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UNIFORM ULTIMATE BOUNDEDNESS AND PERIODICITY IN FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction. In this section we give a brief description of the background of and solution to the problem considered. The detailed assumptions are given in the next section.

It is shown that if solutions of the infinite delay T-periodic system

$$x' = f(t, x_t) \tag{1}$$

are uniformly ultimately bounded (UUB) in the *supremum norm*, then there is a T-periodic solution. This improves known results which have required that solutions of (1) also be uniformly bounded (UB). It was shown by Kato [11] that uniform ultimate boundedness for (1) does not imply uniform boundedness.

This problem goes back to Levinson [12]. It was solved for second order ordinary differential equations by Cartwright [7] and Massera [13]; solutions for general n followed from Browder's fixed point theorem (cf. Browder [2] and Yoshizawa [16; p. 158]). Hale and Lopes [9] show that if (1) has finite delay then UB and UUB imply that (1) has a T-periodic solution. It is known that for periodic ordinary differential equations then UUB implies UB.

When (1) has unbounded delay, an example of Seifert [15] shows that if UUB is expected, then (1) must have some type of fading memory. Moreover, it was believed until very recently that in order to prove that (1) has a T-periodic solution using UB and UUB, then the boundedness must be in terms of a weighted norm on the phase space [1] which allowed unbounded initial functions. If (1) has the special form

$$x' = h(t, x) + \int_{-\infty}^{t} q(t, s, x(s))ds \tag{1*}$$

then a simple fading memory was defined in [3] which enables one to show that if solutions are UB and UUB in the supremum norm, then the same is true for a weighted norm. Investigators have been unsuccessful in extending the stated result for (1)* to (1). The details for this summary are found in [4; pp. 214–324]. The recent survey book by Hale [8] continues the problem to operator equations.

Recently, Burton-Dwiggins-Feng [5] have shown that if (1) has a fading memory

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and if solutions of (1) are UB and UUB in the supremum norm, then (1) has a T-periodic solution. Thus, the present note removes the UB assumption.

2. The setting. Let $(X, \|\cdot\|)$ be the space of bounded continuous functions $\varphi: (-\infty, 0] \to R^n$ with the supremum norm and consider the system

$$x'(t) = f(t, x_t) \tag{1}$$

where $x_t(s) = x(t+s)$ for $-\infty < s \le 0$ and where

$$f: [0, \infty) \times X \to R^n, \qquad f(t+T, \varphi) = f(t, \varphi) \tag{2}$$

for some $T > 0$. It will be supposed that

$$\text{for each } \varphi \in X \text{ there is a unique solution } x(t, 0, \varphi) \text{ satisfying (1) on } 0 \le t < \infty \text{ with } x_0(\cdot, 0, \varphi) = \varphi\,. \tag{3}$$

In the way of fulfillment of (3), Sawano [14] asks that:

$$\text{if } x: (-\infty, A) \to R^n \text{ is bounded and continuous, then } f(t, x_t) \text{ is measurable in } t \in [0, A)\,, \tag{H1}$$

$$\text{for any bounded set } V \subset X \text{ there exists a function } m(t) = m_V(t), \text{ locally integrable on } R^+, \text{ such that } |f(t, \varphi)| \le m(t) \text{ for any } \varphi \in V\,, \tag{H2}$$

and

$$f(t, \varphi) \text{ is continuous in } \varphi \text{ for each } t \in R^+\,. \tag{H3}$$

He then shows that (1) has a solution on some interval $0 \le t \le \alpha$. Moreover,

$$\text{if there is a locally integrable function } \eta(t) = \eta_V(t) \text{ such that } |f(t, \varphi) - f(t, \psi)| \le \eta(t)\|\varphi - \psi\| \text{ on } R^+ \times V\,, \tag{H4}$$

then the solution is unique. Finally, if the solution is defined on $[0, \alpha)$ and is noncontinuable beyond α, then $\limsup_{t \to \alpha^-} |x(t, 0, \varphi)| = \infty$. Since our result asks that solutions be UUB, they will be continuable to $+\infty$.

The following notation will be adopted.

R^n denotes n-dimensional Euclidean space, R^-, R^+, R mean the intervals, $(-\infty, 0]$, $[0, \infty)$, and $(-\infty, \infty)$ respectively.

