INVARIANCE PRINCIPLES AND ASYMPTOTIC CONSTANCY OF SOLUTIONS OF PRECOMPACT FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We apply the method of Liapunov functions to obtain a new invariance principle for a class of nonautonomous functional differential equations. An asymptotic stability result is applied to an equation from population dynamics.

1. Introduction and notation. In 1983, Haddock and Terjéki [6] employed Liapunov-Razumikhin techniques to develop an invariance principle for autonomous functional differential equations (F.D.E.'s) with finite delay. One of the purposes of this paper is to develop a Liapunov-Razumikhin invariance principle which includes nonautonomous F.D.E.'s. In doing so, we employ the notion of precompactness of equations. The basic idea is to use properties of the "limiting equations" of a given equation to determine the asymptotic behavior of solutions of the equation. We obtain generalizations of several of the theorems of [6] above, and we illustrate the applicability of our methods by examining an equation dealing with population dynamics, an autonomous version of which was treated in [5].

The theory of limiting equations for ordinary differential equations is well developed. See, for example, [1]–[3]. Considerable progress has also been made employing the limiting equation concept to functional differential equations. References here include [11]. The definition of limiting equation in the F.D.E. setting tends to be more narrow than that encountered in the O.D.E. literature. This is, no doubt, a consequence of the relative complexity of the phase space in the F.D.E. case. This paper represents the first attempt, as far as we know, to combine the tools of limiting equations and Liapunov functions to investigate the behavior of solutions to F.D.E.'s.

We employ the following notation: For $x \in R^n$, $|x|$ denotes the Euclidean norm of x. For $r > 0$, $C_r = C([-r, 0], R^n)$; that is, C_r is the Banach space of continuous R^n-valued functions on $[-r, 0]$ equipped with the sup norm $|\phi|_r = \sup_{-r \le s \le 0} |\phi(s)|$. In general, if Y is a given topological space, we denote by $C(Y, R^n)$ the space of continuous R^n-valued functions endowed with the compact-open topology. If $x \in C([t_0 - r, \infty), R^n)$, then for each $t \ge t_0$ we denote by x_t the element of C_r defined by $x_t(s) = x(t + s)$, $-r \le s \le 0$. The *positive orbit* of x_t in C_r is the set $\gamma^+(x_t) = \{x_t : t \ge t_0\}$. If $x \in C(R, R^n)$ we define functions x_t as above for all values of t, and we define the *orbit* of x_t as $\gamma(x_t) = \{x_t : t \in R\}$. Next, we define the *ω-limit set* of x_t, denoted by $\Omega(x_t)$, to be the set of all $\psi \in C_r$ for which there exists a sequence $t_k \to \infty$ with $x_{t_k} \to \psi$ in C_r. Finally, for each $f \in C(R \times C_r, R^n)$ and

$t_0 \in \mathbf{R}$, we define a "translated" function $f^{t_0} \in C(\mathbf{R} \times C_r, \mathbf{R}^n)$ by $f^{t_0}(t, \phi) = f(t + t_0, \phi)$.

The differential equations which we consider are of the form

$$(1.1) \qquad\qquad\qquad\qquad x' = f(t, x_t)$$

where $f \in C(\mathbf{R} \times C_r, \mathbf{R}^n)$ and

(1.2) f is bounded and uniformly continuous on sets of the form $\mathbf{R} \times K$ whenever K is a compact subset of C_r, and

(1.3) $f(\mathbf{R} \times H)$ is bounded whenever H is a bounded subset of C_r.

These conditions are sufficient for the following to hold: For any sequence $t_k \to \infty$ there exists a subsequence $\{t_{k'}\}$, such that $f^{t_{k'}} \to g$ uniformly on compact subsets of $\mathbf{R} \times C_r$. For a proof, see [9]. Associated with equation (1.1) are initial value problems

$$(1.4) \qquad\qquad\qquad x' = f(t, x_t), \qquad x_{t_0} = \phi .$$

The above conditions are sufficient to insure the existence of solutions to any initial value problem. We shall assume for the rest of the paper that these solutions are unique, and we denote the unique solution of (1.4) by $x(t_0, \phi, f)$. Equations satisfying (1.2) above are said to be *precompact*. If $x(t_0, \phi, f)$ is a solution of (1.4) defined and bounded on $[t_0 - r, \infty)$, then $\Omega(x_t(t_0, \phi, f))$ is nonempty, connected and compact in C_r, and $\gamma^+(x_t(t_0, \phi, f))$ is connected, and its closure in C_r is compact (cf. [7]). Finally, let Γ denote a collection $\{x' = g(t, x_t): g \in G\}$ of equations where $G \subset C(\mathbf{R} \times C_r, \mathbf{R}^n)$. Then a subset M of C_r is said to be *positively semi-invariant* with respect to Γ if for each $\psi \in M$ and $t_0 \in \mathbf{R}$ there exists an equation $x' = g(t, x_t)$ in the set Γ with a solution $x(t_0, \psi, g)$ defined on $[t_0 - r, \infty)$ such that $\gamma^+(x_t(t_0, \psi, g)) \subset M$. A subset M of C_r is said to be *semi-invariant* with respect to Γ if for each $\psi \in M$ and $t_0 \in \mathbf{R}$ there exists an equation $x' = g(t, x_t)$ in Γ and a solution $x(t_0, \psi, g)$ defined on all of $\mathbf{R}$ with $\gamma(t_0, \psi, f) \subset M$.

