r) \geqq D(4, 4, 4) \quad \text{for} \quad (p, q, r) \quad \text{of type I (i)}, \quad \text{and}$$

$$D(p, q, r) \geqq D(4, 3, 5) \quad \text{for} \quad (p, q, r) \quad \text{of type I (ii), (iii)}.$$

We first evaluate $2\cosh(D(4, 4, 4)/2)$. Consider the hyperbolic triangle Δ with vertices p_A, p_C and p_D. Draw a hyperbolic perpendicular from the midpoint p_1 of the segment $p_A p_C$ to L_A. Then the foot p_2 of the perpendicular lies on the A-side of R, since the angle of all vertices of Δ do not exceed $\pi/2$. Observe that $R = R(4, 4, 4)$ is preserved by the elliptic transformation of order 2 of $PSL(2, \boldsymbol{R})$ with a fixed point p_1. Thus $D(4, 4, 4)/2 = d(p_1, p_2)$. Then by applying the sine rule ([1, 7.12]) to the hyperbolic triangle with vertices p_1, p_2 and p_A, we obtain

$$\frac{\sinh(D(4, 4, 4)/2)}{\sin(\pi/4)} = \frac{\sinh((1/2)\log \lambda(4, 4, 4))}{\sin(\pi/2)}.$$

Since $2\cosh x = 2(1+\sinh^2 x)^{1/2}$, $2\cosh(D(4, 4, 4)/2) = 2[1 + \{\cos(\pi/2) + \cos(\pi/4)\}/2]^{1/2} > 2.32$.

Next we evaluate $2\cosh(D(4, 3, 5)/2)$. *We regard* $L_{C^{-1}}$ as a Euclidean circle. Then it has the center ξ and radius ρ described by

$$\xi = \frac{|p_C|^2 - |p_D|^2}{2\,\mathrm{Re}(p_C - p_D)} \quad \text{and} \quad \rho = \frac{|p_C - p_D|\,|p_C - p_D|}{2|\,\mathrm{Re}(p_C - p_D)\,|}.$$

Since $D(4, 3, 5) = d(L_{A^{-1}}, L_{C^{-1}})$, $(\xi - \rho)(\xi + \rho)^{-1} \tanh^2(D(4, 3, 5)/2) = 1$ ([1, 7.23]). Then $2\cosh(D(4, 3, 5)/2) = [2\rho^{-1}(\rho - \xi)]^{1/2}$. In this case, by (3.3),

$$p_C = \frac{-\sqrt{(2+2\sqrt{2})(1+\sqrt{5})}+\sqrt{2(5-\sqrt{5})}\,i}{1+\sqrt{5}+2\sqrt{2}+\sqrt{(2+2\sqrt{2})(1+\sqrt{5})}},$$

$$p_D = \frac{-\sqrt{(2+2\sqrt{2})(1+\sqrt{5})}+\sqrt{6}\,i}{1+\sqrt{2}+\sqrt{5}}.$$

Hence $2\cosh(D(4,3,5)/2>2.29$. Summing up the results obtained so far, we conclude that $2\cosh(d(A,C)/2)>c_0$ for signatures of type I.

(ii) *Case of type* II *and* III. Draw the hyperbolic perpendicular from p_A to $L_{C^{-1}}$, and let p_3 be the foot of the perpendicular. As in the previous case we can see that $d(A,C)=2d(p_A,p_3)$. Again the sine rule applied to the triangle with vertices p_A, p_B and p_3 yields $2\cosh(d(A,C)/2)=2(\cos^2(\pi/r)+\cos^2(\pi/q))^{1/2}$. Hence $2\cosh(d(A,C)/2)\geqq \min\{2\sqrt{2}\cos(\pi/5),\ (2+4\cos^2(\pi/7))^{1/2}\}>c_0$ for signatures of type II, and $2\cosh d(A,C)=4\cosh^2(d(A,C)/2)-2\geqq\min\{4\cos^2(\pi/5),\ 4\cos^2(\pi/7)-1\}=c_0$ for those of type III. q.e.d.

