Tôhoku Math. J.
42 (1990), 163–193

INVARIANT HYPERFUNCTIONS ON REGULAR PREHOMOGENEOUS VECTOR SPACES OF COMMUTATIVE PARABOLIC TYPE

MASAKAZU MURO*

(Received February 1, 1989, revised December 20, 1989)

Abstract. Let $(G_{\boldsymbol{R}}^{+}, \rho, V)$ be a regular irreducible prehomogeneous vector space defined over the real field $\boldsymbol{R}$. We denote by $P(x)$ its irreducible relatively invariant polynomial. Let $V_1 \cup V_2 \cup \cdots \cup V_l$ be the connected component decomposition of the set $V - \{x \in V;\ P(x) = 0\}$. It is conjectured by [Mr4] that any relatively invariant hyperfunction on V is written as a linear combination of the hyperfunctions $|P(x)|_i^s$, where $|P(x)|_i^s$ is the complex power of $|P(x)|^s$ supported on $\bar{V}_i$. In this paper the author gives a proof of this conjecture when $(G_{\boldsymbol{R}}^{+}, \rho, V)$ is a real prehomogeneous vector space of commutative parabolic type. Our proof is based on microlocal analysis of invariant hyperfunctions on prehomogeneous vector spaces.

CONTENTS

* Partially supported by the Grants-in-Aid for Scientific Research, the Ministry of Education, Science and Culture, Japan.

Key words: prehomogeneous vector space, invariant, hyperfunction, micro-local analysis.

Introduction. Let $P(x)$ be a homogeneous polynomial with real coefficients on a real vector space V. We suppose that the determinant of the Hessian $\det(\partial P/\partial x_i \partial x_j)$ does not vanish identically. We set $G_{\mathbf{R}} := \{g \in GL(V); P(g \cdot x) = \chi(g)P(x)\}$, where $\chi(g)$ is a constant depending only on $g \in G_{\mathbf{R}}$. Then the function $\chi(g)$ is a character of $G_{\mathbf{R}}$. The connected component of $G_{\mathbf{R}}$ containing the neutral element is denoted by $G_{\mathbf{R}}^+$. We let $V_1 \cup V_2 \cup \cdots \cup V_l$ be the connected component decomposition of the set $V - \{x \in V; P(x) = 0\}$. We suppose that each V_i is a $G_{\mathbf{R}}^+$-orbit, i.e., $(G_{\mathbf{R}}^+, \rho, V)$ is a real prehomogeneous vector space. Any relatively invariant polynomial is given by a non-negative integer power of $P(x)$. In this paper, we show that every relatively invariant hyperfunction is necessarily obtained as a linear combination of the complex powers of $P(x)$.

We shall explain our problem more precisely. Let

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & \text{if } x \in V_i, \\ 0 & \text{if } x \notin V_i. \end{cases}$$

Then $|P(x)|_i^s$ is a continuous function when the real part $\mathrm{Re}(s)$ of s is positive, and can be continued to the whole complex plane $s \in \mathbf{C}$ as a hyperfunction with a meromorphic parameter $s \in \mathbf{C}$. A hyperfunction $T(x)$ which is expressed in the form

$$T(x) = \sum_{i=1}^{l} a_i(s) \cdot |P(x)|_i^s \big|_{s=\lambda}, \tag{0.1}$$

satisfies $T(g \cdot x) = \chi(g)^\lambda \cdot T(x)$ if $a_i(s)$'s are meromorphic functions defined near $s = \lambda$ such that the right hand side of (0.1) is holomorphic with respect to s at $s = \lambda$. We call a hyperfunction $T(x)$ a *χ^λ-invariant hyperfunction* if it satisfies $T(g \cdot x) = \chi(g)^\lambda T(x)$ for all $g \in G_{\mathbf{R}}^+$.

Our problem is the converse: *is every χ^λ-invariant hyperfunction $T(x)$ expressed in the form* (0.1)? The purpose of this paper is to give a new approach to this problem via microlocal analysis, and give an affirmative answer for an important class of prehomogeneous vector spaces — the case where $P(x)$ is an irreducible relatively invariant polynomial of a regular prehomogeneous vector space of commutative parabolic type. Our class contains the cases of real symmetric matrix spaces, of Hermitian matrices over complex and quaternion fields and so on. (See the list (4.1)–(4.5).) As a by-product, it follows that the dimension of the space of χ^λ-invariant hyperfunctions coincides with the number of the connected components of $V - \{x \in V; P(x) = 0\}$ (Theorem 5.6, 1) and 2)). Though we shall only deal with the cases of regular prehomogeneous vector spaces of commutative parabolic type, our method is applicable to other examples provided that they satisfy suitable conditions which would be verified by examining microlocal structure of their holonomic system. See [Mr4].

The problem we treat in this paper seems to be dealt with at least implicitly by several authors, for example, Rais [Ra], Rubenthaler [Ru1], Stein [St], Weil [We], and so on. In Ricci and Stein [Ric-St], almost the same problem was dealt with in the case where V is the space of $n \times n$ complex Hermitian matrices and $P(x) = \det(x)$. They proved that the dimension of the space of relatively invariant hyperfunctions corresponding to χ^s equals the number of open orbits. These are all known partial answers to our problem. The results in this paper are new except the cases (4.2) and (4.5).

The author expresses deep appreciation to Professor H. Rubenthaler for his suggestion, encouragement and advice. Professor Kashiwara gave me useful advice. Professor Wright's research [Wr] on prehomogeneous vector spaces from adelic point of view was implicitly stimulating for me. The advice of the referee and the editor was kind, accurate and helpful for improvement of this paper. The author wishes to thank them and their excellent works.

1. Formulation of the main problem. In this section we formulate our problem in an exact form and provide fundamental notions and notation used in this paper.

1.1. Preliminary conditions and some definitions. Let (G_C, ρ, V_C) be a prehomogeneous vector space of dimension n defined over a complex number field C: it means that there exists a Zariski-open orbit in V_C. We put $S_C := V_C - \rho(G_C) \cdot x_0$, where $\rho(G_C) \cdot x_0$ is the necessarily unique open orbit in V_C.

We impose the following three conditions (1.1), 1)–3). The first condition is:

(1.1), 1) *S_C is an irreducible hypersurface in V_C.*

Then S_C is written as $S_C = \{x \in V_C;\ P(x) = 0\}$ with an irreducible polynomial $P(x)$ on V_C. We call S_C the *singular set* and a G_C-orbit in S_C a *singular orbit*. Then the polynomial $P(x)$ is a relatively invariant polynomial with respect to $g \in G_C$: $P(\rho(g) \cdot x) = \chi(g) \cdot P(x)$ with a non-trivial character $\chi(g)$ of G_C. We say that $P(x)$ is a *relatively invariant polynomial corresponding to the character* χ. From the condition (1.1), 1), any relatively invariant polynomial is written as $P(x)^m$ with a non-negative integer m.

The second condition is:

(1.1), 2) *The relatively invariant polynomial $P(x)$ has a non-degenerate Hessian, i.e., $\det(\partial P/\partial x_i \partial x_j)$ does not vanish identically.*

The condition (1.1), 2) guarantees the regularity of the prehomogeneous vector space (G_C, ρ, V_C).

Let (G_R^+, ρ, V) be a real form of (G_C, ρ, V_C). Namely, G_R^+ is the connected component containing the neutral element of a real form G_R of G_C; V is a real form of V_C satisfying $\rho(G_R^+) \subset GL(V)$. We denote $S := S_C \cap V$ and call it the *real singular set*. Let $V_1 \cup V_2 \cup \cdots \cup V_l$ be the connected component decomposition of $V - S$. Then each connected component V_i is a G_R^+-orbit. The final condition is:

(1.1), 3) *The restriction of $P(x)$ on V can be taken as a polynomial with real coefficients.*

We now give some definitions.

(1.2) Definition (Relatively invariant hyperfunction). Let $\nu(g)$ be a character of G_R^+. We call a hyperfunction (resp. microfunction) $T(x)$ on V a *relatively invariant* *hyperfunction* (resp. *microfunction*) *corresponding to* ν, or simply, a ν*-invariant hyperfunction*, (resp. *microfunction*) if it satisfies $T(g\cdot x)=\nu(g)T(x)$ for all $g\in G_{\boldsymbol{R}}^{+}$.

A hyperfunction (resp. microfunction) $u(s, x)$ on $\boldsymbol{C}\times V$ is said to be a hyperfunction (resp. microfunction) with a *holomorphic parameter* $s\in\boldsymbol{C}$ if it satisfies the Cauchy-Riemann equation with respect to $s\in\boldsymbol{C}$: $(\partial/\partial\bar{s})u(s, x)=0$. If $a(s)\cdot u(s, x)$ is a hyperfunction (resp. microfunction) with a holomorphic parameter $s\in\boldsymbol{C}$ for a holomorphic function $a(s)$, $u(s, x)$ is said to be a hyperfunction (resp. microfunction) with a *meromorphic parameter* $s\in\boldsymbol{C}$.

(1.3) Definition (Linear combinations). Let $u_1(s, x), \cdots, u_l(s, x)$ be hyperfunctions with a meromorphic parameter $s\in\boldsymbol{C}$ and let $a_1(s), \cdots, a_l(s)$ be meromorphic functions near $s=\lambda\in\boldsymbol{C}$. If $w(s, x):=\sum_{i=1}^{l} a_i(s)\cdot u_i(s, x)$ is holomorphic at $s=\lambda$, then we call $T(x):=w(s, x)|_{s=\lambda}$ a hyperfunction obtained as a *linear combination of* $u_i(s, x)$ $(i=1, \cdots, l)$ *at* $s=\lambda$.

1.2. Main problem. The hyperfunction $|P(x)|_i^s$ with a meromorphic parameter $s\in\boldsymbol{C}$, which we shall mainly deal with in this paper, is defined in the following way. Let

$$|P(x)|_i^s:=\begin{cases}|P(x)|^s & \text{if } x\in V_i,\\ 0 & \text{if } x\notin V_i,\end{cases} \tag{1.4}$$

for $s\in\boldsymbol{C}$ satisfying $\mathrm{Re}(s)>0$. Then $|P(x)|_i^s$ is a continuous homogeneous function on V and can be viewed as a hyperfunction on V. Clearly, $|P(x)|_i^s$ is a hyperfunction with a holomorphic parameter s if $\mathrm{Re}(s)>0$. It can be continued to the whole $s\in\boldsymbol{C}$ as a hyperfunction with a meromorphic parameter $s\in\boldsymbol{C}$ by the aid of b-function (see for example [Sm-Sh, p. 139]). We also denote by $|P(x)|_i^s$ the hyperfunction with a meromorphic parameter $s\in\boldsymbol{C}$ by the analytic continuation of (1.4) to every $s\in\boldsymbol{C}$.

Then we have:

Proposition 1.1. *Let* $\lambda\in\boldsymbol{C}$. *Any linear combination of* $|P(x)|_i^s$ $(i=1, \cdots, l)$ *at* $s=\lambda$ *in the sense of* (1.3) *is a* χ^λ*-invariant hyperfunction.*

This proposition follows from the analytic continuation of the equation $|P(g\cdot x)|_i^s=\chi(g)^s\cdot|P(x)|_i^s$ from the domain $\{s\in\boldsymbol{C};\ \mathrm{Re}(s)>0\}$. The main problem that we shall treat in this paper is the converse of Proposition 1.1.

Main Problem. *Let* $\lambda\in\boldsymbol{C}$. *Is any* χ^λ*-invariant hyperfunction obtained as a linear combination of* $|P(x)|_i^s$ *at* $s=\lambda$ *in the sense of* (1.3)?

We shall solve this problem by translating it to a problem of estimating the dimension of the solution space of a linear differential equation. Let $\mathscr{G}_{\boldsymbol{C}}$ be the complex Lie algebra of the complex linear algebraic group $G_{\boldsymbol{C}}$. Let $d\rho$ and $\delta\chi$ be the infinitesimal representations of ρ and χ, respectively. Consider the following system of linear differential equations $\mathfrak{M}_s$ with one unknown function $u(x)$ on the complex vector space $V_{\boldsymbol{C}}$:

$$\text{(1.5)} \qquad \mathfrak{M}_s;\left(\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A)\right)u(x)=0 \qquad \text{for all} \quad A\in\mathscr{G}_{\boldsymbol{C}}.$$

Here $\langle\,,\,\rangle$ means the canonical bilinear form on $V_{\boldsymbol{C}}\times V_{\boldsymbol{C}}^*$, where $V_{\boldsymbol{C}}^*$ is the dual space of $V_{\boldsymbol{C}}$.