2. Some invariance principles for F.D.E.'s.

The significance of precompactness of (1.1) lies in the following: given any sequence $t_k \to \infty$, there exists a subsequence $\{t_{k'}\}$ and a $g \in C(\mathbf{R} \times C_r, \mathbf{R}^n)$ with $f^{t_{k'}} \to g$ in this function space. We denote by $\Omega(f)$ the set of all $g \in C(\mathbf{R} \times C_r, \mathbf{R}^n)$ for which there exists a sequence $t_k \to \infty$ with $f^{t_k} \to g$, and we denote by $\mathscr{L}^+(f)$ the collection of *limiting equations* $\{x' = g(t, x_t): g \in \Omega(f)\}$. Finally, for any subset H of C_r, and any collection $\Gamma = \{x' = g(t, x_t): g \in G \subset C(\mathbf{R} \times C_r, \mathbf{R}^n)\}$ of equations, we denote by $M(\Gamma, H)$ the largest subset of H semi-invariant with respect to Γ.

THEOREM 2.1.   *Let* (1.1) *be as above and suppose* $x(t_0, \phi, f)$ *is a solution defined and bounded on* $[t_0 - r, \infty)$ *with* $\psi \in \Omega(x_t(t_0, \phi, f))$, *say* $x_{t_k}(t_0, \phi, f) \to \psi$. *Then there exists a subsequence* $\{t_{k'}\}$ *of* $\{t_k\}$ *such that* $x_{t_{k'}}(t_0, \phi, f)(t)$ *converges in* $C(\mathbf{R}, \mathbf{R}^n)$ *to a solution* $x(t_0, \psi, g)$ *of the equation* $x' = g(t, x_t)$, *where* $f^{t_{k'}} \to g$ *in* $C(\mathbf{R} \times C_r, \mathbf{R}^n)$.

PROOF.   We assume without loss of generality that $t_0 = 0$ and fix our attention first

on a compact interval $[-N, N]$. Now $x(0, \phi, f)$ bounded implies the boundedness of $\gamma^+(x_t(0, \phi, f))$ (in C_r) which in turn insures the boundedness of $x'(0, \phi, f)$ on R^+ by (1.3) above. By the Arzela-Ascoli Theorem, $\{x_{t_k}(0, \phi, f)(t)\}$ is a precompact family in $C([-N, N], R^n)$. Letting $N = 1, 2, \cdots$, and employing a diagonalization process, we choose a subsequence, which we again denote by $\{x_{t_k}(0, \phi, f)(t)\}$, that converges in $C(R, R^n)$ to a bounded function $y(t)$. Clearly $x_{t_k+s}(0, \phi, f) \to y_s$ in C_r for every $s \in R$. By the precompactness of (1.1), there exists yet another subsequence, which we still denote by $\{t_k\}$, and a $g \in C(R \times C_r, R^n)$ with $f^{t_k} \to g$. We claim that $y(t) = x(0, \psi, g)(t)$ for all t. In light of the continuity of the mapping $s \to y_s$, $y'(t) = g(t, y_t)$ for all t is equivalent to the integral equation

$$y(t) = \int_0^t g(s, y_s)ds + y(0) .$$

Now

$$y(t) = \lim_{k \to \infty} x_{t_k}(0, \phi, f)(t) = = \lim_{k \to \infty} \left[\int_0^t f^{t_k}(s, x_{t_k+s}(0, \phi, f))ds + x_{t_k}(0, \phi, f)(0) \right]$$

$$= \int_0^t g(s, y_s)ds + y(0) ,$$

the last equality following from the observation that $f^{t_k}(s, x_{t_k+s}(0, \phi, f))$ converges (pointwise and boundedly) to $g(s, y_s)$, followed by an application of the Lebesgue Dominated Convergence Theorem.

The following three theorems are analogues of results in ordinary differential equations found in [1].

THEOREM 2.2. *Suppose $x(t_0, \phi, f)$ is a bounded solution of* (1.1) *on* $[t_0 - r, \infty)$ *and* $\psi \in \Omega(x_t(t_0, \phi, f))$. *Then there exists a $g \in \Omega(f)$ and a solution $x(t_0, \psi, g)$ of $x' = g(t, x_t)$ defined on R such that $\gamma(x_s(t_0, \psi, g)) \subset \Omega(x_t(t_0, \phi, f))$. That is, $\Omega(x_t(t_0, \phi, f))$ is semi-invariant with respect to $\mathscr{L}^+(f)$.*

PROOF. Suppose $x_{t_k}(t_0, \phi, f) \to \psi$ where $t_k \to \infty$. The existence of an appropriate g and $x(t_0, \psi, g)$ follows from Theorem 2.1. We need only note that for $s \in R$, $x_s(t_0, \psi, g) = \lim_{k' \to \infty} x_{t_{k'}+s}(t_0, \phi, f)$ for some subsequence $\{t_{k'}\}$ of $\{t_k\}$, and hence that $x_s(t_0, \psi, g) \in \Omega(x_t(t_0, \phi, f))$ for every $s \in R$.

THEOREM 2.3. *Consider equation* (1.1) *without the precompactness hypothesis* (1.2). *Suppose, however, that $x' = g(t, x_t)$ is a limiting equation* (1.1). *If $x(t_0, \phi, f)$ is defined and bounded on $[t_0 - r, \infty)$, then there exists a $\psi \in \Omega(x_t(t_0, \phi, f))$ and a solution $x(t_0, \psi, g)$ of the limiting equation $x' = g(t, x_t)$ defined on R with $\gamma(x_s(t_0, \psi, g)) \subset \Omega(x_t(t_0, \phi, f))$.*

PROOF. Suppose $f^{t_k} \to g$ in $C(R \times C_r, R^n)$. The boundedness of $x(t_0, \phi, f)$ implies the precompactness of $\gamma^+(x_t(t_0, \phi, f))$ in C_r. Hence there exists a subsequence $\{t_{k'}\}$ of

$\{t_k\}$ such that $x_{t_k}(t_0, \phi, f) \to \psi$. Employing techniques used in the proof of Theorem 2.1, we see that $x_{t_{k'} + s}(t_0, \phi, f) \to x_s(t_0, \psi, g)$ for every $s \in \mathbf{R}$ where $x(t_0, \psi, g)$ is a solution on $\mathbf{R}$ of the limiting equation, and clearly $x_s(t_0, \psi, g) \in \Omega(x_t(t_0, \phi, f))$ for all s.