For every $t \ge 0$, let $P_t: X \to X$ be defined by

$$(P_t\varphi)(s) = x(t+s, 0, \varphi) \qquad \text{for} \quad -\infty < s \le 0\,.$$

G denotes the set of continuous non-increasing functions $g: (-\infty, 0] \to [1, \infty)$ such that $g(r) \to \infty$ as $r \to -\infty$ and $g(0) = 1$.

For a given $g \in G$, then $(X_g, |\cdot|_g)$ denotes the Banach space of continuous functions $\varphi: R^- \to R^n$ for which

$$|\varphi|_g = \sup_{-\infty < t \le 0} |\varphi(t)/g(t)|$$

exists.

Let $x\colon [a, b] \to R^n$ and define

$$\|x\|^{[a,b]} = \sup\{|x(t)|\colon a \le t \le b\} .$$

DEFINITION 1. Solutions of (1) are *uniformly bounded* at $t=0$ (UB) if for each $B_1 > 0$ there exists $B_2 > 0$ such that $[\|\varphi\| \le B_1, t \ge 0]$ imply that $|x(t, 0, \varphi)| < B_2$. Solutions of (1) are *uniformly ultimately bounded for bound B* at $t=0$ (UUB) if for each $B_3 > 0$ there exists $K > 0$ such that $[\|\varphi\| \le B_3, t \ge K]$ imply that $|x(t, 0, \varphi)| \le B$.

DEFINITION 2. Let $\Omega \subset X$. We say that P_t is *continuous* in (Ω, G) if there is a $g \in G$ and for every $\varphi_1 \in \Omega$, $J > 0$, and $\mu > 0$ there exists a $\delta > 0$ such that $[\varphi_2 \in \Omega, |\varphi_1 - \varphi_2|_g < \delta]$ imply that $|P_J\varphi_1 - P_J\varphi_2|_g < \mu$.

DEFINITION 3. Equation (1) is said to have a *weakly fading memory* in $\Omega \subset X$ if for any $J > 0$, $D > 0$, and $\mu > 0$ there exists a $K > 0$ such that

$$\varphi, \varphi_1 \in \Omega, \quad \|\varphi\| < D, \quad \|\varphi_1\| < D, \quad \varphi(s) = \varphi_1(s) \quad \text{on} \quad [-K, 0], \quad 0 \le t \le J$$

imply that $|f(t, \varphi) - f(t, \varphi_1)| < \mu$.

The following result was proved in [6].

PROPOSITION. *Suppose that* (3) *holds and that*

(i) *equation* (1) *has a weakly fading memory,*

(ii) *$f(t, \varphi)$ is continuous at every (t, φ) of $[0, T] \times X$ with respect to the supremum norm,*

(iii) *for each $M > 0$ and $\alpha > 0$ there exists $L > 0$ such that $[\|\varphi\| \le M, 0 \le t \le \alpha]$ imply that $|f(t, \varphi)| < L$. Then for every $M > 0$ and $\Omega = \{\varphi \in X\colon \|\varphi\| \le M\}$, P_t is continuous in* (Ω, G).

Horn's theorem [10] will now be stated for reference.

THEOREM (Horn). *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space X with S_0 and S_2 compact and S_1 open relative to S_2. Let $P\colon S_2 \to X$ be a continuous function such that for some integer $m > 0$,*

(a) $P^j S_1 \subset S_2 , \qquad 1 \le j \le m-1$

and

(b) $P^j S_1 \subset S_0 , \qquad m \le j \le 2m-1 .$

Then P has a fixed point in S_0.

3. The main result. The proof of the existence of periodic solutions of dissipative systems utilizes a translation map P_t which

(a) must be continuous at least in the supremum norm. In order for solutions of (1) to be defined even locally it is necessary that

(b) f take bounded sets into bounded sets. As f is T-periodic, this takes the form of (ii) in the next theorem. In order for solutions to be UUB, by Seifert's example,

(c) some type of fading memory is required. In view of the referenced proposition of the last section, conditions (ii) and (iii) of the next result are in some sense necessary.

The following theorem yields a T-periodic solution of (1) without asking that solutions be UB.

THEOREM. *Suppose that* (2), (3), *and the following hold*:

(i) *Solutions of* (1) *are* UUB.