Next we consider hyperfunction solutions on the real vector space V of the holonomic system $\mathfrak{M}_s$. We use the same notation x, $\partial/\partial x$ on the real vector space V as on the complex vector space $V_{\boldsymbol{C}}$. Let $\mathscr{G}$ be the real Lie algebra of $G_{\boldsymbol{R}}^+$. Then, since $\mathscr{G}_{\boldsymbol{C}}=\mathscr{G}+\sqrt{-1}\mathscr{G}$ as a real Lie algebra, we have:

$$\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A)$$

$$=\left(\left\langle d\rho(A_1)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A_1)\right)+\sqrt{-1}\left(\left\langle d\rho(A_2)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A_2)\right),$$

where $A=A_1+\sqrt{-1}A_2\in\mathscr{G}_{\boldsymbol{C}}$ with $A_1, A_2\in\mathscr{G}$. Then, if $u(x)$ is a hyperfunction solution on V to $\mathfrak{M}_s$, then $u(x)$ is a solution of the system:

$$\left(\left\langle d\rho(A)\cdot x, \frac{\partial}{\partial x}\right\rangle - s\delta\chi(A)\right)u(x)=0 \qquad \text{for all} \quad A\in\mathscr{G}$$

on the real vector space V. Hence if $u(x)$ is χ^λ-invariant, then $u(x)$ is a solution to $\mathfrak{M}_\lambda$ and vice versa. The vector space $\mathscr{S}\!o\ell(\mathfrak{M}_\lambda)$ of hyperfunction solutions to $\mathfrak{M}_\lambda$ coincides with the vector space of χ^λ-invariant hyperfunctions.

Proposition 1.2. *For any fixed $\lambda\in\boldsymbol{C}$, the dimension of the space of linear combinations of $|P(x)|_i^\lambda$ in the sense of* (1.3) *at $s=\lambda$ is the number l of the connected components of $V-S$. Consequently,* $\dim(\mathscr{S}\!o\ell(\mathfrak{M}_\lambda))\geq l$.

The proof of this propostion is not difficult. See for example Oshima-Sekiguchi [Os-Se], Proposition 2.2.

Our problem is reduced to showing that $\dim(\mathscr{S}\!o\ell(\mathfrak{M}_\lambda))\leq l$. By Proposition 1.1 any χ^λ-invariant hyperfunction is written as a linear combination of $|P(x)|_i^s$ at $s=\lambda$, since the dimension of such linear combinations is $\geq l$. The rest of this paper is thus devoted to the proof of $\dim(\mathscr{S}\!o\ell(\mathfrak{M}_\lambda))\leq l$ for prehomogeneous vector spaces of commutative parabolic type.

2. Regular prehomogeneous vector spaces of commutative parabolic type. In this section we define prehomogeneous vector spaces of commutative parabolic type and give the complex holonomy diagrams of the holonomic systems $\mathfrak{M}_s$ defined by (1.5). All the results in this section were obtained in [Ki].

2.1. Prehomogeneous vector spaces of parabolic type. The notion of prehomogeneous vector spaces of parabolic type was introduced by Rubenthaler [Ru2].

For a semi-simple complex Lie algebra $\mathscr{G}$, he extracted a $\boldsymbol{Z}$-graded structure,

$$\mathscr{G}=\bigoplus_{i\in \boldsymbol{Z}}\mathscr{G}_i \tag{2.1}$$

satisfying $[\mathscr{G}_i, \mathscr{G}_j]\subset\mathscr{G}_{i+j}$. The Lie algebra $\mathscr{G}_0$ acts on $\mathscr{G}_j$ by the adjoint action. Denoting by $\boldsymbol{G}_0$ the exponential group of $\mathscr{G}_0$, we naturally have a representation of $\boldsymbol{G}_0$ on $\mathscr{G}_j$. He gave a general method to get a $\boldsymbol{Z}$-gradation in the form (2.1) by using the root system of the semi-simple Lie algebra $\mathscr{G}$. Then $(\boldsymbol{G}_0, \mathscr{G}_j)$ forms a prehomogeneous vector space by Vinberg [Vi]. In [Ru2] such a pair $(\boldsymbol{G}_0, \mathscr{G}_j)$ is called a prehomogeneous vector space of *parabolic type*. [Ru2] first studied systematically $\boldsymbol{Z}$-gradations of semi-simple Lie algebras and classified regular prehomogeneous vector spaces of parabolic type.

Particularly, consider the case that $\mathscr{G}$ has a $\boldsymbol{Z}$-gradation,

$$\mathscr{G}=\mathscr{G}_{-1}\oplus\mathscr{G}_0\oplus\mathscr{G}_1\,, \tag{2.2}$$

that is to say, $\mathscr{G}_j=\{0\}$ for $|j|\geq 2$. Then, elements of $\mathscr{G}_1$ commute with each other. We call $(\boldsymbol{G}_0, \mathscr{G}_1)$ a prehomogeneous vector space of *commutative parabolic type*, which we are interested in. Any irreducible prehomogeneous vector space of commutative parabolic type is obtained by a $\boldsymbol{Z}$-gradation in the form (2.2) of a simple Lie algebra $\mathscr{G}$. There are several kinds of irreducible prehomogeneous vector spaces of commutative parabolic type, but they have common distinguished properties. We can deal with them in a unified way.

Muller-Rubenthaler-Schiffmann [Mu-Ru-Sc] gave the complete list of irreducible prehomogeneous vector spaces of commutative parabolic type. It consists of seven kinds of prehomogeneous vector spaces. See Table I in [Mu-Ru-Sc]. Among them, type A_n ($n\neq 2k+1$ and $p\neq k+1$) and type E_6 are non-regular prehomogeneous vector spaces. Type B_n and type $D_{n,1}$ are representations of a general orthogonal group of odd and even degree, respectively. We may look upon them as prehomogeneous vector spaces of the same kind. Here is the list of irreducible regular prehomogeneous vector spaces of commutative parabolic type:

(2.3)

1) Type C_m ($m=1, 2, \cdots$). ([Mu-Ru-Sc, Table I, C_m], and [Ki, §2, 2-2]). $\boldsymbol{G}_{\boldsymbol{C}}=GL_m(\boldsymbol{C})$, $\boldsymbol{V}_{\boldsymbol{C}}=\mathrm{Sym}_m(\boldsymbol{C})$. For $(g, x)\in\boldsymbol{G}_{\boldsymbol{C}}\times\boldsymbol{V}_{\boldsymbol{C}}$, $\rho(g): x\mapsto g\cdot x\cdot {}^t g$. An irreducible relatively invariant polynomial $P(x)=\det(x)$. The corresponding character of $P(x)$ is $\chi(g)=\det(g)^2$. $\boldsymbol{G}^1_{\boldsymbol{C}}=SL_m(\boldsymbol{C})$. The dimension of $\boldsymbol{V}_{\boldsymbol{C}}$ is $n=m(m+1)/2$. Here, $\mathrm{Sym}_m(\boldsymbol{C})$ means the space of $m\times m$ complex symmetric matrices and $\det(x)$ is the determinant of x.

2) Type A_k ($k=2m+1$, $m=1, 2, \cdots$). ([Mu-Ru-Sc, Table I, A_k ($k=2m+1$, $p=m+1$)] and [Ki, §2, 2-1]). $\boldsymbol{G}_{\boldsymbol{C}}=GL_m(\boldsymbol{C})\times SL_m(\boldsymbol{C})$, $\boldsymbol{V}_{\boldsymbol{C}}=M_m(\boldsymbol{C})$. For $((g_1, g_2), x)\in\boldsymbol{G}_{\boldsymbol{C}}\times\boldsymbol{V}_{\boldsymbol{C}}$, $\rho(g): x\mapsto g_1\cdot x\cdot {}^t g_2$. An irreducible relatively invariant polynomial is $P(x)=\det(x)$. The corresponding character of $P(x)$ is $\chi(g)=\det(g_1)\det(g_2)$. $\boldsymbol{G}^1_{\boldsymbol{C}}=$

$SL_m(C)\times SL_m(C)$. The dimension of V_C is $n=m^2$.

3) Type $D_{2m,2}$ $(m=1, 2, \cdots)$. ([Mu-Ru-Sc, Table I, $D_{2m,2}$ and [Ki, §2, 2-3]). $G_C=GL_{2m}(C)$, $V_C=\mathrm{Alt}_{2m}(C)$. For $(g, x)\in G_C\times V_C$, $\rho(g): x\mapsto g\cdot x\cdot {}^tg$. An irreducible relatively invariant polynomial is $P(x)=\mathrm{Pff}(x)$. The corresponding character of $P(x)$ is $\chi(g)=\det(g)$. $G_C^1=SL_{2m}(C)$. The dimension of V_C is $n=m(2m-1)$. Here, $\mathrm{Alt}_{2m}(C)$ means the space of $2m\times 2m$ alternating matrices and $\mathrm{Pff}(x)$ is the Pfaffian of $x\in\mathrm{Alt}_{2m}(C)$.

4) Type E_7. ([Mu-Ru-Sc, Table I, E_7] and [Ki, §6]). $G_C=GL_1(C)\times E_{6C}$, $V_C=\mathrm{Her}_3(\mathfrak{C}_C)$. For $((g_1, g_2), x)\in G_C\times V_C$, $\rho(g): x\mapsto g_1(g_2\cdot x)$. An irreducible relatively invariant polynomial is $P(x)=\det(x)$. The corresponding character of $P(x)$ is $\chi(g)=g_1^3$. $G_C^1=E_{6C}$. The dimension of V_C is $n=27$. Here, E_{6C} is the complex exceptional Lie group of type E_6 and $\mathfrak{C}_C$ is the complex Cayley algebra. $\mathrm{Her}_3(\mathfrak{C}_C)$ stands for the space of 3×3 Hermitian matrices over $\mathfrak{C}_C$. The group E_{6C} acts on $\mathrm{Her}_3(\mathfrak{C}_C)$ as the lowest dimensional irreducible representation of E_{6C} and is defined as the connected subgroup of $GL(\mathrm{Her}_3(\mathfrak{C}_C))$ consisting of the elements which leave $P(x)$ invariant. We denote by $g\cdot x$ the action of $g\in E_{6C}$ on $x\in V_C$.

5) Type B_k $(m=2k+1)$ and $D_{k+1,1}$ $(m=2k)$ with $k=1, 2, \cdots$. ([Mu-Ru-Sc, Table I, B_k and $D_{k+1,1}$] and [Sm-Ka-Ki-Os, Example 9.2]). $G_C=GL_1(C)\times SO_m(C)$, $V_C=C^m$. For $((g_1, g_2), x)\in G_C\times V_C$, $\rho(g): x\mapsto g_1(g_2x)$. An irreducible relatively invariant polynomial is $P(x)={}^tx\cdot x$. The corresponding character of $P(x)$ is $\chi(g)=g_1^2$. $G_C^1=SO_m(C)$. The dimension of V_C is $n=m$.

Although [Mu-Ru-Sh] investigated their structure from a unified view point, we rather follow [Ki] and [Sm-Ka-Ki-Os] which studied them on a case-by-case basis, since we need individual information found in the latter. It is easily checked that they satisfy the conditions (1.1), 1) and 2).

2.2. Holonomic systems $\mathfrak{M}_s$ for prehomogeneous vector spaces of commutative parabolic type. The prehomogeneous vector space (2.3), 1) (resp. (2.3), 2), (2.3), 3), (2.3), 4), (2.3), 5)) were treated in [Ki, §2, 2-2] (resp. [Ki, §2, 2-1], [Ki, §2, 2-3], [Ki, §6], [Sm-Ka-Ki-Os, Example 9.2 for $m=1$]) and its complex holonomy diagram and its b-function were computed there. We shall quote from them required results in a slightly different form in Propositions 2.1 and 2.2. Since the proofs can be found in [Ki] or [Sm-Ka-Ki-Os], or can be easily checked after direct computations, we omit the proof.

PROPOSITION 2.1. (i) *The prehomogeneous vector spaces* (G_C, ρ, V_C) *in* (2.3), 1), 2), 3) *and* 4) *have the* G_C*-orbit decompositions* $\bigcup_{i=0}^m S_{iC}=V_C$ *with* $S_{iC}=\{x\in V_C;\ \mathrm{rank}(x)=m-i\}$. *In particular,* $S_{0C}=V_C-S_C$ *with* $S_C=\{x\in V_C;\ P(x)=0\}$ *and* $S_C=\bigcup_{i=1}^m S_{iC}$. *Here we let* $m=3$ *in the case of* (2.3), 4).