For a signature (p,q,r) of type I, the segment p_Ap_C intersects perpendicularly to the segment p_Bp_D. Consider the hyperbolic triangle which their point of intersection makes with p_A and p_B. Then the sine rule yields

$$2\cosh(d(p_B,p_D)/2)=2\left\{1+\frac{\left(\cos\left(\frac{\pi}{r}\right)+\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\right)^2-\sin^2\left(\frac{\pi}{p}\right)\sin^2\left(\frac{\pi}{q}\right)}{\sin^2\left(\frac{\pi}{q}\right)}\right\}^{1/2}.$$

Thus,

$$2\cosh(d(p_B,p_D)/2)>c_0 \quad \text{if} \quad (p,q,r) \quad \text{is of type I}. \tag{3.5}$$

In a similar way we obtain

$$2\cosh(d(p_C,A(p_C))/2)>c_0 \quad \text{if} \quad (p,q,r) \quad \text{is of type II or type III}. \tag{3.6}$$

3.3. We fix a signature (p,q,r) of type I, II or III. Let $\Gamma=\Gamma(p,q,r)$. Let $\mathcal{N}$ be the set of images of ∂R under Γ; its vertices are elliptic fixed points of Γ and its edges are equivalent to the sides of R under Γ. We call $\gamma(R)$ with $\gamma\in\Gamma$ simply a *copy* of R. For a hyperbolic element γ of Γ we regard the axis a_γ as a directed line which tends to the attracting fixed point.

LEMMA 3.3. *Let γ be a hyperbolic element of Γ. If a_γ passes through a vertex in $\mathcal{N}$, then $|\mathrm{tr}\,\gamma|>c_0$.*

PROOF. Assume that a_γ passes through a vertex v in $\mathcal{N}$. Consider the edges in $\mathcal{N}$ opposite to v in some copies of R and let $\mathcal{I}$ be their union. Suppose that a_γ meets $\mathcal{I}$ in two points w_1 and w_2 in this order. The segment $s_i=w_iv$ $(i=1,2)$ is equivalent under

Γ to a segment in R connecting a vertex of R to one of its opposite sides. Hence $d(w_i, v) > d(A, C)$. If $\gamma(s_1) \neq s_2$, we obtain by Lemma 3.2 that $|\operatorname{tr} \gamma| > c_0$. If otherwise, we have $\gamma(w_1) = v$. Then, for an element η of Γ, $\eta(s_1)$ connects two equivalent vertices of R under Γ. This means $\eta(s_1) = p_B p_D$ if $p \geqq 3$ and $\eta(s_1) = p_C A(p_C)$ or $p_B p_D$ if $p = 2$. For the first two cases (3.5) and (3.6) yield the desired result. For the last case, either $\eta\gamma\eta^{-1}$ or $\eta\gamma^{-1}\eta^{-1}$ equals AB^k for some k, $1 \leqq k < q$. By (3.1) and (3.2), we have $|\operatorname{tr} AB^k| = 2|\cos(\pi/r)\sin(k\pi/q)|(\sin(\pi/q))^{-1}$. Hence if (p, q, r) is of type III (i) and $2 \leqq k \leqq q-2$, $|\operatorname{tr} AB^k| \geqq 2\sqrt{2}\cos(\pi/5) > c_0$. For other cases AB^k cannot be hyperbolic. q.e.d.