(ii) *For each* $M>0$ *there exists* $L>0$ *such that* $[\|\varphi\|\leq M,\ t\geq 0]$ *imply that* $|f(t,\varphi)|<L$.

(iii) *For every bounded (in the supremum norm) set* $\Omega\subset X$, P_t *is continuous in* (Ω, G).

Then (1) *has a T-periodic solution.*

PROOF. We first prepare the sets for Horn's theorem. Since solutions of (1) are UUB, there is an $N>0$ such that

$$[\varphi\in X,\ \|\varphi\|\leq 2B,\ t\geq N] \quad \text{imply that} \quad |x(t,0,\varphi)|\leq B\,.$$

By (ii), there is an $L_B>0$ such that

$$\|\varphi\|\leq 2B \quad \text{implies that} \quad |f(t,\varphi)|<L_B \qquad \text{for all} \quad t\geq 0\,. \tag{4}$$

Let

$$S_B=\{\varphi\in X\colon\ \|\varphi\|\leq 2B,\ |\varphi(u)-\varphi(v)|\leq L_B|u-v| \text{ for } u, v\in R^-\}\,. \tag{5}$$

By (iii) there exists a $g^*\in G$ such that P_N is continuous in the g^*-norm on S_B. Also, S_B is compact in the g^*-norm; hence, $P_N(S_B)$ is bounded in the g^*-norm and, being bounded for $t\leq -N$, is bounded in the supremum norm: there exists $B^*>0$ such that

$$\varphi\in S_B \quad \text{implies that} \quad \|P_N(\varphi)\|\leq B^* \quad \text{and} \quad |P_N(\varphi)|_{g^*}\leq B^*\,. \tag{6}$$

In particular, there exists a $B_1>B$ such that

$$\varphi\in S_B \quad \text{implies that} \quad \|x(\cdot,0,\varphi)\|^{[0,N]}\leq B_1\,. \tag{7}$$

Let

(8) $B_2=B_1+B$ and find $L>L_B$ with $|f(t,\varphi)|<L$ if $t\geq 0$ and $\|\varphi\|\leq B_2$.

By (iii), there is a $g\in G$ such that P_t is continuous in the g-norm on

$$\Omega=\{\varphi\in X\colon\ \|\varphi\|\leq B_2,\ |\varphi(u)-\varphi(v)|\leq L|u-v| \text{ for } u, v\in R^-\} \tag{9}$$

where B_2 and L are defined in (8).

By the UUB, for the B_2 of (8) there exists $K>0$ such that

$$(10) \qquad [\varphi \in X,\ \|\varphi\| \le B_2,\ t \ge K] \quad \text{imply that} \quad |x(t, 0, \varphi)| \le B\,.$$

As P_K is continuous on the compact set Ω in the g-norm, it is uniformly continuous; thus, there exists a $\delta > 0$ such that

$$(11) \quad [\varphi_1, \varphi \in \Omega,\ |\varphi - \varphi_1|_g < 2\delta] \quad \text{imply that} \quad \sup_{0 \le t \le K} |x(t, 0, \varphi) - x(t, 0, \varphi_1)| < B/2\,.$$

In view of (7) and (11),

$$[\varphi \in S_B,\ \varphi_1 \in \Omega,\ |\varphi - \varphi_1|_g < 2\delta] \quad \text{imply that}$$

$$\sup_{0 \le t \le K} |x(t, 0, \varphi_1)| \le \sup_{0 \le t \le K} |x(t, 0, \varphi)| + \sup_{0 \le t \le K} |x(t, 0, \varphi) - x(t, 0, \varphi_1)| \le B_1 + (B/2) < B_2\,.$$

Now define

$$S_2 = \{\varphi \in X:\ \|\varphi\| \le B_2,\ |\varphi(u) - \varphi(v)| \le L|u - v|,\ u, v \in R^-\}\,,$$

$$Q_1 = \bigcup_{\varphi_1 \in S_B} \{\varphi \in X_g\colon |\varphi - \varphi_1|_g < 2\delta\}\,,$$

$$Q_0 = \bigcup_{\varphi_1 \in S_B} \{\varphi \in X_g\colon |\varphi - \varphi_1|_g \le \delta\}\,,$$

$$S_1 = Q_1 \cap S_2\,,$$

$$S_0 = Q_0 \cap S_2\,.$$

Now S_2 is compact in $(X_g, |\cdot|_g)$, while Q_1 is open in $(X_g, |\cdot|_g)$. We will show that Q_0 is closed in $(X_g, |\cdot|_g)$ since S_B is a compact set. Thus, S_1 is open relative to S_2 and S_0 is compact. Moreover, it can be verified that $S_0 \subset S_1 \subset S_2$ are all convex.