(ii) *The prehomogeneous vector space* (G_C, ρ, V_C) *in* (2.3), 5) *has the orbit decomposition* $\bigcup_{i=0}^2 S_{iC}=V_C$, *with* $S_{0C}=\{x\in V_C;\ P(x)\neq 0\}$, $S_{1C}=\{x\in V_C;\ P(x)=0\}-\{0\}$, *and* $S_{2C}=\{0\}$.

Proposition 2.2. *Let* $\mathfrak{M}_s$ $(s\in \boldsymbol{C})$ *be the holonomic system defined by* (1.5) *for one of the prehomogeneous vector spaces* (2.3), 1)–5). *Then*:

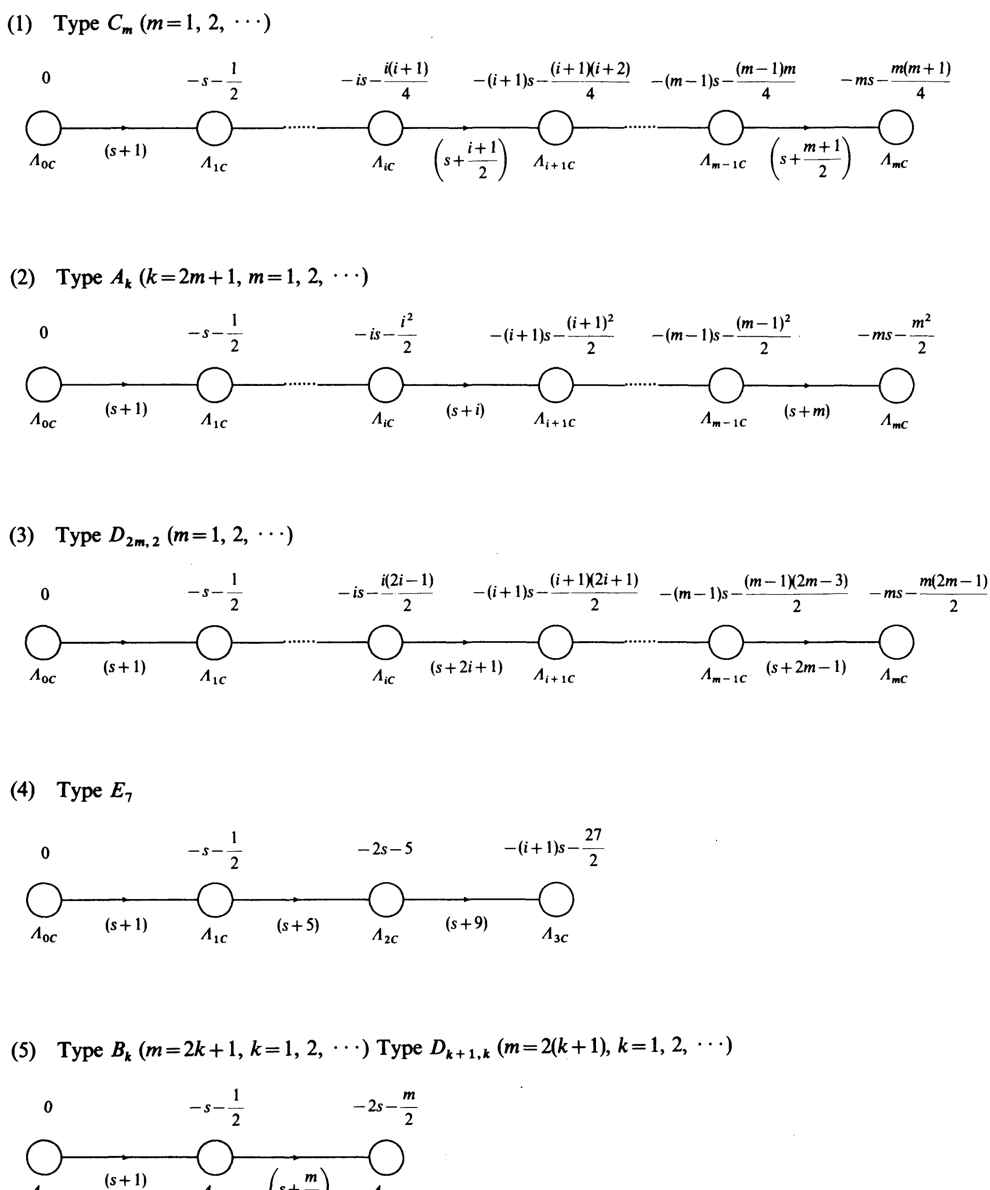

Figure 1.

(i) *The characteristic variety* $\mathrm{ch}(\mathfrak{M}_s)$ *is given by*

$$\mathrm{ch}(\mathfrak{M}_s)=\bigcup_{i=0}^{m} \Lambda_{iC}\,, \tag{2.4}$$

with

$$\Lambda_{iC}=\overline{T^*_{S_iC}V_C}\,.$$

where, we let $m=3$ *for the case* 4) *and* $m=2$ *for the case* 5). *Each* Λ_{iC} *is a Lagrangian irreducible component of* $\mathrm{ch}(\mathfrak{M}_s)$, *hence* (2.4) *gives the irreducible component decomposition of* $\mathrm{ch}(\mathfrak{M}_s)$.

(ii) *Their holonomy diagrams are as in Figure* 1. *Type* C_m $(m=1, 2, \cdots)$ *is Figure* 1, (1); *Type* A_k $(k=2m+1,\ m=1, 2, \cdots)$ *is Figure* 1, (2); *Type* $D_{2m,2}$ $(m=1, 2, \cdots)$ *is Figure* 1, (3); *Type* E_7 *is Figure* 1, (4); *Type* B_k $(m=2k+1)$ *and Type* $D_{k+1,1}$ $(m=2(k+1))$ *with* $k=1, 2, \cdots$ *are Figure* 1, (5).

PROOF. (i) is a direct consequence of the argument in [Sm-Ka-Ki-Os] and Proposition 2.1.

(ii) See [Ki, §2, 2-2], [Ki §2, 2-1], [Ki, §2, 2-3], [Ki, §6] and [Sm-Ka-Ki-Os, Example 9.2 ($m=1$)], respectively. We add arrows for convenience, although the original holonomy diagrams in [Ki] or [Sm-Ka-Ki-Os] do not contain them. The orders and the "factors of b-functions" are computed from the definition of [Sm-Ka-Ki-Os]. q.e.d.

[Sm-Ka-Ki-Os] and [Ki] computed the b-functions of the complex powers of the relatively invariant polynomials of some regular irreducible prehomogeneous vector spaces by utilizing this holonomy diagrams. A b-function is, by definition, a polynomial $b(s)$ satisfying $Q(\partial/\partial x)\cdot P(x)^{s+1}=b(s)\cdot P(x)^s$ where $P(x)$ and $Q(y)$ are irreducible relatively invariant polynomials on V_C and the dual space V_C^*, respectively. The b-function is a polynomial in s, and is determined uniquely up to constant multiple. One of the main theorems of [Sm-Ka-Ki-Os] is that b-functions of prehomogeneous vector spaces are obtained as the products of all "factors of b-functions". We give the b-functions of the prehomogeneous vector spaces in (2.3) for later reference. See [Ki] and [Sm-Ka-Ki-Os].

PROPOSITION 2.3. *The b-functions of regular prehomogeneous vector spaces* (2.3), 1)–5) *are given by*:

(2.5)

1) $$b(s)=\prod_{i=1}^{m}\left(s+\frac{i+1}{2}\right).$$

2) $$b(s)=\prod_{i=1}^{m}(s+i)\,.$$

3) $$b(s)=\prod_{i=1}^{m}(s+(2i-1))\,.$$

4) $b(s)=\prod_{i=1}^{3}(s+(4i-3))=(s+1)(s+5)(s+9)$.

5) $b(s)=(s+1)\left(s+\frac{m}{2}\right)$.

3. Holonomic systems on the real locus and its solutions. In this section we study real micro-local strucrure of $\mathfrak{M}_s$ near a normal intersection of two Lagrangian subvarieties of codimension one. There is a simple relation between microfunction solutions on the two Lagrangian subvarieties (Proposition 3.3). It will help the determination of hyperfunction solutions of $\mathfrak{M}_s$ in § 5.

3.1. Solutions with a holomorphic parameter s. Let $u(s, x)$ be a hyperfunction or microfunction solution to $\mathfrak{M}_s$ with a holomorphic parameter $s \in \boldsymbol{C}$. Then $u(s, x)$ can be restricted to the subset $\{(s, x) \in \boldsymbol{C} \times V; s=\lambda, x \in V\}$ and the restriction $u(s, x)|_{s=\lambda}$ is a solution to $\mathfrak{M}_\lambda$. If $u(s, x)$ is a solution with a meromorphic parameter $s \in \boldsymbol{C}$ and if $u(s, x)$ has a pole at $s=\lambda$ of order m, then $(s-\lambda)^m u(s, x)|_{s=\lambda}$ is well-defined and is a solution to $\mathfrak{M}_\lambda$. Namely, the lowest order coefficient of the Laurent expansion of $u(s, x)$ at $s=\lambda$ is a solution to $\mathfrak{M}_\lambda$. For example, $|P(x)|_i^s$ is a hyperfunction with a meromorphic parameter $s \in \boldsymbol{C}$.

We now consider the support or the singular spectrum of the solutions.

Proposition 3.1. *Let λ be a fixed point in $\boldsymbol{C}$. Let $f(x)$ be a hyperfunction (resp. microfunction) solution to the holonomic system $\mathfrak{M}_\lambda$ on V (resp. on T^*V). Then we have*:

$$\widehat{S.S.}(f(x)) \subset \mathrm{ch}(\mathfrak{M}_\lambda) \cap T^*V, \ (\textit{resp. } \mathrm{supp}(f(x)) \subset \mathrm{ch}(\mathfrak{M}_\lambda) \cap T^*V), \tag{3.1}$$

where $\widehat{S.S.}$ stands for the singular spectrum on T^*V. In particular, if $f(s, x)$ is a hyperfunction (resp. microfunction) solution with a holomorphic parameter $s \in \boldsymbol{C}$, then the hyperfunction (resp. microfunction) $f(x):=f(\lambda, x)$ for each $\lambda \in \boldsymbol{C}$ satisfies (3.1).

This porposition is well known. We omit the proof. See [Ka2], [Ka3] or [Ka4].

The real locus $\mathrm{ch}(\mathfrak{M}_s) \cap T^*V$ is denoted by $\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}}$ and is called the *real characteristic variety*. The characteristic variety $\mathrm{ch}(\mathfrak{M}_s)$ has the irreducible component decomposition: $\mathrm{ch}(\mathfrak{M}_s)=\bigcup_{i=1}^m \Lambda_{i\boldsymbol{C}}$. We denote by $\Lambda_{i\boldsymbol{R}}$ the real locus $\Lambda_{i\boldsymbol{C}} \cap T^*V$. Then $\Lambda_{i\boldsymbol{R}}$ may not be of real dimension n while $\Lambda_{i\boldsymbol{C}}$ is always of complex dimension n. In other words, it may not be a *real* conic Lagrangian subvariety in T^*V, i.e., a subvariety of dimension n in T^*V on which the real canonical 2-form $\sum_{i=1}^n dx_i \wedge d\xi_i$ vanishes. So we have to assume the following condition:

$$\text{Each } \Lambda_{i\boldsymbol{R}} \text{ is a real Lagrangian subvariety in } T^*V. \tag{3.2}$$

Then each $\Lambda_{i\boldsymbol{R}}$ is a real form of $\Lambda_{i\boldsymbol{C}}$.