COROLLARY 2.1.   *If* $x(t_0, \phi, f)(t) \to c \in \mathbf{R}$, *then* $y(t) = c$ *is a solution for any limiting equation of* (1.1).

THEOREM 2.4.   *Suppose* $x(t_0, \phi, f)$ *is a bounded solution of* (1.1) *on* $[t_0 - r, \infty)$, *and* $x_t(t_0, \phi, f) \to H \subset C_r$ *as* $t \to \infty$. *Then* $x_t(t_0, \phi, f) \to M(\mathcal{L}^+(f), H)$.

PROOF.   We have $x_s(t_0, \phi, f) \to \Omega(x_t(t_0, \phi, f))$ as $s \to \infty$, so $x_s(t_0, \phi, f) \to H$ entails $x_s(t_0, \phi, f) \to H \cap \Omega(x_t(t_0, \phi, f)) \subset M(\mathcal{L}^+(f), H)$, the inclusion following from Theorem 2.2.

3. A fundamental invariance theorem.   In this section, we develop and illustrate an invariance principle for nonautonomous F.D.E.'s with finite delay. The afore-mentioned paper of Haddock and Terjéki is used as a guideline in the development of the main theorems. We are particularly interested in establishing results regarding asymptotic constancy of solutions.

Throughout the remainder of this paper, we make the following convention: Characterization of a subset H of C_r as positively invariant without further clarification means positively invariant with respect to (1.1).

DEFINITION 3.1.   Let V denote a function from $\mathbf{R}^{n+1}$ into $\mathbf{R}$. The *upper right-hand derivative* of V with respect to (1.1) is given by

$$V'_f[t, \phi] = \lim_{h \to 0^+} (V[t+h, x(t, \phi, f)(t+h)] - V[t, \phi(0)])/h .$$

A continuously differentiable function V is called a *Liapunov* (or *Razumikhin*) *function* for equation (1.1) if the following holds:

$$V'_f[t, \phi] \le 0 \text{ whenever } \phi \in C_r \text{ with } V[t, \phi(0)] = \max_{-r \le s \le 0} V[t+s, \phi(t+s)] .$$

In applications, we often restrict ϕ to some subset of C_r positively invariant with respect to (1.1). Throughout the remainder of this paper, we make the following assumption: V is bounded and uniformly continuous on sets of the form $\mathbf{R} \times B$ whenever B is a bounded subset of $\mathbf{R}^n$. The Ascoli Theorem assures us that for any sequence $\{t_k\}$ there exists a subsequence $\{t_{k'}\}$ and a function $\hat{V} \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ such that $V^{t_{k'}} \to \hat{V}$ in $C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ where $V^p[t, x] \equiv V[t+p, x]$. Set $\Omega(V) = \{\hat{V} \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}): \hat{V} = \lim_{k \to \infty} V^{t_k}, t_k \to \infty\}$, the limit being in $C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$. For our purposes, Liapunov functions V for which $\Omega(V)$ consists of a single element $\hat{V}$ are of particular importance.

Consider again equation (1.1). For any Liapunov function V for (1.1), $\hat{V} \in \Omega(V)$, $c, t_0 \in \mathbf{R}$, $g \in \Omega(f)$, and closed $D \subset C_r$ positively invariant with respect to (1.1), set

$$M_{\hat{V}}(t_0, g, D; c) =$$

$$\{\psi \in D: x(t_0, \psi, g)(t) \text{ is defined on } \boldsymbol{R} \text{ and } \max_{-r \le s \le 0} \hat{V}[t+s, x(t_0, \psi, f)(t+s)] \equiv c\}\,.$$

THEOREM 3.1. *Let V be a Liapunov function for (1.1) and $D \subset C_r$ a closed, positively invariant set. Then for any $\phi \in D$ such that $x(t_0, \phi, f)$ if defined and bounded on $[t_0 - r, \infty)$ and every $\psi \in \Omega(x_t(t_0, \phi, f))$, we have $\psi \in M_{\hat{V}}(t_0, g, D; c)$ for some $c \in \boldsymbol{R}$, $\hat{V} \in \Omega(V)$, and $g \in \Omega(f)$. Then number c, but not necessarily $\hat{V}$ and g, are independent of our choice of $\psi \in \Omega(x_t(t_0, \phi, f))$.*

PROOF. From the boundedness of $x(t_0, \phi, f)$, and the assumptions on (1.1), we see that $\Omega(x_t(t_0, \phi, f))$ is nonempty and compact. By a Razumikhin-type argument, it is clear that $\max_{-r \le s \le 0} V[t+s, x(t_0, \phi, f)(t+s)]$ is nonincreasing (as a function of t) on $\boldsymbol{R}$. Details can be found in [10]. Suppose $\psi \in \Omega(x_t(t_0, \phi, f))$, say $\phi = \lim_{k \to \infty} x_{t_k}(t_0, \phi, f)$ where $t_k \to \infty$. In light of the precompactness of (1.1), Theorem 2.1, and the assumptions regarding V, there exists a subsequence $\{t_{k'}\}$ such that

(i) $f^{t_{k'}} \to g$ in C_r for some $g \in C_r$ and $x_{t_{k'}}(t_0, \phi, f)(t) \to x(t_0, \phi, g)(t)$ in $C(\boldsymbol{R}, \boldsymbol{R}^n)$,

and

(ii) $V^{t_{k'}} \to \hat{V}$ in $C(\boldsymbol{R} \times \boldsymbol{R}^n, \boldsymbol{R})$ for some $\hat{V} \in C(\boldsymbol{R} \times \boldsymbol{R}^n, \boldsymbol{R})$.