To see that Q_0 is closed in $(X_g, |\cdot|_g)$, let $\{\psi_n\} \subset Q_0$ and $|\psi_n - \psi|_g \to 0$ as $n \to \infty$ for some $\psi \in X_g$. For each ψ_n, there exists a $\varphi_n \in S_B$ such that $|\psi_n - \varphi_n|_g \le \delta$. Since S_B is compact in $(X_g, |\cdot|_g)$, there exists a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ and a $\varphi \in S_B$ such that $|\varphi_{n_k} - \varphi|_g \to 0$ as $k \to \infty$. Now

$$|\psi - \varphi|_g \le |\psi - \psi_{n_k}|_g + |\psi_{n_k} - \varphi_{n_k}|_g + |\varphi_{n_k} - \varphi|_g \le \delta + |\psi - \psi_{n_k}|_g + |\varphi - \varphi_{n_k}|_g\,.$$

Letting $k \to \infty$ yields $|\psi - \varphi|_g \le \delta$. This implies that $\psi \in Q_0$ and Q_0 is closed in $(X_g, |\cdot|_g)$.

Define $P\colon S_2 \to X_g$ by

$$(12) \qquad P(\varphi) = x_T(\cdot, 0, \varphi) \qquad \text{for} \quad \varphi \in S_2\,.$$

That is, $P = P_T$ in terms of the notation of Section 2. In preparation for part (a) of Horn's theorem, we now show that $P^j S_1 \subset S_2$ for $j = 1, 2, \cdots$.

For every $\varphi \in S_1$ there is a $\varphi_1 \in S_B$ such that $|\varphi - \varphi_1|_g < 2\delta$. Thus, by (7), (11), and the fact that $B_1 > B$ we have

$$\sup_{0\le t\le K} |x(t,0,\varphi)| \le \sup_{0\le t\le K} |x(t,0,\varphi_1)| + \sup_{0\le t\le K} |x(t,0,\varphi)-x(t,0,\varphi_1)| \le B_1+(B/2)<B_2 .$$

Also, $\varphi\in S_1$ implies that $\|\varphi\|\le B_2$ which, together with (10), yields $|x(t,0,\varphi)|\le B$ for $t\ge K$. Moreover, $|f(t,P_t(\varphi))|\le L$ for $t\ge 0$ by (8). As $P^j(\varphi)=P_{jT}(\varphi)=x_{jT}(\cdot,0,\varphi)$, it is clear that $P^j(\varphi)\in S_2$ for $j=1,2,\cdots$.

Next, we find an m and J with $P^j(S_1)\subset S_0$ for $m+J\le j\le 2(m+J)-1$. First, there is a $J>0$ such that $4B_2<\delta g(-JT)$ where δ is defined just before (11). Use the fact that P_{JT} is continuous on the compact set Ω (see (9)) to find a $\mu>0$ such that

$$\begin{aligned}&[\varphi,\varphi_1\in\Omega,\ |\varphi-\varphi_1|_g<\mu,\ 0\le t\le JT] \quad \text{imply that}\\ &|x(t,0,\varphi)-x(t,0,\varphi_1)|\le\min\{\delta,B\}/2 .\end{aligned} \tag{13}$$

Find $H>0$ such that $4B_2<\mu g(-HT)$. By (10) we have

$$\begin{aligned}&[\varphi\in\Omega,\ P_{kT}(\varphi)\in\Omega \text{ for } k=0,1,2,\cdots, mT>K+HT,\ -HT\le\theta\le 0]\\ &\qquad\text{imply that}\quad |x(mT+\theta,0,\varphi)|\le B .\end{aligned} \tag{14}$$

Define

$$\bar{x}(\theta)=\begin{cases} x(mT+\theta,0,\varphi) & \text{if}\quad -HT\le\theta\le 0\\ x(mT-HT,0,\varphi) & \text{if}\quad -\infty<\theta\le -HT .\end{cases}$$