Recall that the set of *generic points* of $\Lambda_{i\boldsymbol{C}}$ in $\mathrm{ch}(\mathfrak{M}_s)$ is denoted by $\Lambda_{i\boldsymbol{C}}^{\circ}:=\{p \in \Lambda_{i\boldsymbol{C}}; (1)$

Λ_{iC} is non-singular near p, (2) p is not contained in any other irreducible components Λ_{jC} $(j\neq i)\}$. Since Λ_{iC}^{o} is a non-singular open dense subvariety in Λ_{iC}, its real locus $\Lambda_{iC}^{o}:=\Lambda_{iC}^{o}\cap T^{*}V$ is a non-singular open dense subvariety in Λ_{iR}. The subvariety Λ_{iR}^{o} decomposes into a finite number of connected components. Let $\Lambda_{iR}^{o}=\coprod_{j=1}^{k_i}\Lambda_i^j$ be the connected component decomposition. Let U be an open set in $T^{*}V$ such that $U\cap \mathrm{ch}(\mathfrak{M}_s)_R=U\cap\Lambda_i^j$ with a connected component Λ_i^j of Λ_{iR}^{o}. Then the support of a microfunction solution on U is contained in $\Lambda_i^j\cap U$. We call it a *microfunction solution on* Λ_i^j by abuse of language. We have the following theorem on a microfunction solution on Λ_i^j:

PROPOSITION 3.2. *For each fixed $\lambda\in C$ and for any point $p\in\Lambda_i^j$, there is a one-dimensional microfunction solution space to $\mathfrak{M}_\lambda$ near p. In particular, if there exists a non-trivial global microfunction solution on Λ_i^j, then it is uniquely determined up to constant multiple, and non-vanishing on Λ_i^j.*

PROOF. By definition, $\mathfrak{M}_\lambda$ is a simple holonomic system on Λ_i^j. Therefore its microfunction solution space on Λ_i^j is one-dimensional. For a detailed proof, see for example the proof of Theorem 4.2.5 in [Sm-Kw-Ka]. q.e.d.

3.2. Real holonomy diagrams. The aim of this subsection is to introduce the real holonomy diagrams of the holonomic system $\mathfrak{M}_s$ on V. We have given the complex holonomy diagram of a holonomic system $\mathfrak{M}_s$ in order to see the geometric configuration of intersections of codimension one among the Lagrangian irreducible components of $\mathrm{ch}(\mathfrak{M}_s)$. We would like to do the same for the real locus $\mathrm{ch}(\mathfrak{M}_s)_R$. Since, it is too complicated to describe all the intersections of all the real Lagrangian subvarieties in $\mathrm{ch}(\mathfrak{M}_s)_R$, however, we confine ourselves to writing down intersections between two irreducible components in $\mathrm{ch}(\mathfrak{M}_s)_R$. Let Λ_{aC} and Λ_{bC} have a regular intersection of dimension $n-1$. The intersection is necessarily transversal. Let Σ_C be an irreducible component of $\Lambda_{aC}\cap\Lambda_{bC}$. Then we have the complex holonomy subdiagram Figure 2, (1). In Figure 2, (1), $(p(s)+1)=q_a(s)-q_b(s)+(1/2)$. Here $q_a(s)$ and $q_b(s)$ are the *orders* of $\mathfrak{M}_s$ on Λ_{aC} and Λ_{bC}, respectively, and $(p(s)+1)$ is the *factor of b-function* from Λ_{aC} to Λ_{bC}. See [Sm-Ka-Ki-Os].

Recall that we denote by Λ_{aR}^{o} and Λ_{bR}^{o} the sets of generic points of Λ_{aR} and Λ_{bR} in $\mathrm{ch}(\mathfrak{M}_s)_R$, respectively. Let $\coprod_p\Lambda_a^p=\Lambda_{aR}^{o}$ and $\coprod_q\Lambda_b^q=\Lambda_{bR}^{o}$ be the connected component decompositions of Λ_{aR}^{o} and Λ_{bR}^{o}, respectively. We denote by $(\Sigma_C)_{\mathrm{reg}}$ the set of non-singular points of Σ_C. Then $(\Sigma_C)_{\mathrm{reg}}$ is an $(n-1)$-dimensional non-singular complex algebraic subvariety and its real locus $(\Sigma_R)_{\mathrm{reg}}:=(\Sigma_C)_{\mathrm{reg}}\cap T^{*}V$ is an $(n-1)$-dimensional real algebraic subvariety by the condition (3.2). Let $\coprod_\varepsilon\Sigma^\varepsilon=(\Sigma_R)_{\mathrm{reg}}$ be the connected component decomposition of $(\Sigma_R)_{\mathrm{reg}}$.

Take a connected component Σ^ε and let $p\in\Sigma^\varepsilon$. Then we have $T_p\Lambda_{aR}\cap T_p\Lambda_{bR}=T_p\Sigma^\varepsilon$, that is to say, Λ_{aR} and Λ_{bR} have the $(n-1)$-dimensional regular intersection Σ^ε in a neighborhood of p, which is transversal. Since Λ_{aR} and Λ_{bR} are n-dimensional, Σ^ε divides

$\Lambda_{a\mathbf{R}}$ and $\Lambda_{b\mathbf{R}}$ into two connected parts near p, i.e., $\Lambda_{a\mathbf{R}}-\Sigma^{\varepsilon}$ and $\Lambda_{b\mathbf{R}}-\Sigma^{\varepsilon}$ have two connected components in a sufficiently small neighborhood of p. On the other hand, we have $\Lambda_{a\mathbf{R}}-\Sigma^{\varepsilon}=\Lambda^{o}_{a\mathbf{R}}$ and $\Lambda_{b\mathbf{R}}-\Sigma^{\varepsilon}=\Lambda^{o}_{b\mathbf{R}}$. Hence there exist two connected components, Λ^{α}_{a} and Λ^{β}_{a} in $\Lambda^{o}_{a\mathbf{R}}$, and, Λ^{γ}_{b} and Λ^{δ}_{b} in $\Lambda^{o}_{b\mathbf{R}}$, which are the connected components near p. Namely,

$$\Lambda_{a\mathbf{R}}-\Sigma^{\varepsilon}=\Lambda^{\alpha}_{a}\coprod\Lambda^{\beta}_{a}, \quad \text{and} \quad \Lambda_{b\mathbf{R}}-\Sigma^{\varepsilon}=\Lambda^{\gamma}_{b}\coprod\Lambda^{\delta}_{b} \tag{3.4}$$

in a neighborhood of p. The indices α, β, γ and δ do not depend on the choice of the point $p\in\Sigma^{\varepsilon}$. In order to describe such a geometric situation in $\mathrm{ch}(\mathfrak{M}_{s})_{\mathbf{R}}$, we express each connected component of $\Lambda^{o}_{a\mathbf{R}}$ and $\Lambda^{o}_{b\mathbf{R}}$ by a circle and write the situation (3.4) by Figure 2, (2). In the diagram Figure 2, (2), each circle stands for a connected component in $\Lambda^{o}_{a\mathbf{R}}$ or $\Lambda^{o}_{b\mathbf{R}}$ and the cross means an $(n-1)$-dimensional intersection in $(\Sigma_{\mathbf{R}})_{\mathrm{reg}}$. Thus by representing each connected component by a circle and each connected component of $(\Sigma_{\mathbf{R}})_{\mathrm{reg}}$ by a cross, and by connecting circles, we obtain a diagram consisting of circles and crosses like Figure 2, (2).

(3.5) Definition (Real holonomy diagram). We call the diagram thus obtained

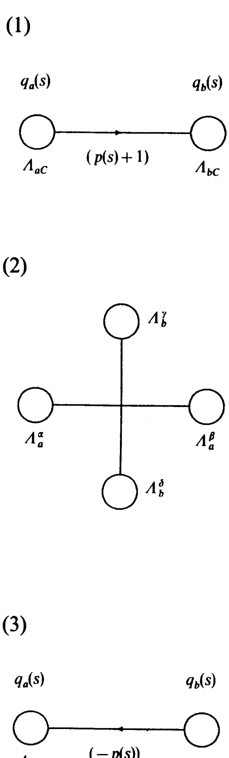

Figure 2.

a *real holonomy diagram* of the intersection of $\Lambda_{a\boldsymbol{R}}$ and $\Lambda_{b\boldsymbol{R}}$ at $\Sigma_{\boldsymbol{R}}$, or of the complex holonomy diagram Figure 2, (1).

In § 4.2 we draw real holonomy diagrams of $\mathfrak{M}_s$ for several real forms, which were partly obtained in [Mr1].

3.3. Relations of microfunction solutions. Now we prove that there exist some linear relations among the microfunction solutions on Λ_a^α, Λ_a^β and Λ_b^γ, Λ_b^δ.

(3.6) DEFINITION (Critical points of $\mathfrak{M}_s$). Suppose that $\mathfrak{M}_s$ has a holonomy subdiagram Figure 2, (1). We say that $\lambda \in \boldsymbol{C}$ is a *critical point of* $\mathfrak{M}_s$, or $\mathfrak{M}_s$ is *critical at* $s=\lambda$, *from* $\Lambda_{a\boldsymbol{C}}$ *to* $\Lambda_{b\boldsymbol{C}}$, if $p(\lambda)$ is a negative integer. Otherwise, we say that λ is *non-critical from* $\Lambda_{a\boldsymbol{C}}$ *to* $\Lambda_{b\boldsymbol{C}}$.

REMARK. When we look upon the above holonomy diagram Figure 2, (1) as the one with the inverse arrow Figure 2, (3), the critical points of $\mathfrak{M}_s$ are $\lambda \in \boldsymbol{C}$ satisfying $-p(\lambda) \in \{0, -1, -2, \cdots\}$. Namely the set of critical points from $\Lambda_{a\boldsymbol{C}}$ to $\Lambda_{b\boldsymbol{C}}$ and those form $\Lambda_{b\boldsymbol{C}}$ to $\Lambda_{a\boldsymbol{C}}$ are disjoint and their union is $\{s \in \boldsymbol{C}; p(s) \in \boldsymbol{Z}\}$.

PROPOSITION 3.3. *Let* $\lambda \in \boldsymbol{C}$. *Let* $\Lambda_{a\boldsymbol{C}}$ *and* $\Lambda_{b\boldsymbol{C}}$ *be two irreducible Lagrangian subvarieties in* $\mathrm{ch}(\mathfrak{M}_s)$ *having the complex holonomy diagram Figure* 2, (1). *Let* Λ_a^α, Λ_a^β *and* Λ_b^γ, Λ_b^δ *be two pairs of connected components of* $\Lambda_{a\boldsymbol{R}}^\circ$ *and* $\Lambda_{b\boldsymbol{R}}^\circ$, *respectively, having the real holonomy diagram Figure* 2, (2).

(1) *For each* $s \in \boldsymbol{C}$, *the space of microfunction solutions to* $\mathfrak{M}_s$ *near* p *is two-dimensional.*

(2) *If* $\mathfrak{M}_s$ *is not critical at* $s=\lambda$ *from* $\Lambda_{a\boldsymbol{C}}$ *to* $\Lambda_{b\boldsymbol{C}}$, *i.e.,* $p(\lambda) \neq -1, -2, -3, \cdots$, *then the microfunction solutions to* $\mathfrak{M}_\lambda$ *on* Λ_b^γ *and* Λ_b^δ *are determined by the microfunction solutions on* Λ_a^α *and* Λ_a^β. *If* $\mathfrak{M}_s$ *is not critical at* $s=\lambda$ *from* $\Lambda_{b\boldsymbol{C}}$ *to* $\Lambda_{a\boldsymbol{C}}$, *i.e.,* $p(\lambda) \neq 0, 1, 2, \cdots$, *then the microfunction solutions to* $\mathfrak{M}_\lambda$ *on* Λ_a^α *and* Λ_a^β *are determined by the microfunction solutions on* Λ_b^γ *and* Λ_b^δ.

(3) *Suppose that* $\mathfrak{M}_s$ *is critical at* $s=\lambda$ *from* $\Lambda_{a\boldsymbol{C}}$ *to* $\Lambda_{b\boldsymbol{C}}$, *i.e.,* $p(\lambda) = -1, -2, -3, \cdots$. *Let* $v(x)$ *be a microfunction solution to* $\mathfrak{M}_s$. *The* $v(x)|_{\Lambda_a^\alpha}$ *is determined by* $v(x)|_{\Lambda_a^\beta}$ *and vice versa. If the support of* $v(x)$ *is contained in* $\Lambda_{b\boldsymbol{R}}$, *then* $v(x)|_{\Lambda_b^\delta}$ *is determined by* $v(x)|_{\Lambda_b^\gamma}$ *and vice versa.*

PROOF. The holonomic system $\mathfrak{M}_s$ is transformed to the following holonomic system $\mathfrak{L}_{r(s),p(s)}$ through a real quantized contact transformation.

$$
(3.7) \qquad \mathfrak{L}_{r(s),p(s)} \begin{cases} \left(x_1 \dfrac{\partial}{\partial x_1} - r(s)\right) u(x) = 0\,, \\[2ex] \left(x_2 \dfrac{\partial}{\partial x_2} - p(s)\right) u(x) = 0\,, \end{cases}
$$

$$\left\{ \frac{\partial}{\partial x_3}u(x)=\frac{\partial}{\partial x_4}u(x)=\cdots\frac{\partial}{\partial x_n}u(x)=0\,,\right.$$

with

$$\Lambda_{aR}=\{(x, y)\in T^*V;\, x_1=y_2=y_3=\cdots=y_n=0\}\,,$$
$$\Lambda_{bR}=\{(x, y)\in T^*V;\, x_1=x_2=y_3=\cdots=y_n=0\}\,,$$
$$p=(0,\ +dx_1)\,,$$
$$\Sigma^{\varepsilon}=\{(x, y)\in T^*V;\, x_1=x_2=y_2=y_3=\cdots=y_n=0\}\,.$$

Here, $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ means real coordinates of T^*V. This fact is proved as a special case of Theorem 6.3 in [Sm-Ka-Ki-Os]. Though the proof given in [Sm-Ka-Ki-Os] is the one for holonomic systems in the complex domain, it works well in the real domain by real analytic contact transformation instead of holomorphic contact transformation. Namely, that the real version of Theorem 6.3 in [Sm-Ka-Ki-Os] is easily justified.