Now

$$\max_{-r \le s \le 0} \hat{V}[t+s, x(t_0, \phi, g)(t+s)] = \max_{-r \le s \le 0} \lim_{k' \to \infty} V^{t_{k'}}[t+s, x(t_0, \phi, f)(t+s)]$$

$$= \max_{-r \le s \le 0} \lim_{k' \to \infty} V\left[t+s+t_{k'}, \lim_{k' \to \infty} x(t_0, \phi, f)(t+s+t_{k'})\right]$$

$$= \lim_{k' \to \infty} \lim_{-r \le s \le 0} V[t+s+t_{k'}, x(t_0, \phi, f)(t+s+t_{k'})] \equiv c = c(t_0, \phi)\,.$$

Hence $\psi \in M_{\hat{V}}(t_0, g, D; c)$, and the theorem is proved.

Let $x(t_0, \phi, f)$ be a bounded solution of (1.1). It is a fundamental theorem that $x_s(t_0, \phi, f) \to \Omega(x_t(t_0, \phi, f))$ as $s \to \infty$. (See [7].) If we set

$$M_V(t_0, f, D; c) = \bigcup_{g \in \Omega(f)} \bigcup_{\hat{V} \in \Omega(V)} M_{\hat{V}}(t_0, g, D; c)\,,$$

then we have

$$x_s(t_0, \phi, f) \to \Omega(x_t(t_0, \phi, f)) \subset M_V(t_0, f, D; c)\,.$$

Next, set $M_V(t_0, f, D) = \bigcup_{c \in \boldsymbol{R}} M_V(t_0, f, D; c)$. The following is an easy corollary:

COROLLARY 3.1. *Suppose there exists a constant function $K \in C_r$ such that $M_V(t_0, f, D) = \{K\}$. Then every bounded solution $x(t_0, \phi, f)$ of (1.1) with $\phi \in D$ tends to the constant value K as $t \to \infty$.*

Consider again equation (1.1). Suppose D is a closed subset of C_r positively invariant with respect to (1.1). Suppose as well that $u \in C(\mathbf{R}, \mathbf{R}^n)$ is bounded and $\bigcup_{t \in \mathbf{R}} B_r(u_t, r_1) \subset D$ where $B_r(u_t, r_1)$ is the open ball in C_r of radius r_1 centered at u_t. We say that the function u is *eventually stable* with respect to (1.1) if for each $\varepsilon > 0$ there is an $\alpha = \alpha(\varepsilon)$ and a $\delta = \delta(t_0, \varepsilon)$ with $0 < \delta < r_1$ such that for any $t_0 \geq \alpha$, $|u_{t_0} - \phi|_r < \delta$ implies $|x_t(t_0, \phi, f) - u_t|_r < \varepsilon$ for all $t \geq t_0$. We say that u is *eventually D-globally asymptotically stable* with respect to (1.1) if it is eventually stable with respect to (1.1) and for any $\phi \in D$, $|x_t(t_0, \phi, f) - u_t|_r \to 0$ as $t \to \infty$ for every t_0.

To simplify the statement of the next theorem, we introduce some notation. Suppose that K is the constant function whose value is $K \in \mathbf{R}^n$, $\phi \in D$, D a closed, positively invariant subset of C_r, and that V is a Liapunov function for (1.1). We can think of the triple (ϕ, f, V) as an element of $X \equiv C([-r, 0] \times \mathbf{R}^{n+1} \times D, \mathbf{R}^{2n+1})$. For any $t_0 \in \mathbf{R}$, we define a semi-group $T_{t_0} : \mathbf{R}^+ \to X$ by

$$T_{t_0}(t) = (x_{t+t_0}(t_0, \phi, f), f^{t+t_0}, V^{t+t_0}) .$$

We set $\Omega_D(t_0; \phi, f, V) = \{(\psi, g, \hat{V}) : T_{t_0}(t_k)(\phi, f, V) \to (\psi, g, \hat{V})$ for some $t_k \to \infty\}$. That is, $\Omega_D(t_0; \phi, f, V)$ is just the ω-limit set of (ϕ, f, V) under the semi-group T_{t_0}.

A *wedge* is a continuous, strictly increasing function $W : \mathbf{R}^+ \to \mathbf{R}^+$ with $W(0) = 0$.

THEOREM 3.2. *Suppose that V is a Liapunov function for (1.1) and K is the constant function whose value is $K \in \mathbf{R}^n$. Suppose also that W is a wedge and that*

   (i) $V[t, x] \geq W(|x - K|)$ *for every* $x \in \mathbf{R}^n$, $t \in \mathbf{R}$;

   (ii) $\lim_{r \to \infty} W(r) = \infty$;

   (iii) $\lim_{t \to \infty} V[t, K] \to 0$; *and*

   (iv) $\hat{V}'_g[t, \psi] < 0$ *whenever* $(\psi, g, \hat{V}) \in \Omega_D(t; \phi, f, V)$ *with* $\psi \neq K$ *and*

$$\hat{V}[t, \psi(0)] = \max_{-r \leq s \leq 0} \hat{V}[t+s, x(t_0, \psi, g)(t+s)] .$$

Then K is eventually D-globally asymptotically stable.