Any conjugacy class of a hyperbolic element in Γ contains an element whose axis passes through R. Hence by Lemma 3.3 we need only to consider hyperbolic elements whose axes pass through R and meet no vertices in $\mathscr{N}$. Let T be the collection of such elements. Let γ be an element of T. Suppose that its axis a_γ meets the edges $E_1, \cdots, E_{n-1}$, $E_n = \gamma(E_1)$ in $\mathscr{N}$ in succession. Here E_1 and E_2 are sides of R. We call $(E_1, \cdots, E_n)$ the edges associated to γ. We shall also associate to γ a sequence of pairs of the letters A, A^{-1}, C and C^{-1}:

$$w = w(\gamma) = (X_1^{-1}, X_2)(X_2^{-1}, X_3) \cdots (X_{n-1}^{-1}, X_n)\,. \tag{3.7}$$

Here we use the convention $(A^{-1})^{-1} = A$, $(C^{-1})^{-1} = C$. Let s_i $(i = 1, \cdots, n-1)$ be the subarc of a_γ which connects E_i and E_{i+1}. Then s_1 is contained in R. Knowing that s_1 goes from the X_1^{-1}-side to X_2-side of R, we obtain the first pair (X_1^{-1}, X_2). Let γ_1 be the transformation of Γ which sends the X_2-side to its corresponding side, namely the X_2^{-1}-side. Then $\gamma_1(s_2)$ is contained in R. Then the second pair (X_2^{-1}, X_3) means that $\gamma_1(s_2)$ goes from the X_2^{-1}-side to the X_3-side of R. Next we choose the transformation γ_2 sending the X_3-side to the X_3^{-1}-side and consider $\gamma_2\gamma_1(s_3)$ contained in R. Continuing in this manner, we obtain the sequence w in (3.7). We call w the *word associated to* γ. Let w_0 be some sequence of pairs of the letters. If w contains w_0 k times in a row, we contract this part by writing w_0^k. Set $P = \{(A, C), (C, A), (A^{-1}, C^{-1}), (C^{-1}, A^{-1})\}$. Then the word w satisfies:

If $p \geqq 3$, (a) $X_i^{-1} \neq X_{i+1}$ for $i = 1, \cdots, n-1$,

(b) $X_1 = X_n$, and

(c) w contains no subsequences of the forms:

$(A, A^{-1})^k$, $(A^{-1}, A)^k$ with $k \geqq p/2$,

$(C, C^{-1})^k$, $(C^{-1}, C)^k$ with $k \geqq r/2$,

$[(X^{-1}, Y)(Y^{-1}, X)]^k$ with $k \geqq q/2$, where $(X, Y) \in P$.

If $p = 2$, (a)′ $X_i^{-1} \neq X_{i+1}$ for $i = 1, \cdots, n-1$,

(b)′(1) $X_1 = X_n$ or (2) $(X_1, X_n) \in P$, and (c)′ w contains no subsequences of the forms:

$(A, A^{-1})^k, (A^{-1}, A)^k, (C, C^{-1})^k, (C^{-1}, C)^k$ with $k \geqq q/2$,

$[(X^{-1}, Y)(Y^{-1}, X)]^k$ with $2k \geqq r/2$ and

$[(X^{-1}, Y)(Y^{-1}, X)]^k(X^{-1}, Y)$ with $2k+1 \geqq r/2$,

where $(X, Y) \in P$.

For the case $p=2$, let $A^*=C$, $(A^{-1})^*=C^{-1}$, $C^*=A$ and $(C^{-1})^*=A^{-1}$. Then the condition (2) in (b)′ can be replaced by

$$(2)' \quad X_1^*=X_n .$$

The conditions (c) and (c)′ are due to the fact that a_γ is a geodesic so that the shortest pass between two points on it lies in a_γ. For the case $p=2$, the A-side and the C-side, and the A^{-1}-side and the C^{-1}-side of R are equivalent under the action of A of Γ. Hence the condition (2) in (b)′ arises. We call a sequence w of the form (3.7) with the above conditions a *word* even if it is not associated to a hyperbolic element. The *inverse* of w in (3.7) is the word $w^{-1}=(X_n, X_{n-1}^{-1})\cdots(X_2, X_1^{-1})$.