Therefore the problem is reduced to showing Proposition 3.3 for the holonomic system $\mathfrak{L}_{r(s),p(s)}$ defined in (3.7). Namely what we have to show is the following; let $u(x)$ be a microfunction solution to $\mathfrak{L}_{r(\lambda),p(\lambda)}$ defined near the point $p=(0,\ +dx_1)\in T^*V$; if $u(x)$ is zero on $\Lambda^o_{aR}:=\{(x, y)\in T^*V;\, x_1=y_2=\cdots=y_n=0,\, x_2\neq 0\}$, then $u(x)$ is zero near p. We need the following obvious Lemma 3.3.1.

LEMMA 3.3.1. *Let X_1 and X_2 be two real analytic manifolds. Let*

$$\mathfrak{M}_1:\ P_i\left(x_1, \frac{\partial}{\partial x_1}\right)u(x_1)=0 \qquad (i=1, \cdots, k_1)$$

$$\mathfrak{M}_2:\ Q_j\left(x_2, \frac{\partial}{\partial x_2}\right)v(x_2)=0 \qquad (i=1, \cdots, k_2)$$

be holonomic systems on X_1 and X_2, respectively. We denote by $\mathscr{Sol}(\mathfrak{M}_1)$, $\mathscr{Sol}(\mathfrak{M}_2)$ the spaces of hyperfunction or microfunction solutions. Consider the holonomic system on $X_1\times X_2$:

$$\mathfrak{M}_1\hat{\otimes}\mathfrak{M}_2:\ P_i\left(x_1, \frac{\partial}{\partial x_1}\right)u(x_1, x_2)=0\,,$$

$$Q_j\left(x_2, \frac{\partial}{\partial x_2}\right)u(x_1, x_2)=0\,, \quad (i=1, \cdots, k_1, j=1, \cdots, k_2)\,.$$

Then the solution space $\mathscr{Sol}(\mathfrak{M}_1\hat{\otimes}\mathfrak{M}_2)$ is given by $\mathscr{Sol}(\mathfrak{M}_1)\otimes\mathscr{Sol}(\mathfrak{M}_2)$.

We set $p(\lambda):=\nu$ and $r(\lambda):=\mu$. By Lemma 3.3.1, the holonomic system $\mathfrak{L}_{\mu,\nu}$ is given by $\mathfrak{L}_{\mu,\nu}=\mathfrak{M}_1\hat{\otimes}\mathfrak{M}_2\hat{\otimes}\mathfrak{M}_3$ with

$$\mathfrak{M}_1: \left(x_1\frac{\partial}{\partial x_1}-\mu\right)u(x_1)=0 \quad \text{near} \quad (0, +dx_1)\in T^*\boldsymbol{R},$$

$$\mathfrak{M}_2: \left(x_2\frac{\partial}{\partial x_2}-\nu\right)v(x_2)=0 \quad \text{near} \quad (0, 0)\in T^*\boldsymbol{R},$$

$$\mathfrak{M}_3: \frac{\partial}{\partial x_3}w(x_3, \cdots, x_n)=\cdots=\frac{\partial}{\partial x_n}w(x_3, \cdots, x_n)=0 \quad \text{near} \quad (0, 0)\in T^*\boldsymbol{R}^{n-2}.$$

We now examine the space of microfunction solutions to $\mathfrak{L}_{\mu,\nu}$ near p. We have

(3.8) $\mathcal{S}ol(\mathfrak{M}_1)=a\cdot(x_1+i0)^{\mu}\cdot\Gamma(-\mu)$, with $a\in\boldsymbol{C}$.

$\mathcal{S}ol(\mathfrak{M}_2)=a\cdot|x_2|_+^{\nu}+b\cdot|x_2|_-^{\nu}$, with $a, b\in\boldsymbol{C}$ when $\nu\neq-1, -2, -3, \cdots$.

$=a\cdot(x_2+i0)^{\nu}+b\cdot(x_2-i0)^{\nu}$ with $a, b\in\boldsymbol{C}$ when $\nu\neq 0, 1, 2, \cdots$.

$\mathcal{S}ol(\mathfrak{M}_3)=a$, with a constant function $a\in\boldsymbol{C}$.

The microfunction $(x_1+i0)^{\mu}\cdot\Gamma(-\mu)$ defined near $(0, dx_1)\in T^*V$ is the boundary value of the holomorphic function $\Gamma(-\mu)\cdot z_1^{\mu}$ from the upper half plane $\{z_1=x_1+\sqrt{-1}y_1; x_1, y_1\in\boldsymbol{R}, y_1>0\}$. Here we take a suitable branch of the holomorphic function z_1^{μ} on it. Regarded as a microfunction defined near $(0, dx_1)\in T^*V$, it is well defined for all $\mu\in\boldsymbol{C}$. The microfunction $|x_2|_+^{\nu}$ (resp. $|x_2|_-^{\nu}$) is a hyperfunction on $\boldsymbol{R}$ defined by

$$|x_2|_+^{\nu}:=\begin{cases}|x_2|^{\nu} & \text{on} \quad x_2>0\\ 0 & \text{on} \quad x_2<0\end{cases}$$

$$\left(\text{resp. } |x_2|_-^{\nu}:=\begin{cases}|x_2|^{\nu} & \text{on} \quad x_2<0\\ 0 & \text{on} \quad x_2>0\end{cases}\right)$$

which is a hyperfunction with a holomorphic parameter $\nu\in\boldsymbol{C}$ obtained by the analytic continuation from $\mathrm{Re}(\nu)\gg 0$ and is well-defined for $\nu\neq-1, -2, \cdots$.

We set

$$\begin{cases}u_+(x)=(x_1+i0)^{\mu}\cdot\Gamma(-\mu)\cdot|x_2|_+^{\nu},\\ u_-(x)=(x_1+i0)^{\mu}\cdot\Gamma(-\mu)\cdot|x_2|_-^{\nu},\end{cases} \quad \text{for} \quad \nu\neq-1, -2, \cdots.$$

$$\begin{cases}u^+(x)=(x_1+i0)^{\mu}\cdot\Gamma(-\mu)\cdot(x_2+i0)^{\nu},\\ u^-(x)=(x_1+i0)^{\mu}\cdot\Gamma(-\mu)\cdot(x_2-i0)^{\nu},\end{cases} \quad \text{for} \quad \nu\neq 0, 1, 2, \cdots.$$

When ν is not an integer, the vector space spanned by $u^+(x)$ and $u^-(x)$ coincides with that generated by $u_+(x)$ and $u_-(x)$. The pairs of microfunctions $\{u_+(x), u_-(x)\}$ or $\{u^+(x), u^-(x)\}$ give a basis for the space of microfunction solutions for $\mathfrak{L}_{\mu,\nu}$ near p by Lemma 3.3.1. Thus the proof of (1) is completed.

Next we prove (2). First we suppose that $\nu\neq-1, -2, \cdots$. Let $u(x)$ be a microfunction solution of $\mathfrak{L}_{\mu,\nu}$ defined near p. Then $u(x)$ is written as $a_+\cdot u_+(x)+a_-\cdot u_-(x)$ with

$a_+, a_- \in \boldsymbol{C}$. We set $\Lambda_a^+ := \Lambda_{a\boldsymbol{R}} \cap \{x_2 > 0\}$ and $\Lambda_a^- := \Lambda_{a\boldsymbol{R}} \cap \{x_2 < 0\}$. Then $\Lambda_{a\boldsymbol{R}}^o = \Lambda_a^+ \cup \Lambda_a^-$ near the point p. Since $u_+(x)|_{\Lambda_a^-} = 0$, $u_-(x)|_{\Lambda_a^+} = 0$ and $u_+(x)|_{\Lambda_a^+} \neq 0$, $u_-(x)|_{\Lambda_a^-} \neq 0$, we have $u(x)|_{\Lambda_a^+} = a_+ \cdot u_+(x)|_{\Lambda_a^+}$ and $u(x)|_{\Lambda_a^-} = a_- \cdot u_-(x)|_{\Lambda_a^-}$. Thus the values of a_+ and a_- are determined by the restrictions $u(x)|_{\Lambda_a^+}$ and $u(x)|_{\Lambda_a^-}$. This means that a microfunction solution $u(x)$ of $\mathfrak{M}_\lambda$ is determined by the data of $u(x)$ on $\Lambda_{a\boldsymbol{R}}^o$. Thus the data of $u(x)$ on $\Lambda_{b\boldsymbol{R}}^o$ are determined by those on $\Lambda_{a\boldsymbol{R}}^o$ if $\nu = p(\lambda) \neq -1, -2, \cdots$. Next we suppose that $\nu \neq 0, 1, 2, \cdots$. Then $u(x)$ is written as $b_+ \cdot u^+(x) + b_- \cdot u^-(x)$ with $b_+, b_- \in \boldsymbol{C}$. We put $\Lambda_b^+ := \Lambda_{b\boldsymbol{R}} \cap \{y_2 > 0\}$ and $\Lambda_b^- := \Lambda_{b\boldsymbol{R}} \cap \{y_2 < 0\}$. Since $u^+(x)|_{\Lambda_b^-} = 0$, $u^-(x)|_{\Lambda_b^+} = 0$ and $u^+(x)|_{\Lambda_b^+} \neq 0$, $u^-(x)|_{\Lambda_b^-} \neq 0$, we have $u(x)|_{\Lambda_b^+} = b_+ \cdot u^+(x)|_{\Lambda_b^+}$ and $u(x)|_{\Lambda_b^-} = b_- \cdot u^-(x)|_{\Lambda_b^-}$. Thus the values of b_+ and b_- are determined by $u(x)|_{\Lambda_b^+}$ and $u(x)|_{\Lambda_b^-}$. This means that a microfunction solution $u(x)$ to $\mathfrak{M}_\lambda$ is determined by the data of $u(x)$ on $\Lambda_{b\boldsymbol{R}}^o$. Thus the data of $u(x)$ on $\Lambda_{a\boldsymbol{R}}^o$ are determined by those on $\Lambda_{b\boldsymbol{R}}^o$ if $\nu = p(\lambda) \neq 0, 1, 2, \cdots$.

As in the proof of (3), suppose that $\nu = -1, -2, -3, \cdots$. By (3.8), the space of microfunction solutions of $\mathfrak{M}_2$ is given by

$$\mathcal{S}ol(\mathfrak{M}_2) = a \cdot (x_2 + i0)^\nu + b \cdot \delta^{(-\nu+1)}(x_2),$$

where $\delta^{(i)}(x_2)$ is the i-th derivative of the delta-function for the variable x_2. Indeed, we have (const.) $\times\, \delta^{(-\nu+1)}(x_2) = (x_2 + i0)^\nu - (x_2 - i0)^\nu$. The space of microfunction solutions of $\mathfrak{M}_1$ and $\mathfrak{M}_3$ are the same as those of (3.8). Then, by Lemma 3.3.1, the microfunctions $u_1(x) = (x_1 + i0)^\mu \cdot \Gamma(-\mu) \cdot \delta^{(-\nu+1)}(x_2)$ and $u_2(x) = (x_1 + i0)^\mu \cdot \Gamma(-\mu) \cdot (x_2 + i0)^\nu$ give a basis of the space of microfunction solutions of $\mathfrak{L}_{\mu,\nu}$ near p. It is clear by the definition that $\operatorname{supp}(u_1(x)) = \Lambda_b^\gamma \cup \Lambda_b^\delta$ and $\operatorname{supp}(u_2(x)) \supset \Lambda_a^\alpha \cup \Lambda_a^\beta$ near p. In particular, $u_1(x)$ and $u_2(x)$ generate the solution space. Therefore, for any microfunction solution $v(x)$, $v(x)|_{\Lambda_a^\alpha \cup \Lambda_a^\beta}$ is written as $a \cdot u_2(x)|_{\Lambda_a^\alpha \cup \Lambda_a^\beta}$ with $a \in \boldsymbol{C}$. Thus the value of $v(x)$ on Λ_a^α is determined by the value on Λ_a^β and vice versa. If the support of $v(x)$ is contained in $\Lambda_{b\boldsymbol{R}}$, then $v(x)$ is a constant multiple of $u_1(x)$ and hence the value of $v(x)$ on Λ_b^γ is determined by the value of $v(x)$ on Λ_b^δ and vice versa. q.e.d.