PROOF. We first establish the eventual stability of K. Let $\varepsilon > 0$ be given. Then by (ii) above, there is an $\alpha > 0$ such that $V[t, K] < W(\varepsilon)/2$ whenever $t \geq \alpha - r$. For any $t_0 \geq \alpha$, there exists a $\delta > 0$ such that $\max_{-r \leq s \leq 0} |V[t_0 + s, \phi(s)] - V[t_0 + s, K]| < W(\varepsilon)/2$ whenever $|\phi - K|_r < \delta$. Now for any $t_0 \geq \alpha$ and $\phi \in D$ with $|\phi - K|_r < \delta$, $t \geq t_0$ implies that

$$W(|x(t_0, \phi, f)(t) - K|) \leq V[t, x(t_0, \phi, f)(t)] \leq \max_{-r \leq s \leq 0} V[t+s, x(t_0, \phi, f)(t+s)]$$

$$\leq \max_{-r \leq s \leq 0} V[t_0 + s, \phi(s)] < W(\varepsilon) .$$

It follows that $|x_t(t_0, \phi, f) - K|_r < \varepsilon$ whenever $|\phi - K|_r < \delta$ and $t \geq t_0$. Next, we note that all solutions of (1.1) are bounded. For any $t_0 \in \mathbf{R}$ and $\phi \in D$, we have again that $W(|x(t_0, \phi, f)(t) - K|) \leq \max_{-r \leq s \leq 0} V[t_0 + s, \phi]$ for all $t \geq t_0$, and the boundedness of

$\gamma^+(x_t(t_0, \phi, f))$ follows from (ii) above. Consider a solution $x(t_0, \phi, f)$ of (1.1), and suppose $(\psi, g, \hat{V}) \in \Omega_D(t_0; \phi, f, V)$. If $\psi \neq K$, it follows from (iv) above and standard comparison results that $\max_{-r \leq s \leq 0} \hat{V}[t+s, x(t_0, \phi, f)(t+s)] < \max_{-r \leq s \leq 0} V[t_0 + s, \phi(s)]$ as soon as $t > t_0 + r$, a contradiction to Theorem 3.1. We conclude that $\psi = K$, and hence that all solutions of (1.1) tend to K as $t \to \infty$.

We note that all theorems proved so far have "local" analogues. That is, we could replace C_r with an invariant open subset of C_r. In fact, the example appearing in the next section involves an equation $x' = f(t, x_t)$ where f is not defined on all of $\boldsymbol{R} \times C_r$. However, we believe that more is gained by stating the above results globally and subsequently adapting them for applications than would be achieved by giving local versions of the theorems.

4. An equation from population dynamics.

In a recent paper, Freedman and Gopalsamy [5] established criteria under which three types of equations modelling single species population dynamics have globally asymptotically stable positive equilibria. Such stability is shown, in these models, to be independent of the (finite) delay occuring in these equations. All equations treated in their paper are autonomous. In particular, the authors assume the existence of a unique and constant carrying capacity. We treat an equation which is more general in form than related ones treated in [5]. We allow for a variable "instantaneous carrying capacity" and for somewhat greater flexibility regarding the fashion in which current behavior of the population depends on its past history.

In the following, $C_r^+ \equiv \{\phi \in C_r : \phi(s) > 0 \text{ for all } s \in [-r, 0]\}$, and $\boldsymbol{R}^+ = (0, \infty)$. Suppose $h: \boldsymbol{R} \times \boldsymbol{R}^+ \times \boldsymbol{R}^+ \to \boldsymbol{R}$ is continuous and satisfies the following properties:

(H1) There exists a constant $m > 0$ such that $h(t, 0, 0) \geq m$ for all t.

(H2) There exists a bounded, continuous function $\xi : \boldsymbol{R} \to \boldsymbol{R}^+$ such that for every t we have $h(t, \xi(t), \xi(t)) = 0$ and $(z - \xi(t))h(t, z, z) < 0$ for $z \neq \xi(t)$.

(H3) $\xi(t) \to K$ as $t \to \infty$ for some constant $K > 0$.

(H4) For each $t \in \boldsymbol{R}$ and $x \in \boldsymbol{R}^+$, the mapping $y \to h(t, x, y)$ is increasing.

We consider an equation

(4.1) $$x'(t) = f(t, x_t) = B(t, x_t)h(t, x(t), D(t, x_t))$$

where

(C1) B and D are bounded on sets of the form $\boldsymbol{R} \times H$ whenever H is a bounded subset of C_r^+ and are uniformly continuous on sets $\boldsymbol{R} \times S$ whenever $S \subset C_r$ is compact.

(C2) $B(t, \phi)$, $\hat{B}(t, \phi) > 0$ for every $\phi \in C_r^+$ and $\hat{B} \in \Omega(B)$.

(C3) $\min_{-r \leq s \leq 0} \phi(s) \leq D(t, \phi) \leq \max_{-r \leq s \leq 0} \phi(s)$ for all $\phi \in C_r^+$ and $t \in \boldsymbol{R}$.

We assume also that all initial value problems associated with (4.1) have unique solutions defined for all times t greater than the initial time.

We choose a strictly decreasing, continuously differentiable function $b: \mathbf{R} \to \mathbf{R}^+$ such that

$$K - b(t) < \xi(t) < K + b(t)$$

for all t, where $\lim_{t \to \infty} b(t) = 0$. Set

$$V[t, x] = \max\left\{\frac{(x - K)^2}{2}, \max_{-r \leq s \leq 0} \frac{b^2(t + s)}{2}\right\},$$

and suppose $V[t, \phi(0)] = \max_{-r \leq s \leq 0} V[t + s, \phi(s)]$. Then either

(i) $\phi(0) > K + b(t)$, in which case $\phi(0) > \xi(t)$, and hence

$$V'_f[t, \phi] = (\phi(0) - K)B(t, \phi)h(t, \phi(0), D(t, \phi)) \leq (\phi(0) - K)B(t, \phi)h(t, \phi(0), \phi(0)) < 0,$$

or

(ii) $\phi(0) < K - B(t)$, in which case $\phi(0) < \xi(t)$, and hence

$$V'_f[t, \phi] = (\phi(0) - K)B(t, \phi)h(t, \phi(0), D(t, \phi)) < 0.$$

We see that V is a Liapunov function for (4.1). It is immediate that V satisfies (i)–(iii) of Theorem 3.1 with $W(x) \equiv x^2/2$. Also, it is clear that conditions (C1)–(C3) above guarantee that $\hat{V}'_g[t, \psi] < 0$ whenever $g \in \Omega(f)$ and $\psi \in C_r^+$ with $\psi \neq K$ and $\hat{V}[t, \psi(0)] = \max_{-r \leq s \leq 0} \hat{V}[t + s, \psi(t + s)]$. We remark that $\hat{V}[t, x] = (x - K)^2/2$. Eventual C_r^+-global asymptotic stability of K will follow as soon as we establish the invariance of C_r^+ with respect to (4.1). Suppose $\phi \in C_r^+$ with $\phi(0) = \min_{-r \leq s \leq 0} \phi(s) < K - b(t)$. Then $x'(t_0, \phi, f)(t_0) \geq B(t, \phi)h(t, \phi(0), \phi(0)) > 0$, and the desired invariance follows.