Let $w(\gamma)$ be a word as in (3.7) associated to a hyperbolic element γ. Let γ_1 be the transformation of Γ which sends the X_2-side to the X_2^{-1}-side of R. Then the conjugation $\gamma \to \gamma_1\gamma\gamma_1^{-1}$ causes the change of the words such as

$$\text{(A)} \quad (X_1^{-1}, X_2)(X_2^{-1}, X_3)\cdots(X_{n-1}^{-1}, X_1) \longrightarrow (X_2^{-1}, X_3)\cdots(X_{n-1}^{-1}, X_1)(X_1^{-1}, X_2)$$

or, for $w(\gamma)$ satisfying (2) in (b)′,

$$\text{(B)} \quad (X_1^{-1}, X_2)(X_2^{-1}, X_3)\cdots(X_{n-1}^{-1}, X_n) \longrightarrow (X_2^{-1}, X_3)\cdots(X_{n-1}^{-1}, X_n)(X_n^{-1}, X_2^*) .$$

We regard (A) and (B) as operations on the set of words. We also consider the operation $w \mapsto w^{-1}$, that is,

$$\text{(C)} \quad (X_1^{-1}, X_2)(X_2^{-1}, X_3)\cdots(X_{n-1}^{-1}, X_n) \longrightarrow (X_n, X_{n-1}^{-1})\cdots(X_3, X_2^{-1})(X_2, X_1^{-1}) .$$

If we can deform a word w_1 into another one w_2 by a finite number of operations (A), (B), (C) and their inverses, then we say that w_1 and w_2 are *equivalent* and write $w_1 \sim w_2$.

We consider the case $p=2$. For a pair (X, Y) of letters we define an element $\Phi(X, Y)$ of Γ by

$$\Phi(X, Y)=\begin{cases} ABA & \text{if } Y=A , \\ AB^{-1}A & \text{if } Y=A^{-1} , \\ B & \text{if } Y=C , \\ B^{-1} & \text{if } Y=C^{-1} . \end{cases}$$

Note that $\Phi(X, Y)$ is the transformation which sends the Y^{-1}-side of R to the Y-side. For a word $w=(X_1^{-1}, X_2)\cdots(X_{n-1}^{-1}, X_n)$ define $\Phi(w)=\Phi(X_1^{-1}, X_2)\cdots\Phi(X_{n-1}^{-1}, X_n)$ if

$X_n = X_1$ and $\Phi(w) = \Phi(X_1^{-1}, X_2) \cdots \Phi(X_{n-1}^{-1}, X_n)A$ if $X_n = X_1^*$. For a γ in T, let $(E_1, \cdots, E_n)$ and w be the edges and the word associated to γ. Then by the definition of Φ we see that $\Phi(w)$ sends E_1 to E_n. Hence $\gamma = \Phi(w)$. We remark that if $w_1 \sim w_2$, then $\Phi(w_1)$ is conjugate to either $\Phi(w_2)$ or $\Phi(w_2)^{-1}$. For the case $p \geqq 3$, we can define a similar function of words into Γ by setting $\tilde{\Phi}((X_1^{-1}, X_2) \cdots (X_{n-1}^{-1}, X_n)) = X_2 \cdots X_n$. However we do not need $\tilde{\Phi}$ for the rest of this paper.

3.4. Let $P = \{(A, C), (A^{-1}, C^{-1}), (C, A), (C^{-1}, A^{-1})\}$. Let $(E_1, \cdots, E_n)$ and $(X_1^{-1}, X_2) \cdots (X_{n-1}^{-1}, X_n)$ be the edges and word associated to an element γ of T. Then two consecutive edges E_i and E_{i+1} do not have a common vertex if and only if $(X_i^{-1}, X_{i+1}) \in P$. In this case the part of a_γ connecting E_i and E_{i+1} has length not less than $d(A, C)$. Let $\mathscr{F}(E_i)$ denote the hyperbolic polygon made up of copies of R which have a common vertex with E_i. Then the interior of $\mathscr{F}(E_i)$ contains all points of $\boldsymbol{H}$ which are at a distance $< d(A, C)$ from E_i.