4. Real forms of prehomogeneous vector spaces of commutative parabolic type. In this section we give the list of the real forms of prehomogeneous vector spaces of commutative parabolic type and write down their real holonomy diagrams.

4.1. The list of real forms. We consider the following real forms.

(4.1) Type C_m $(m = 1, 2, \cdots)$.

i) $(G_{\boldsymbol{R}}^+, V) = (GL_m(\boldsymbol{R})^+, \operatorname{Sym}_m(\boldsymbol{R}))$.

Here, $\operatorname{Sym}_m(\boldsymbol{R})$ stands for the space of $m \times m$ symmetric matrices over $\boldsymbol{R}$.

(4.2) Type A_k $(k = 2m + 1, m = 1, 2, \cdots)$.

i) $(G_{\boldsymbol{R}}^+, V) = (GL_1(\boldsymbol{R})^+ \times SL_m(\boldsymbol{C}), \operatorname{Her}_m(\boldsymbol{C}))$.

Here $\operatorname{Her}_m(\boldsymbol{C})$ is the space of $m \times m$ complex Hermitian matrices.

ii) $(G_{\boldsymbol{R}}^+, V) = (GL_m(\boldsymbol{R})^+ \times SL_m(\boldsymbol{R}), M_m(\boldsymbol{R}))$.

Here $M_m(\boldsymbol{R})$ is the space of $m \times m$ real matrices.

(4.3) Type $D_{2m,2}$ $(m=1, 2, \cdots)$.

i) $(G_{\boldsymbol{R}}^{+}, V)=(GL_1(\boldsymbol{R})^{+}\times SL_m(\boldsymbol{H}), \mathrm{Her}_m(\boldsymbol{H}))$.
Here $\boldsymbol{H}$ stands for the quaternion field over $\boldsymbol{R}$ and $\mathrm{Her}_m(\boldsymbol{H})$ is the space of $m\times m$ quaternion Hermitian matrices.

ii) $(G_{\boldsymbol{R}}^{+}, V)=(GL_{2m}(\boldsymbol{R})^{+}, \mathrm{Alt}_{2m}(\boldsymbol{R}))$.
Here $\mathrm{Alt}_{2m}(\boldsymbol{R})$ stands for the space of $2m\times 2m$ alternating matrices over $\boldsymbol{R}$.

(4.4) Type E_7.

i) $(G_{\boldsymbol{R}}^{+}, V)=(GL_1(\boldsymbol{R})^{+}\times E_6^d, \mathrm{Her}_3(\mathfrak{C}_{\boldsymbol{R}}^d))$.
Here, $\mathfrak{C}_{\boldsymbol{R}}^d$ is the space of division Cayley number field over $\boldsymbol{R}$ and $\mathrm{Her}_3(\mathfrak{C}_{\boldsymbol{R}}^d)$ means the space of 3×3 Hermitian matrices over $\mathfrak{C}_{\boldsymbol{R}}^d$. The group E_6^d is the subgroup of $GL^{+}(\mathrm{Her}_3(\mathfrak{C}_{\boldsymbol{R}}^d))$ consisting of the elements which leave $P(x)=\det(x)$ invariant.

ii) $(G_{\boldsymbol{R}}^{+}, V)=(GL_1(\boldsymbol{R})^{+}\times E_6^s, \mathrm{Her}_3(\mathfrak{C}_{\boldsymbol{R}}^s))$.
Here, $\mathfrak{C}_{\boldsymbol{R}}^s$ is the space of split Cayley number algebra over $\boldsymbol{R}$ and $\mathrm{Her}_3(\mathfrak{C}_{\boldsymbol{R}}^s)$ means the space of 3×3 Hermitian matrices over $\mathfrak{C}_{\boldsymbol{R}}^s$. The group E_6^s is the subgroup of $GL^{+}(\mathrm{Her}_3(\mathfrak{C}_{\boldsymbol{R}}^s))$ consisting of the elements which leave $P(X)=\det(x)$ invariant.

(4.5) Type B_k $(m=2k+1)$ and $D_{k+1,1}$ $(m=2(k+1))$ with $k=1, 2, \cdots$.

i) $(G_{\boldsymbol{R}}^{+}, V)=(GL_1(\boldsymbol{R})^{+}\times SO(p, q; \boldsymbol{R}), \boldsymbol{R}^m)$, $(p, q>0$ and $p+q=m)$.

It is easy to check that the above real forms satisfy the condition (1.1), 3) in addition to (1.1.), 1) and 2). The restriction of $P(x)$ to V can be taken to be a polynomial with real coefficients.

REMARK. 1) T. Kimura determined all the real forms of irreducible regular prehomogeneous vector spaces in 1975, although the result was not published.

2) For the cases (4.1), (4.2) and (4.5), there are other real forms which do not satisfy the condition (3.2).

4.2. Real holonomy diagrams of $\mathfrak{M}_s$. For the real forms listed in (4.1)–(4.5), we now give the real holonomy diagrams of all the intersections of codimension one in the complex holonomy diagrams of $\mathfrak{M}_s$ in Proposition 2.2. The same result was obtained by [Su].

Let $\mathfrak{M}_s$ be the holonomic system defined in (1.5) for one of the prehomogeneous vector spaces (2.3), 1)–5). The number m is defined there. We set n to be the dimension of $V_{\boldsymbol{C}}$. By Proposition 2.2, 1) the characteristic variety $\mathrm{ch}(\mathfrak{M}_s)$ has the irreducible component decomposition given by (2.4): $\mathrm{ch}(\mathfrak{M}_s)=\bigcup_{i=0}^{m}\Lambda_{i\boldsymbol{C}}$ with $\Lambda_{i\boldsymbol{C}}=\overline{T_{S_{i\boldsymbol{C}}}^{*}V_{\boldsymbol{C}}}$ where $S_{i\boldsymbol{C}}$ are $G_{\boldsymbol{C}}$-orbits defined in Proposition 2.1. Note that each $\Lambda_{i\boldsymbol{C}}$ is a $G_{\boldsymbol{C}}$-invariant subset in $T^{*}V_{\boldsymbol{C}}$ when we identify $T^{*}V_{\boldsymbol{C}}$ with $V_{\boldsymbol{C}}\times V_{\boldsymbol{C}}^{*}$. The action of $G_{\boldsymbol{C}}$ on the dual space $V_{\boldsymbol{C}}^{*}$ is through the contragredient representation. Let $(G_{\boldsymbol{R}}^{+}, \rho, V)$ be a real form of the prehomogeneous vector space $(G_{\boldsymbol{C}}, \rho, V_{\boldsymbol{C}})$. In the same way as in the complex case, $T^{*}V$ is naturally identified with $V\times V^{*}$, on which $G_{\boldsymbol{R}}^{+}$ acts. The real locus $\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}}$ of the characteristic variety $\mathrm{ch}(\mathfrak{M}_s)$ is given by

(4.6)
$$\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}}=\bigcup_{i=0}^{m} \Lambda_{i\boldsymbol{R}}$$

with $\Lambda_{i\boldsymbol{R}}=\Lambda_{i\boldsymbol{C}}\cap T^*V$. Each $\Lambda_{i\boldsymbol{R}}$ is a $G_{\boldsymbol{R}}^+$-invariant subset.

In particular, we suppose that:

(4.7) each orbit $S_{i\boldsymbol{R}}=S_{i\boldsymbol{C}}\cap V$ is a real form of $S_{i\boldsymbol{C}}$.

Naturally, we have

$$\Lambda_{i\boldsymbol{R}}=\overline{T^*_{S_{i\boldsymbol{C}}}V_{\boldsymbol{C}}}\cap T^*V=\overline{T^*_{S_{i\boldsymbol{C}}\cap V}V}=\overline{T^*_{S_{i\boldsymbol{R}}}V}\,.$$

Here, $T^*_{S_{i\boldsymbol{R}}}V$ is the *real conormal bundle* of the subvariety $S_{i\boldsymbol{R}}$ in V. Thus the real locus of $\Lambda_{i\boldsymbol{C}}$ is a real form of $\Lambda_{i\boldsymbol{C}}$, and hence the condition (3.2) is satisfied if so is (4.7). We show that the condition (4.7) is satisfied in all real forms of (4.1)–(4.5) by the case-by-case calculations in the following. Furthermore, we construct $\Lambda_{i\boldsymbol{R}}$ as a union of some $G_{\boldsymbol{R}}^+$-orbits in $V\times V^*$ and calculate the real holonomy diagrams.

The first case. Consider the cases i) in (4.1)–(4.4). The vector space V is (4.1) $\mathrm{Sym}_m(\boldsymbol{R})$, (4.2) $\mathrm{Her}_m(\boldsymbol{C})$, (4.3) $\mathrm{Her}_m(\boldsymbol{H})$ or (4.4) $\mathrm{Her}_3(\mathfrak{C}^d_{\boldsymbol{R}})$, respectively. The real locus $S_{i\boldsymbol{R}}$ of the $G_{\boldsymbol{R}}$-orbit $S_{i\boldsymbol{C}}$ in $V_{\boldsymbol{C}}$ is

$$S_{i\boldsymbol{R}}=S_{i\boldsymbol{C}}\cap V=\{x\in V;\ \mathrm{rank}(x)=m-i\}\,,\qquad (i=0, 1, \cdots, m)\,.$$

The subset $S_{i\boldsymbol{R}}$ is a $G_{\boldsymbol{R}}^+$-invariant subset and decomposes into the following $G_{\boldsymbol{R}}^+$-orbits: $S_{i\boldsymbol{R}}=\bigcup_{j=1}^{m-i}S_i^j$, where S_i^j is the $G_{\boldsymbol{R}}^+$-orbit generated by

$$\begin{bmatrix} I_j & & \\ & -I_{m-i-j} & \\ & & 0_i \end{bmatrix}.$$

Each S_i^j is a real form of $S_{i\boldsymbol{C}}$ because the real dimension of S_i^j coincides with the complex dimension of $S_{i\boldsymbol{C}}$. Therefore, the condition (4.7) is satisfied.

By the inner product $\langle x, y\rangle:=\mathrm{Re}\,\mathrm{tr}(x\cdot{}^t\bar{y})$ for $x, y\in V$, we identify V with its dual space V^*. The group $G_{\boldsymbol{R}}^+$ acts on V as the dual space by the contragredient representation and the orbit decomposition of V as the dual space is the same as that for V. We denote by $\Sigma_{i,j}^{p,q}=$ the $G_{\boldsymbol{R}}^+$-orbit in $V\times V^*$ generated by the point

$$\left(\begin{bmatrix} I_p & & \\ & -I_{m-i-p} & \\ & & 0_i \end{bmatrix}, \begin{bmatrix} 0_j & & \\ & I_q & \\ & & -I_{m-j-q} \end{bmatrix}\right)\in V\times V^*$$

where $i+j\geq m$, $0\leq p\leq m-i$ and $0\leq q\leq m-j$. Then we have:

PROPOSITION 4.1.

(4.8)
$$\Lambda_{i\boldsymbol{R}}=\bigcup_{\substack{m\geq k\geq i,\,0\leq p\leq m-k\\ m\geq j\geq n-i,\,0\leq q\leq m-j}}\Sigma_{k,j}^{p,q}\,.$$

(4.9) $$\Lambda^{o}_{i\boldsymbol{R}} = \bigcup_{\substack{0\le p\le m-i \\ 0\le q\le i}} \Sigma^{p,q}_{i,m-i}\,.$$

These propositions can be verified by a routine but a little complicated computation. See the method in [Sm-Ka-Ki-Os]. In [Mr1], the author has carried out the orbit decomposition (4.8) in the cases i)–iii). We omit the proof here.