Then

$$|\bar{x}-P_{mT}(\varphi)|_g=\sup_{\theta\le -HT}|\bar{x}(\theta)-P_{mT}(\varphi)|/g(\theta)\le 2B_2/g(-HT)<\mu/2 \tag{15}$$

by choice of H. This implies that

$$|x(t,0,\bar{x})-x(t,0,P_{mT}(\varphi))|<\min\{\delta,B\}/2 \qquad \text{on}\quad [0,JT] \tag{16}$$

by (13) since (15) holds, $\|\bar{x}\|\le B$, $B_2>2B$, and so $P_{mT}(\varphi)$ and $\bar{x}$ are both in Ω. This yields

$$|x(t,0,\bar{x})|\le(B/2)+|x(t,0,P_{mT}(\varphi))|<2B \qquad \text{on}\quad [0,JT] \tag{17}$$

since $x(t,0,P_{mT}(\varphi))=x(t+mT,0,\varphi)$. Hence,

$$|x'(t,0,\bar{x})|=|f(t,P_t(\bar{x}))|\le L_B \qquad \text{on}\quad [0,JT] \tag{18}$$

by (4).

Let

$$y(t)=\begin{cases}\bar{x}(0) & \text{for}\quad t\le 0\\ x(t,0,\bar{x}) & \text{for}\quad 0\le t\le JT .\end{cases} \tag{19}$$

It follows that $y_{JT}\in S_B$ by (17) (see (5)) and that for $\varphi\in S_2$ then

$$|P_{(m+J)T}(\varphi)-y_{JT}|_g=|P_{JT}(P_{mT}(\varphi))-y_{JT}|_g=\sup_{\theta\le 0}|x(JT+\theta,0,P_{mT}(\varphi))-y(JT+\theta)|/g(\theta)$$

$$\leq \sup_{\theta \leq -JT} |x(JT+\theta, 0, P_{mT}(\varphi)) - y(JT+\theta)|/g(\theta)$$

$$+ \sup_{-JT \leq \theta \leq 0} |x(JT+\theta, 0, P_{mT}(\varphi)) - y(JT+\theta)|/g(\theta)$$

$$\leq 2B_2/g(-JT) + \sup_{-JT \leq \theta \leq 0} |x(JT+\theta, 0, P_{mT}(\varphi)) - x(JT+\theta, 0, \bar{x})|$$

$$\leq (\delta/2) + \sup_{0 \leq t \leq JT} |x(t, 0, P_{mT}(\varphi)) - x(t, 0, \bar{x})| \leq \delta$$

by choice of J (see the material just before (13)) and by (16). This proves that if $\varphi \in \Omega$ and $x_{kT}(\cdot, 0, \varphi) = P_{kT}(\varphi) \in \Omega$ for $k = 1, 2, \cdots$, then

$$x_{(m+J)T}(\cdot, 0, \varphi) = P_{(m+J)T}(\varphi) \in S_0 \tag{20}$$

by definition of Q_0.

In particular, if $\varphi \in S_1$, then $P_{(m+J)T}(\varphi) \in S_0$. Now consider $P_{(m+J+1)T}(\varphi)$ for $\varphi \in S_1$. It follows that $P_T(\varphi) \in S_2$ and $P_{kT}(P_T(\varphi)) \in S_2$ for $k = 1, 2, \cdots$. By (20) we have $P_{(m+J)T}(P_T(\varphi)) \in S_0$. But $P_{(m+J+1)T}(\varphi) = P_{(m+J)T}(P_T(\varphi))$. Thus, $P_{(m+J+1)T}(\varphi) \in S_0$. In this way we argue that

$$P^j S_1 \subset S_2 \qquad \text{for} \quad 1 \leq j \leq m+J-1\,,$$

$$P^j S_1 \subset S_0 \qquad \text{for} \quad m+J \leq j \leq 2(m+J)-1\,.$$

Also, P is continuous in the g-norm by (iii). By Horn's theorem, there is a $\varphi \in S_0$ with $P\varphi = \varphi$. Since $x(t, 0, \varphi)$ and $x(t+T, 0, \varphi)$ are both solutions of (1) with the same initial function, by uniqueness, they are equal. This completes the proof.

REMARK. Many examples of UUB are to be found in [1], [4], and [5]. In the example of Kato [11], solutions are UUB, but not UB, and 0 is the unique periodic solution.

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