Equation (4.1) represents a generalization of equations I and II of [5]. The more general form of (4.1) suggests that we could allow for certain refinements in our model. Fortunately, any allowable refinements would result in an equation whose solutions approach the "asymptotic carrying capacity" of the environment. The uniform stability of the equilibrium K present in [5] is sacrificed, however, and we must settle for eventual stability.

Next, we consider the effects of weakening the assumption that $\xi(t) \to K$. Suppose instead that ξ is *slowly varying* in the sense that for each $T > 0$,

$$\lim_{t \to \infty}\left[\sup_{t \leq s \leq t+T} \xi(s) - \inf_{t \leq s \leq t+T} \xi(s)\right] = 0.$$

We still assume that $\xi(t) > 0$ for all t, $\liminf_{t \to \infty} \xi(t) > 0$, and that (H1) above holds. With the assumption that ξ is slowly varying, it is still true that any limiting equation

$$(4.2) \qquad x' = g(t, x_t) = \hat{B}(t, x_t)\hat{h}(t, x(t), \hat{D}(t, x_t))$$

of (4.1) will have a (unique) eventually C_r^+-asymptotically stable equilibrium. Let $I = (\liminf_{t \to \infty} \xi(t), \limsup_{t \to \infty} \xi(t))$, and suppose $c \in I$, where $\xi(t_k) \to c$ for a sequence

$t_k \to \infty$. For any $t_0 \in R$ and $\phi \in C_r^+$, there exists a subsequence of $\{t_k\}$, which we again denote by $\{t_k\}$, a $g \in \Omega(f)$ and a $\psi \in \Omega(x_t(t_0, \phi, f))$ such that $f^{t_k} \to g$ and $\gamma(x_t(t_0, \psi, g)) \subset \Omega(x_t(t_0, \phi, f))$. This follows from Theorem 2.3. But $\lim_{t \to \infty} x(t_0, \psi, g)(t) = c$, and hence $c \in \Omega(x_t(t_0, \phi, f))$. It follows that any solution to (4.1) oscillates into I. Such behavior is not unexpected, for even slow variation of the carrying capacity can be destabilizing. See [8].

5. Asymptotic constancy of solutions.

Perhaps the most common application of the limiting equation concept arises in the study of systems whose limiting equations are of a simpler form than the equation itself. For example, equations of the form

$$(5.1) \qquad x' = f(x_t) + g(t, x_t) ,$$

where $g \to 0$ in some sense, are of this type. While the idea of gaining information concerning the qualitative behavior of solutions to complex equations by examining the properties exhibited by the solutions of less complicated ones has great appeal, the role of limiting equations is broader. In this final section, we consider instead equations whose limiting equations retain the same general form as the original equation. Equations which are autonomous save the slow deviation of their arguments sometimes fall into this catagory. We prove an asymptotic constancy result extending work of Haddock and Terjéki [6]. In this case, a Liapunov function for the original equation serves as a Liapunov function for all limiting equations as well. For the remainder of this paper, all Liapunov functions will be autonomous, i.e., $V[t, x] = V[x]$. With this restriction, the theorems of [6] have very natural extensions to precompact nonautonomous F.D.E.'s. We choose to show how Theorem 3.1 of [6] can be generalized, and we present a proof somewhat simpler than the one found there.

We consider again (1.1). Suppose V is a Liapunov function for (1.1), and that $H \subset C_r$ is invariant. In the following, $K_V = \{\phi \in H: V[\phi(s)] = V[\phi(0)]$ for all $s \in [-r, 0]\}$.

LEMMA 5.1. *Suppose there exists a Liapunov function V and a closed set H positively invariant with respect to (1.1) such that whenever $\phi \in H$ with $x(t_0, \phi, f)$ defined and bounded on $[t_0 - r, \infty)$, we have $\Omega(x_t(t_0, \phi, f)) \subset K_V$. Then $x(t_0, \phi, f)(t) \to c$ as $t \to \infty$ where $c = \lim_{t \to \infty} \max_{-r \le s \le 0} V[x(t_0, \phi, f)(t + s)]$.*

The proof of this lemma is practically indentical to that of Lemma 2.2 of [6].

For any nonempty set $S \subset [-r, 0)$, let $K_V[S] = \{\phi \in H: V[\phi(s)] = V[\phi(0)]$ for all $s \in S\}$.

LEMMA 5.2. *Let V, H, and S be as above with S nonempty. Suppose the following conditions hold:*

(i) *$[V[\psi(0)] = \max_{-r \le s \le 0} V[\psi(s)]$ and $V'_g[t, \psi] = 0$ for some $t \in R$ and $g \in \Omega(f)]$ implies that $\psi \in K_V[S]$.*

(ii) *If $\phi \in H$ with $x(t_0, \phi, f)$ defined and bounded on $[t_0 - r, \infty)$ and $\psi = \lim_{k \to \infty} x_{t_k}(t_0, \phi, f)$ where $t_k \to \infty$, then there exists a subsequence $\{t_{k'}\}$ and a*

$g \in \Omega(f)$ with $f^{t_{k'}} \to g$ such that $\psi \in M_V(t_0, V, H; c)$ and $V[x(t_0, \psi, g)(t)]$ is eventually constant.