We denote by $\Lambda^{p,q}_i$ the $G^+_{\boldsymbol{R}}$-orbit $\Sigma^{p,q}_{i,m-i}$, which is a connected component of $\Lambda^o_{i\boldsymbol{R}}$. By computing the action of the Lie algebra $\mathscr{G}_{\boldsymbol{R}}$, we see that $\Lambda^{p,q}_i$ $(0\le p\le m-i, 0\le q\le i)$ are $G^+_{\boldsymbol{R}}$-orbits in $V\times V^*$ and hence real Lagrangian subvarieties. The other orbits in $\Lambda_{i\boldsymbol{R}}$ are strictly less dimensional than n. In particular, we may write

(4.10) $$\Lambda^{o}_{i\boldsymbol{R}} = \bigcup_{\substack{0\le p\le m-i \\ 0\le q\le i}} \Lambda^{p,q}_{i}\,.$$

By Proposition 4.1, we have

$$\Lambda_{i\boldsymbol{R}} \cap \Lambda_{i+1\boldsymbol{R}} = \bigcup_{\substack{m\ge k\ge i+1,\, 0\le p\le m-k \\ m\ge j\ge n-i,\, 0\le q\le m-j}} \Sigma^{p,q}_{k,j}\,.$$

The non-singular locus of $\Lambda_{i\boldsymbol{R}} \cap \Lambda_{i+1\boldsymbol{R}}$ is given by

$$(\Lambda_{i\boldsymbol{R}} \cap \Lambda_{i+1\boldsymbol{R}})_{\mathrm{reg}} = \bigcup_{\substack{0\le p\le m-i-1 \\ 0\le q\le i}} \Sigma^{p,q}_{i+1,m-1}\,.$$

By computing the action of the Lie algebra $\mathscr{G}_{\boldsymbol{R}}$, we see that each orbit $\Sigma^{p-1,q}_{i+1,m-i+1}$ $(0\le p\le m-i-1, 0\le q\le i)$ is a connected component of $(\Lambda_{i\boldsymbol{R}} \cap \Lambda_{i+1\boldsymbol{R}})_{\mathrm{reg}}$ and is an $(n-1)$-dimensional $G^+_{\boldsymbol{R}}$-orbit. Thus we have the following proposition.

PROPOSITION 4.2. *The $(n-1)$-dimensional intersection of $\Lambda_{i\boldsymbol{R}}$ and $\Lambda_{i+1\boldsymbol{R}}$ is represented by the real holonomy diagram as in Figure* 3, (1). *Here* $1\le p\le n-i, 0\le q\le i$.

The second case. Next we consider the cases ii) in (4.2)–(4.4). Then V is (4.2) $M_n(\boldsymbol{R})$, (4.3) $\mathrm{Alt}_{2n}(\boldsymbol{R})$ and (4.4) $\mathrm{Her}_3(\mathfrak{C}^s_{\boldsymbol{R}})$, respectively. The real locus $S_{i\boldsymbol{R}}$ of the orbit $S_{i\boldsymbol{C}}$ is $S_{i\boldsymbol{R}} = S_{i\boldsymbol{C}} \cap V = \{x\in V; \mathrm{rank}\, x = n-i\}$, for (4.2) and (4.4), $S_{i\boldsymbol{R}} = S_{i\boldsymbol{C}} \cap V = \{x\in V; \mathrm{rank}\, x = 2(n-i)\}$ for (4.3). The subset $S_{i\boldsymbol{R}}$ is $G^+_{\boldsymbol{R}}$-invariant and decomposes into the following $G^+_{\boldsymbol{R}}$-orbits: $S_{0\boldsymbol{R}} = S^+_0 \cup S^-_0$ with $S^+_0 = \{x\in V; P(x)>0\}$ and $S^-_0 = \{x\in V; P(x)<0\}$, and $S_{i\boldsymbol{R}}$ $(i>1)$ is a single $G^+_{\boldsymbol{R}}$-orbit. By the inner product $\langle x, y\rangle := \mathrm{tr}(x\cdot {}^t y)$ for $x, y\in V$ on V, we identify V with its dual space V^*. The group $G^+_{\boldsymbol{R}}$ acts on V^* by the contragredient action. The vector space V^* has the same orbit decomposition by the contragredient action of $G^+_{\boldsymbol{R}}$. We denote by Λ^{ε}_i the $G^+_{\boldsymbol{R}}$-orbit in $V\times V^*$ generated by the point,

$$\left(\begin{bmatrix} I_{m-i} & \\ & 0_i \end{bmatrix}, \begin{bmatrix} 0_{m-i} & & \\ & \varepsilon & \\ & & I_{i-1} \end{bmatrix}\right) \qquad \text{for} \quad (4.2) \text{ and } (4.4)\,,$$

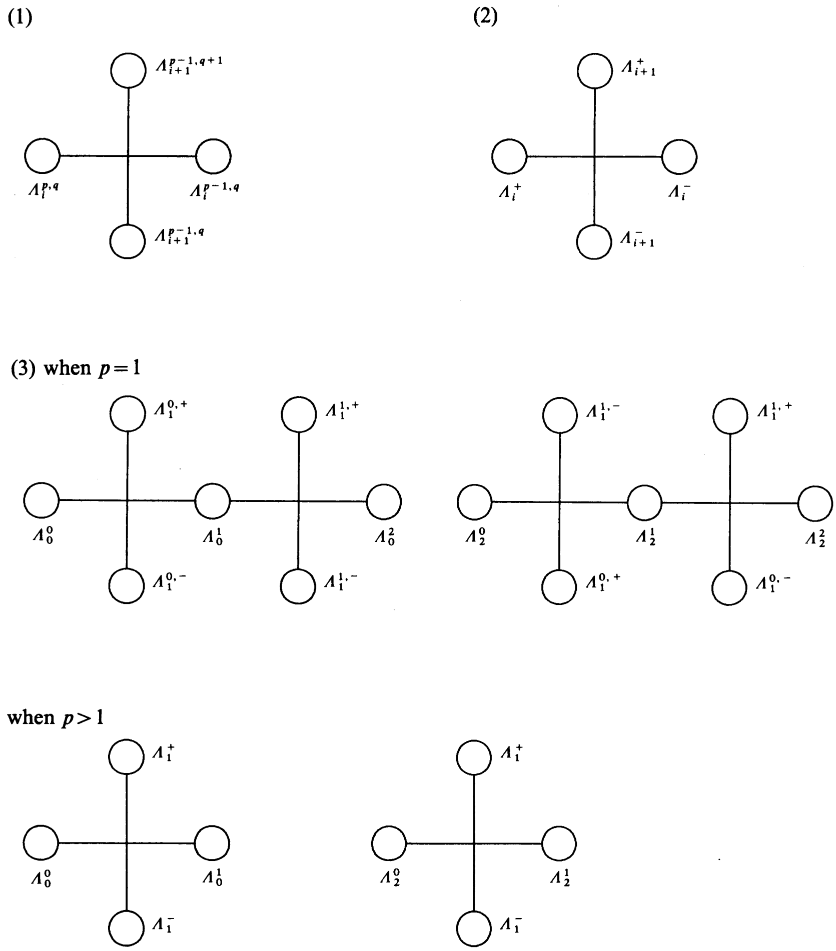

FIGURE 3.

and

$$\left(\begin{bmatrix} I_{m-i} & \\ & 0_i \end{bmatrix} \otimes J,\ \begin{bmatrix} 0_{m-i} & & \\ & \varepsilon & \\ & & I_{i-1} \end{bmatrix} \otimes J\right), \qquad \text{for} \quad (4.3),$$

with $i=0, 1, \cdots, m$ and $\varepsilon=\pm 1$. Here

$$J=\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$$

and $\otimes$ means the tensor product of matrices. Let $\Sigma_{i,j}$ be the $G_{\boldsymbol{R}}^+$-orbit in $V\times V^*$ generated by the point,

$$\left(\begin{bmatrix} I_{m-i} & \\ & 0_i \end{bmatrix}, \begin{bmatrix} 0_j & \\ & I_{m-j} \end{bmatrix}\right) \quad \text{for (4.2) and (4.4)},$$

and

$$\left(\begin{bmatrix} I_{m-i} & \\ & 0_i \end{bmatrix}\otimes J, \begin{bmatrix} 0_j & \\ & I_{m-j} \end{bmatrix}\otimes J\right) \quad \text{for (4.3)},$$

with $i+j>m$, $m\geq i\geq 0$ and $m\geq j\geq 0$. Then we have:

PROPOSITION 4.3.

$$\Lambda_{i\boldsymbol{R}}=(\Lambda_i^+ \cup \Lambda_i^-)\cup\Bigl(\bigcup_{\substack{k\geq i\\ j\geq m-i}} \Sigma_{k,j}\Bigr). \tag{4.11}$$

$$\Lambda_{i\boldsymbol{R}}^o=\Lambda_i^+ \cup \Lambda_i^- . \tag{4.12}$$

We omit the easy proof. The $\boldsymbol{G}_{\boldsymbol{R}}^+$-orbits Λ_i^+ and Λ_i^- are n-dimensional. The other orbits in $\Lambda_{i\boldsymbol{R}}$ are of dimension strictly less than n. By Proposition 4.3, we have $\Lambda_{i\boldsymbol{R}}\cap\Lambda_{i+1\boldsymbol{R}}=\bigcup_{\substack{k\geq i+1\\ j\geq m-i}}\Sigma_{k,j}$. The non-singular locus of $\Lambda_{i\boldsymbol{R}}\cap\Lambda_{i+1\boldsymbol{R}}$ is $\Sigma_{i+1,m-i}$, which is an $(n-1)$-dimensional $\boldsymbol{G}_{\boldsymbol{R}}^+$-orbit.

PROPOSITION 4.4. *The $(n-1)$-dimensional intersections of $\Lambda_{i\boldsymbol{R}}$ and $\Lambda_{i+1\boldsymbol{R}}$ are represented by the real holonomy diagrams Figure* 3, (2).

The third case. Finally we consider the case i) of (4.5). We may suppose that $q\geq p>0$. The vector space V is $\boldsymbol{R}^n$. Without loss of generality we may assume that V is a vector space having the inner product:

$$\langle x, y\rangle={}^t x\cdot I_{pq}\cdot y \quad \text{where} \quad I_{pq}=\begin{bmatrix} I_p & \\ & -I_q \end{bmatrix} \quad \text{with} \quad p+q=n. \tag{4.13}$$

The group $SO(p, q; \boldsymbol{R})$ is the subgroup of $GL(V)$ consisting of elements leaving the inner product invariant. We can identify V^* with V by the inner product. Thus the real contangent space T^*V is naturally viewed as $V\times V^*$. We set:

$$\begin{aligned} &p_0^0=(1, 0, \cdots, 0; 0, \cdots, 0), \quad p_0^1=(0, \cdots 0; 1, 0, \cdots, 0),\\ &p_0^2=(-1, 0, \cdots, 0; 0, \cdots, 0),\\ &p_1^0=(1, 0, \cdots, 0; 1, 0, \cdots, 0), \quad p_1^1=(1, 0, \cdots, 0; -1, 0, \cdots, 0),\\ &p_2^0=(0, \cdots, 0; 0, \cdots, 0). \end{aligned} \tag{4.14}$$

The expression $(x_1; x_2)$ means the coordinate in $\boldsymbol{R}^n=V$ with $x_1\in\boldsymbol{R}^p$ and $x_2\in\boldsymbol{R}^q$. When $p=1$, the points in (4.14) generate mutually different $\boldsymbol{G}_{\boldsymbol{R}}^+$-orbits. When $p>1$, the orbits generated by p_0^0 and p_0^2 (resp. p_1^0 and p_1^1) are the same. We denote by $\Sigma_{i,j}^{p,q,\varepsilon}$ = the $G_{\boldsymbol{R}}^+$-orbit in $V\times V^*$ generated by the point $(p_i^p, \varepsilon p_j^q)$, ($\varepsilon=\pm$). Then we have:

PROPOSITION 4.5.