PROOF OF PROPOSITION 3.1. By Lemma 3.3 we need only to consider the hyperbolic elements of T. First we consider Γ with signature (p, q, r) of type I or II. Let $\gamma \in T$ and let $(E_1, \cdots, E_n)$ be the edges associated to γ.

If E_i and E_{i+1} do not have a common vertex for some i, then the part of a_γ connecting E_i and E_{i+1} has length $\geqq d(A, C)$. Hence by Lemma 3.2 we obtain $|\operatorname{tr} \gamma| > c_0$.

If E_i and E_{i+1} have a common vertex for all $i = 1, \cdots, n-1$, we divide the edges into groups

$$\varepsilon_1 = (E_1, \cdots, E_{j_1}), \varepsilon_2 = (E_{j_1}, \cdots, E_{j_2}), \cdots, \varepsilon_a = (E_{j_{a-1}}, \cdots, E_n)$$

so that the edges in the same group ε_b have a common vertex v_b. We remark that $a \geqq 2$, since otherwise E_1 and E_n have a common vertex, which means that γ is elliptic. By the condition (c) or (c)′, E_{j_b+1} lies in $\partial\mathscr{F}(E_{j_b-1})$, $1 \leqq b \leqq a-1$, and hence the part of a_γ connecting E_{j_b-1} and E_{j_b+1} has length $\geqq d(A, C)$. Therefore, by Lemma 3.2 we obtain $|\operatorname{tr} \gamma| > c_0$. Now we conclude that $|\operatorname{tr} \gamma| > c_0$ for every hyperbolic element γ of $\Gamma(p, q, r)$, if (p, q, r) is of type I or II. Figure 3.2 is not correct in view of hyperbolic geometry, but we can conceive the idea of the proof from it.

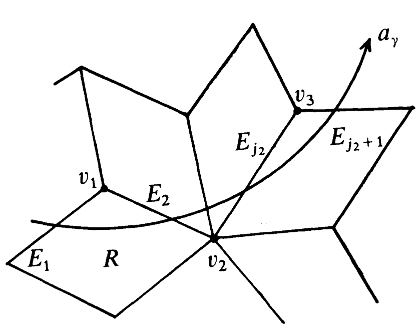

FIGURE 3.2

Next we consider γ with signature $(2, q, r)$ of type III. Let $\gamma \in T$ and $(E_1, \cdots, E_n)$ be the edges associated to γ.

Assume that E_i and E_{i+1} do not have a common vertex for some i. Let η be the transformation of Γ which sends the copy of R whose sides contain E_i and E_{i+1} to R. Then the edges associated to $\eta\gamma\eta^{-1}$ are $(\eta(E_i), \cdots, \eta(E_n), \eta\gamma(E_1), \cdots, \eta\gamma(E_i))$ and $\eta(E_i)$ and $\eta(E_{i+1})$ do not have a common vertex. Hence by replacing γ by $\eta\gamma\eta^{-1}$ we may assume that $i=1$. If there exists another pair of consecutive edges E_j and E_{j+1} with no common vertex, then the part of a_γ connecting E_1 and E_{j+1} has length $\geqq 2d(A, C)$. Then, by Lemma 3.2 we obtain $|\operatorname{tr}\gamma| \geqq c_0$. On the other hand, if E_i and E_{i+1} have a common vertex for all $i=2, \cdots, n-1$, we divide the edges into groups each of which contains edges with a common vertex:

$$\varepsilon_1=(E_2, \cdots, E_{j_1}), \varepsilon_2=(E_{j_1}, \cdots, E_{j_2}), \cdots, \varepsilon_a=(E_{j_{a-1}}, \cdots, E_n) .$$

If $a \geqq 2$, the length of the part of a_γ connecting E_{j_1-1} and E_{j_1+1} is not less than $d(A, C)$. Since the part of a_γ connecting E_1 and E_2 has already length $\geqq d(A, C)$, we obtain $|\operatorname{tr}\gamma| \geqq c_0$. The first pair (X, Y) of the word w associated to γ belongs to P. Thus, if $a=1$, w would be either $(X, Y)(Y^{-1}, Y)^k$, $k \geqq 1$, $(X, Y)[(Y^{-1}, X)(X^{-1}, X)]^k$, $k \geqq 1$ or $(X, Y)[(Y^{-1}, X)(X^{-1}, Y)]^k(Y^{-1}, X)$, $k \geqq 1$. However none of these words satisfy the condition (b)$'$ and hence cannot be associated to γ.