Then for any $\phi \in H$ such that $x(t_0, \phi, f)$ is defined and bounded on $[t_0 - r, \infty)$, we have $\Omega(x_t(t_0, \phi, f)) \subset K_V$. Hence $\lim_{t \to \infty} V[x(t_0, \phi, f)(t)] = c = \lim_{t \to \infty} \max_{-r \le s \le 0} V[x(t_0, \phi, f)(t+s)]$.

PROOF. Suppose $\phi \in H$ such that $x(t_0, \phi, f)$ is bounded and suppose $\psi \in \Omega(x_t(t_0, \phi, f))$. The above hypotheses insure that there exists a $g \in \Omega(f)$ such that $\psi \in M_V(t_0, g, H; c)$, where $c = \lim_{t \to \infty} \max_{-r \le s \le 0} V[x(t_0, \phi, f)(t+s)]$, and that we actually have $V[x(t_0, \psi, g)(t)] = c$ for all t sufficiently large. We claim that $\psi \in K_V$. For if not, there is a minimal $t_1 \ge t_0$ such that $V[x(t_0, \psi, g)(t)] = c$ for all $t \ge t_1$. Note that $V[x_{t_1}(t_0, \psi, g)(s)]$ is not a constant function of s. Let $-r_1$ be an element of S. In light of condition (i) above, we have $x_t(t_0, \psi, g) \in K_V[S]$ for every $t \ge t_1$. Hence $V[x(t_0, \psi, g)(t - r_1)] = V[x(t_0, \psi, g)(t)] = c$ for every $t \ge t_1$, a contradiction to our choice of t_1. So $\Omega(x_t(t_0, \phi, f)) \subset K_V$, and the result follows from Lemma 5.1.

THEOREM 5.1. *Suppose there exists a Liapunov function V for* (1.1), *a closed set H positively invariant with respect to* (1.1), *and a nonempty set $S \subset [-r, 0)$ such that the following conditions hold*:
  (i) $[V[\psi(0)] = \max_{-r \le s \le 0} V[\psi(s)]$ *and* $V'_g[t, \psi] = 0$ *for some* $t \in R$ *and* $g \in \Omega(f)]$ *implies that* $\psi \in K_V[S]$.
  (ii) *Given any* $\varepsilon < 0$ *there exist* $-r_1, -r_2 \in S$ *and nonnegative integers p and q such that* $0 < qr_2 - pr_1 < \varepsilon$.
Then for any $\phi \in H$ such that $x(t_0, \phi, f)$ is defined and bounded on $[t_0 - r, \infty)$, we have $\Omega(x_t(t_0, \phi, f)) \subset K_V$, and hence $\lim_{t \to \infty} V[x(t_0, \phi, f)(t)] = c \equiv \lim_{t \to \infty} \max_{-r \le s \le 0} V[x(t_0, \phi, f)(t+s)]$.

PROOF. Suppose $\phi \in H$ with $x(t_0, \phi, f)$ defined and bounded on $[t_0 - r, \infty)$ and that $\psi \in \Omega(x_t(t_0, \phi, f))$. By Theorem 3.1, we have $\psi \in M_V(t_0, g, H; c)$ for some $g \in \Omega(f)$, where c is as defined above. We want to show that $V[x(t_0, \psi, g)(t)] = c$ for all $t \ge t_0$. Let $\varepsilon > 0$ and $t^* > t_0$ be given, and suppose that p and q are nonnegative integers and $-r_1$ and $-r_2$ elements of S such that $0 < qr_2 - pr_1 < \varepsilon$. For ease of notation, set $Q = qr_2$ and $P = pr_1$. Choose an integer N so large that $N(Q - P) - t^* > r$. Since $\psi \in M_V(t_0, g, H; c)$, there exists a t^{**} such that $V[x(t_0, \psi, g)(t^{**})] = c$ where t^{**} satisfies the inequalities $t^* + NP < t^{**} < NQ$. Cleary $[t_0, t^*] \subset [t_0 + t^{**} - NQ, t_0 + t^{**} - NP]$, and the partition $F_N \equiv \{t_0 + t^{**} - NQ, \cdots, t_0 + t^{**} - ((N-k)Q + kP), \cdots, t_0 + t^{**} - NP\}$ has norm less than ε. As a consequence of hypothesis (i), $V[x(t_0, \psi, g)(t)] = c$ for each $t \in F_N \cap [t_0, t^*]$. Now $\hat{F}_N \equiv (F_N \cap [t_0, t^*]) \cup \{t_0, t^*\}$ is a partition of $[t_0, t^*]$ of norm less than ε with $V[x(t_0, \psi, g)(t)] = c$ for every $t \in \hat{F}_N$ with the possible exceptions $t = t_0$ and $t = t^{**}$. As $\varepsilon > 0$ and $t^* > t_0$ were chosen arbitrarily, we have $V[x(t_0, \psi, g)(t)] = c$ for every $t \ge t_0$. It follows that if $\phi \in H$ such that $x(t_0, \phi, f)$ is defined and bounded on $[t_0 - r, \infty)$ and $\psi \in \Omega(x_t(t_0, \phi, f))$, then there exists a $g \in \Omega(f)$ such that $x(t_0, \psi, g)$ is eventually constant.

The theorem follows from Lemma 5.2.

NOTE: Hypothesis (ii) holds if S is infinite, or, as a consequence of a theorem of Dirichlet, if there exist s_1, $s_2 \in S$ with s_1/s_2 irrational.

The equation examined in the next example represents a generalization of an equation studied in [4].

EXAMPLE 5.1. Consider the equation

$$(5.2) \qquad x' = f(t, x_t) = -x(t) + x(t - r_1(t)) - x(t)x(t - r_1(t))G(x(t), x(t - r_2(t)), x_t)$$

where $G \in C(R \times R \times C_r, [0, \infty))$ is continuous with $G(x, y, \phi) = 0$ only if $\phi = 0$ and $r_i: R \to [-r, 0]$ is uniformly continuous for $i = 1, 2$. Now $H \equiv C([-r, 0], [0, \infty))$ is positively invariant with respect to (5.2), and it can be shown that any limiting equation for (5.2) has the form

$$(5.3) \qquad x' = g(t, x_t) = -x(t) + x(t - \hat{r}_1(t)) - x(t)x(t - \hat{r}_1(t))G(x(t), x(t - \hat{r}_2(t)), x_t)$$

where $\hat{r}_i$ is uniformly continuous, $i = 1, 2$. Set $V[x] = x^2/2$. Suppose $\phi \in H$ with $\max_{-r \le s \le 0} \phi(s) = \phi(0)$. Then $V'_f[t, \phi] \le 0$ for each t, and we see that V is a Liapunov function for (5.2). If $g \in \Omega(f)$ and $\psi \in H$ with $\max_{-r \le s \le 0} \psi(s) = \psi(0)$, then $V'_g[t, \psi] \le 0$ for all t, and in the case of equality for some t_0, we conclude that $-\psi^2(0)\psi(-\hat{r}_1(t_0)) = 0$. Clearly if $\psi(0) = 0$, we have $\psi = 0$. If $\psi(-\hat{r}_1(t_0)) = 0$, it follows that $\psi(0) = 0$ as well, and again that $\psi = 0$. The conclusion is that any solution of (5.2) with initial condition $\phi \in H$ which is defined and bounded for all future times tends to a constant. We note that since $V[x] \to \infty$ as $x \to \infty$ and V is a Liapunov function for (5.2), all solutions $x(t_0, \phi, f)$ with $\phi \in H$ are bounded.

EXAMPLE 5.2. Consider an n-dimensional system

$$(5.4) \qquad x' = f(t, x_t) = F(x(t)) - \sum_{i=1}^{\infty} A_i(t)F(x(t - r_i(t))) - \int_{-r(t)}^{0} G(t, s)F(x(t + s))ds$$

where

    (i) $F[x] = [-\text{grad } V[x]]$ for all $x \in R^n$ where $V: R^n \to [0, \infty)$ is continuously differentiable;

    (ii) $|\text{grad } V[u]| < |\text{grad } V[w]|$ whenever $V[u] < V[w]$;

    (iii) Each A_i is an $n \times n$ matrix of uniformly continuous real-valued function on R and $r: R \to [0, r]$ is uniformly continuous;

    (iv) For each i, $r_i: R \to [0, r]$ is continuous and $r_i(t) \to r_i \in [0, r]$ with $r_i \ne r_j$ if $i \ne j$;

    (v) G is an $n \times n$ matrix of continuous real-valued functions uniformly continuous on $R \times [-r, 0]$ and

$$\sum_{i=1}^{\infty} \sup_{t \in R} \|A_i(t)\| + \int_{-r}^{0} \sup_{t \in R} \|G(t, s)\| ds \le 1$$

where

$$\|B\| \equiv \sup_{|x|=1} |Bx| \qquad \text{for} \quad B \in R^{n^2}.$$

The above conditions are sufficient to insure the precompactness of (5.4), and it is not hard to show that any limiting equation of (5.4) has the form

$$(5.5) \qquad x' = g(t, x_t) = F(x(t)) - \sum_{i=1}^{\infty} \hat{A}_i(t) F(x(t - r_i(t))) - \int_{-\hat{r}(t)}^{0} \hat{G}(t, s) F(x(t + s)) ds$$

where $\hat{G}(t, \cdot)$ and each $\hat{A}_i$ is an $n \times n$ matrix of uniformly continuous functions and $\hat{r}: R \to [r, 0]$ is uniformly continuous. Note that

$$V'_f[t, \phi] \le -|\operatorname{grad} V[\phi(0)]|^2 + |\operatorname{grad} V[\phi(0)]| \left[\sum_{i=1}^{\infty} \left(\sup_{s \in R} \|A_i(s)\| \right) |\operatorname{grad} V[\phi(-r_i(t))]| \right]$$

$$+ \int_{-r}^{0} \|G(t, s)\| |\operatorname{grad} V[\phi(s)]| ds .$$

Hence $V'_f[t, \phi] \le 0$ for every t whenever $V[\phi(s)] = \max_{-r \le s \le 0} V[\phi(s)]$, and an analogous result holds for V'_g whenever $g \in \Omega(f)$. Set

$$S = \left\{ -r_i : \sup_{t \in R} \|A_i(t)\| > 0 \right\} \cup \left(\bigcap_{\hat{G} \in \Omega(G)} \bigcap_{t \in R} \{ s \in [-r, 0] : \hat{G}(t, s) \ne 0 \} \right).$$

S satisfies hypothesis (i) of Theorem 5.3. It follows that all solutions $x(t_0, \phi, f)$ of (5.4) tend to a constant if any of the following conditions are present:

    (vi)   Infinitely many of the $A_{i's}$ are not identically the zero matrix;

    (vii) There exist i and j and distinct, nonzero A_i and A_j with r_i/r_j irrational;

    (viii) For some $s \in [-r, 0]$ there is a $t_1 \in R$ and an $\varepsilon > 0$ such that $G(t, s) > \varepsilon$ for all $t \ge t_1$.

Hence if conditions (i)–(v) and any of (vi)–(viii) hold, we can conclude that $V[x(t_0, \phi, f)(t)]$ tends to a constant value as $t \to \infty$ for every t_0 and $\phi \in C_r$.

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