Finally we assume that E_i and E_{i+1} have a common vertex for all $i=1, \cdots, n-1$. Again we divide the edges into groups $\varepsilon_1=(E_1, \cdots, E_{j_1}), \cdots, \varepsilon_a=(E_{j_{a-1}}, \cdots, E_n)$ so that the edges in the same group ε_b have a common vertex v_b. As in the previous argument we have $a \geqq 2$. We consider the cases.

(1) *Case* $a \geqq 4$. Consider two parts of a_γ; one connecting E_{j_1-1} and E_{j_1+1} and the other connecting E_{j_3-1} and E_{j_3+1}. Since both parts have length $\geqq d(A, C)$, by Lemma 3.2 we obtain $|\operatorname{tr}\gamma| \geqq c_0$.

(2) *Case* $a=3$. In this case v_1 and v_3 are equivalent under the action of Γ. Hence for a suitable transformation η of Γ, the edges associated to $\eta\gamma\eta^{-1}$ are divided into two groups $(\eta(E_{j_2}), \cdots, \eta(E_n), \eta\gamma(E_1), \cdots, \eta\gamma(E_{j_1}))$ and $(\eta\gamma(E_{j_1}), \cdots, \eta\gamma(E_{j_2}))$. So we transfer the argument to the case of $a=2$.

(3) *Case* $a=2$. The possibilities are the equivalence classes of the following words:

$$[(X, Y^{-1})(Y, X^{-1})]^k(X, X^{-1}) \quad \text{with} \quad (X, Y) \in P, \quad 1 \leqq k < r/4,$$

(3.8)

$$[(X, Y^{-1})(Y, X^{-1})]^k(X, Y^{-1})(Y, Y^{-1}) \quad \text{with} \quad (X, Y) \in P, \quad 1 \leqq k < (r-2)/4 .$$

The images of the words in (3.8) under Φ are conjugate to AC^k $(2 \leqq k < r/2+2)$ or their inverses. The transformation C^k has the following expression:

$$\frac{1}{\operatorname{Im} p_C}\begin{bmatrix} (\operatorname{Im} p_C)\cos\left(\frac{k\pi}{r}\right)+(\operatorname{Re} p_C)\sin\left(\frac{k\pi}{r}\right) & -|p_C|^2\sin\left(\frac{kn}{r}\right) \\ \sin\left(\frac{k\pi}{r}\right) & (\operatorname{Im} p_C)\cos\left(\frac{k\pi}{r}\right)-(\operatorname{Re} p_C)\sin\left(\frac{k\pi}{r}\right) \end{bmatrix}$$

For the present case, $\operatorname{Im} p_C=\cos(\pi/q)^{-1}\sin(\pi/r)$ and $|p_C|=1$. Hence $|\operatorname{tr} AC^k|=2\sin(\pi/r)^{-1}\cos(\pi/q)|\sin(k\pi/r)|$. If $(p, q, r)=(2, 4, 6)$ or $(p, q, r)=(2, 4, 5)$ *and* $k=2, 3$, $|\operatorname{tr} AC^k|\geqq 2\sqrt{2}\cos(\pi/5)>c_0$. If $(p, q, r)=(2, 4, 5)$, $AC^4=B$ is elliptic. For signatures $(2, 3, r)$, $r\geqq 7$, $AC^2=(ACA)C^{-1}(ACA)^{-1}$ and $AC^{r-1}=AC^{-2}=(ACA)^{-1}C(ACA)$ are elliptic. If $3\leqq k\leqq r-3$, then $|\operatorname{tr} AC^k|\geqq \sin(3\pi/7)\sin(\pi/7)^{-1}=c_0$. Therefore we conclude that $|\operatorname{tr}\gamma|\geqq c_0$ for every hyperbolic element γ of $\Gamma(2, q, r)$, if $(2, q, r)$ is of type III. Now the proof of the proposition is completed. We remark that $|\operatorname{tr} AC^3|=c_0$ for AC^3 in the group $\Gamma(2, 3, 7)$.

4. Completion of the proof of the theorem. It is not difficult to show that there are no simple closed geodesics on $\boldsymbol{H}/\Gamma$ if Γ is a triangle group, from which the sharpness of the inequality of the theorem follows. However we conclude the theorem by a direct computation. Let us consider the following elements of $\Gamma(2, 3, 7)$ both of which have the absolute value of trace c_0:

$$CBA=-\frac{1}{4}\begin{bmatrix} 1+3\lambda^{-2} & -(\lambda-\lambda^{-1})\sqrt{3} \\ -(\lambda-\lambda^{-1})\sqrt{3} & 1+3\lambda^2 \end{bmatrix}$$

$$BAC=-\frac{1}{4}\begin{bmatrix} 1+3\lambda^{-2} & (\lambda-\lambda^{-1})\sqrt{3} \\ (\lambda-\lambda^{-1})\sqrt{3} & 1+3\lambda^2 \end{bmatrix}$$

where $\lambda=\lambda(2, 3, 7)=\{2\cos(\pi/7)+(4\cos^2(\pi/7)-3)^{1/2}\}/\sqrt{3}$. We set $D=\{3(\lambda^2+\lambda^{-2})+2\}^2-4^3$. Then the fixed points $(3(\lambda^2-\lambda^{-2})\pm\sqrt{D})/2\sqrt{3}(\lambda-\lambda^{-1})$ of CBA separates the fixed points $(-3(\lambda^2-\lambda^{-2})\pm\sqrt{D})/2\sqrt{3}(\lambda-\lambda^{-1})$ of BAC in $\boldsymbol{R}$, since

$$-3(\lambda^2-\lambda^{-2})-\sqrt{D}<\sqrt{3}(\lambda^2-\lambda^{-2})-\sqrt{D}<-3(\lambda^2-\lambda^{-2})+\sqrt{D}<3(\lambda^2-\lambda^{-2})+\sqrt{D}.$$

Thus CBA satisfies the condition (∞) with respect to $\Gamma(2, 3, 7)$ and now we can conclude the theorem.

We remark that the axis a_{CBA} of CBA is projected under ϕ: $\boldsymbol{H}\to\boldsymbol{H}/\Gamma(2, 3, 7)$ onto a geodesic segment which connects the projection of the elliptic fixed points of order 2 to itself. This fact follows, since CBA and $ACB=A(CBA)A^{-1}$ have the same fixed points and thus the same axes. The set $\phi(a_{CBA})$ is topologically a simple closed curve. However, as mentioned in §1, $\phi(a_{CBA})$ is regarded here as a degenerate closed curve. From this point of view we can say that $\phi(a_{CBA})$ has self-intersections.

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Added in proof on November 9, 1989. After this paper was submitted, Professors A. F. Beardon and Ch. Pommerenke informed the author that the theorem on p. 527 was already obtained by Pommerenke and Purzitski [14]. Their proof was based on computation of commutators defined in an iterational manner. Our proof is more geometric. Professor Beardon also pointed out the following: Using Theorem 11.6.8 in [1] it can be shown that, if a hyperbolic element γ satisfies the condition (∞) in a Fuchsian group Γ, then $|\operatorname{tr}\gamma| \geqq 2\sqrt{2} > c_0$ except when Γ has one of the signatures $(0; 2, 3, q)$, $(0; 2, 4, q)$ and $(0; 3, 3, 4)$.

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