1) *When $p=1$, we have the following disjoint decompositions of $\Lambda_{i\boldsymbol{R}}$ and $\Lambda_{i\boldsymbol{R}}^o$.*

(4.15)
$$\Lambda_{0\boldsymbol{R}}=\left(\bigcup_{p=0,1,2}\Sigma_{0,2}^{p,0,+}\right)\cup\left(\bigcup_{p=0,1}\Sigma_{1,2}^{p,0,+}\right)\cup\Sigma_{2,2}^{0,0,+}.$$
$$\Lambda_{1\boldsymbol{R}}=\left(\bigcup_{\substack{p=0,1\\ \varepsilon=\pm}}\Sigma_{1,1}^{p,p,\varepsilon}\right)\cup\left(\bigcup_{p=0,1}\Sigma_{1,2}^{p,0,+}\right)\cup\left(\bigcup_{q=0,1}\Sigma_{2,1}^{0,q,+}\right)\cup\Sigma_{2,2}^{0,0,+}.$$
$$\Lambda_{2\boldsymbol{R}}=\left(\bigcup_{q=0,1,2}\Sigma_{2,0}^{0,q,+}\right)\cup\left(\bigcup_{q=0,1}\Sigma_{2,1}^{0,q,+}\right)\cup\Sigma_{2,2}^{0,0,+}.$$

(4.16)
$$\Lambda_{0\boldsymbol{R}}^o=\left(\bigcup_{p=0,1,2}\Sigma_{0,2}^{p,0,+}\right).$$
$$\Lambda_{1\boldsymbol{R}}^o=\left(\bigcup_{\substack{p=0,1\\ \varepsilon=\pm}}\Sigma_{1,1}^{p,p,\varepsilon}\right).$$
$$\Lambda_{2\boldsymbol{R}}^o=\left(\bigcup_{q=0,1,2}\Sigma_{2,0}^{0,q,+}\right).$$

2) *When $p>1$, we have the following disjoint decompositions of $\Lambda_{i\boldsymbol{R}}$ and $\Lambda_{i\boldsymbol{R}}^o$* ($i=0, 1, 2$).

(4.17)
$$\Lambda_{0\boldsymbol{R}}=\left(\bigcup_{p=0,1}\Sigma_{0,2}^{p,0,+}\right)\cup\Sigma_{1,2}^{0,0,+}\cup\Sigma_{2,2}^{0,0,+}.$$
$$\Lambda_{1\boldsymbol{R}}=\left(\bigcup_{\varepsilon=\pm}\Sigma_{1,1}^{0,0,\varepsilon}\right)\cup\Sigma_{1,2}^{0,0,+}\cup\Sigma_{2,1}^{0,0,+}\cup\Sigma_{2,2}^{0,0,+}.$$
$$\Lambda_{2\boldsymbol{R}}=\left(\bigcup_{q=0,1}\Sigma_{2,0}^{0,q,+}\right)\cup\Sigma_{2,1}^{0,0,+}\cup\Sigma_{2,2}^{0,0,+}.$$

(4.18)
$$\Lambda_{0\boldsymbol{R}}^o=\left(\bigcup_{p=0,1}\Sigma_{0,2}^{p,0,+}\right).$$
$$\Lambda_{1\boldsymbol{R}}^o=\left(\bigcup_{\varepsilon=\pm}\Sigma_{1,1}^{0,0,\varepsilon}\right).$$
$$\Lambda_{2\boldsymbol{R}}^o=\left(\bigcup_{q=0,1}\Sigma_{2,0}^{0,q,+}\right).$$

We omit the easy proof. We set

(4.19)
$$\Lambda_0^i=\Sigma_{0,2}^{i,0,+} \qquad (i=0,1,2),$$
$$\Lambda_1^{i,\varepsilon}=\Sigma_{1,1}^{i,i,\varepsilon} \qquad (i=0,1 \text{ and } \varepsilon=\pm),$$
$$\Lambda_2^i=\Sigma_{2,0}^{0,i,+} \qquad (i=0,1,2),$$

when $p=1$ and set

(4.20)
$$\Lambda_0^i=\Sigma_{0,2}^{i,0,+} \qquad (i=0,1),$$
$$\Lambda_1^{\varepsilon}=\Sigma_{1,1}^{0,0,\varepsilon} \qquad (\varepsilon=\pm),$$
$$\Lambda_2^i=\Sigma_{2,0}^{0,i,+} \qquad (i=0,1),$$

when $p>1$. The orbits in (4.19) and (4.20) are n-dimensional and the other orbits in $\Lambda_{i\boldsymbol{R}}$ are of dimension strictly less than n. By Proposition 4.5, we have

$$\Lambda_{0\boldsymbol{R}}\cap\Lambda_{1\boldsymbol{R}}=\left(\bigcup_{p=0,1}\Sigma_{1,2}^{p,0,+}\right)\cup\Sigma_{2,2}^{0,0,+},$$
$$\Lambda_{1\boldsymbol{R}}\cap\Lambda_{2\boldsymbol{R}}=\left(\bigcup_{q=0,1}\Sigma_{2,1}^{0,q,+}\right)\cup\Sigma_{2,2}^{0,0,+},$$

where $p=1$ and we have

$$\Lambda_{0\boldsymbol{R}}\cap\Lambda_{1\boldsymbol{R}}=\Sigma_{1,2}^{0,0,+}\cup\Sigma_{2,2}^{0,0,+},$$
$$\Lambda_{1\boldsymbol{R}}\cap\Lambda_{2\boldsymbol{R}}=\Sigma_{2,1}^{0,0,+}\cup\Sigma_{2,2}^{0,0,+},$$

where $p>1$. The real holonomy diagrams are given by the following proposition.

PROPOSITION 4.6. *The $(n-1)$-dimensional intersections of $\Lambda_{i\boldsymbol{R}}$ and $\Lambda_{i+1\boldsymbol{R}}$ are represented by the real holonomy diagrams Figure* 3, (3).

5. Proof of the main theorem. In this section we prove the main theorem for the real forms listed in (4.1)–(4.5) of regular prehomogeneous vector spaces of commutative parabolic type.

5.1. Critical points for $P(x)^s$. Let $(G_{\boldsymbol{R}}^+, \rho, V)$ be one of the real forms of the prehomogeneous vector spaces in (4.1)–(4.5). We always suppose that a relatively invariant polynomial $P(x)$ on V is taken to be with real coefficients. Let $b(s)$ be the b-function of the complex form $(G_{\boldsymbol{C}}, \rho, V_{\boldsymbol{C}})$ of $(G_{\boldsymbol{R}}^+, \rho, V_{\boldsymbol{R}})$. The explicit form is given in (2.5).

(5.1) DEFINITION (Critical points). We set $\mathrm{Crit}(P(x)^s):=\{\lambda\in\boldsymbol{C};\ b(\lambda+k)=0$ with some non-negative integer $k\}$. We call an element of $\mathrm{Crit}(P(x)^s)$ a *critical point* for $P(x)^s$.

We may express the b-functions as $b(s)=\prod_{i=1}^m(s-\lambda_i)$ with $0>\lambda_1=-1>\lambda_2>\cdots>\lambda_m$ where $\lambda_1,\cdots,\lambda_m$ are negative integers or negative half-integers by Proposition 2.3.

Definition (5.1) says that $\lambda=\lambda_i-p$ with a non-negative integer p and an integer i if $\lambda\in\mathrm{Crit}(P(x)^s)$.

PROPOSITION 5.1. *Let λ be a complex number.*

(1) *If $\lambda\notin\mathrm{Crit}(P(x)^s)$, then $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{iC} to Λ_{i+1C} for $i=0, 1, \cdots, m-1$.*

(2) *If $\lambda\in\mathrm{Crit}(P(x)^s)$, then $\lambda\leq\lambda_1$.*

(3) *If $\lambda\leq\lambda_m$, then $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{i+1C} to Λ_{iC} for $i=0, 1, \cdots, m-1$.*

(4) *Suppose that $\lambda\in\mathrm{Crit}(P(x)^s)$ and $\lambda_m<\lambda\leq\lambda_1$. Let k be a positive integer $\leq m-1$. If $\lambda_{k+1}<\lambda\leq\lambda_k$, then $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{iC} to Λ_{i+1C} for any $i\geq k$ and $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{i+1C} to Λ_{iC} for any $i\leq k-1$.*

PROOF. (1) By Definition (5.1), if $\lambda\in\mathrm{Crit}(P(x)^s)$, then there exist a root λ_i of $b(s)$ and a non-negative integer p such that $\lambda=\lambda_i-p$. Each $(s-\lambda_i)$ is the factor of b-function of $\mathfrak{M}_s$ from Λ_{i-1C} to Λ_{iC}. We set $p(s)+1=(s-\lambda_i)$. Then $p(\lambda)=-p-1$ and it is a negative integer. By Definition (3.6), $\mathfrak{M}_s$ is critical from Λ_{i-1C} to Λ_{iC} at $s=\lambda$.

(2) If $\lambda\in\mathrm{Crit}(P(x)^s)$, then there exist λ_i and a non-negative integer p such that $\lambda=\lambda_i-p$. Thus $\lambda=\lambda_i-p\leq\lambda_i\leq\lambda_1$.

(3) Note that the factor of b-function of $\mathfrak{M}_s$ from Λ_{i+1C} to Λ_{iC} is $-s+\lambda_{i+1}+1$. Then we have $(-s+\lambda_{i+1}+1)|_{s=\lambda}=-\lambda+\lambda_{i+1}+1\geq-\lambda_m+\lambda_{i+1}+1\geq 1$, since $\lambda\leq\lambda_m\leq\lambda_{i+1}$ for all i. The $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{i+1C} to Λ_{iC}.

(4) The factor of b-function of $\mathfrak{M}_s$ from Λ_{iC} to Λ_{i+1C} (resp. from Λ_{i+1C} to Λ_{iC}) is $(s-\lambda_{i+1})$ (resp. $(-s+\lambda_{i+1}+1)$). Thus, if $i\geq k$, then we have $s-\lambda_{i+1}|_{s=\lambda}=\lambda-\lambda_{i+1}>\lambda_{k+1}-\lambda_{i+1}\geq 0$, and hence $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{iC} to Λ_{i+1C}. If $i\leq k-1$, then we have $(-s+\lambda_{i+1}+1)|_{s=\lambda}=-\lambda+\lambda_{i+1}+1\geq-\lambda_k+\lambda_{i+1}+1\geq-\lambda_k+\lambda_k+1=1$, and hence $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{i+1C} to Λ_{iC}. q.e.d.

5.2. Proof of the main theorem at non-critical points.

PROPOSITION 5.2. *Let $\lambda\notin\mathrm{Crit}(P(x)^s)$. Then the dimension of the space of χ^λ-invariant hyperfunctions is the number l of the connected components of $V-S_{\mathbf{R}}$.*

PROOF. It suffices to prove that the dimension of the space of χ^λ-invariant hyperfunctions is at most l, since it is at least l by Proposition 1.2.

Let $u(x)$ be a χ^λ-invariant hyperfunction. Then $u(x)$ is a solution to the holonomic system $\mathfrak{M}_\lambda$ (see § 1). The function $u(x)$ is real analytic, since $\mathfrak{M}_\lambda$ is an elliptic system on $V-S$, i.e., the characteristic variety is $(V-S)\times\{0\}$. Thus we have $u(x)|_{V-S}=\sum_{i=1}^{l}a_i\cdot|P(x)|_i^\lambda|_{V-S}$, because any χ^λ invariant real analytic function on a connected component V_i is written as a constant multiple of $|P(x)|_i^\lambda$.

Consider the hyperfunction $v(x):=u(x)-\sum_{i=1}^{l}a_i\cdot|P(x)|_i^\lambda$ on V. Then $v(x)$ is a hyperfunction solution of $\mathfrak{M}_\lambda$ and is zero on $V-S$. Now look upon $v(x)$ as the microfunction $sp(v(x))$ on T^*V. Then the support of $sp(v(x))$ is contained in

$\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}} = \mathrm{ch}(\mathfrak{M}_s) \cap T^*V$. The real characteristic variety $\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}}$ has the irreducible component decomposition $\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}} = \bigcup_{i=0}^{m} \Lambda_{i\boldsymbol{R}}$ (see (4.6)). Among the irreducible components, $\Lambda_{0\boldsymbol{R}}$ is the zero section $V_{\boldsymbol{R}} \times \{0\}$. The set $\Lambda_{0\boldsymbol{R}}^{\circ}$ of generic points has the connected component decomposition $\Lambda_{0\boldsymbol{R}}^{\circ} = V_1 \times \{0\} \cup V_2 \times \{0\} \cup \cdots \cup V_l \times \{0\}$. Since the hyperfunction $v(x)$ is zero on each connected component V_i, the microfunction $sp(v(x))$ is zero on each $V_i \times \{0\}$ $(i = 1, \cdots, l)$.

LEMMA 5.2.1. *For an arbitrary complex number λ, let $v(x)$ be a hyperfunction solution to $\mathfrak{M}_\lambda$. Suppose that $\mathfrak{M}_s$ is not critical at $s = \lambda$ from $\Lambda_{i\boldsymbol{C}}$ to $\Lambda_{i+1\boldsymbol{C}}$ (resp. $\Lambda_{i+1\boldsymbol{C}}$ to $\Lambda_{i\boldsymbol{C}}$). If the microfunction $sp(v(x))$ is zero on $\Lambda_{i\boldsymbol{R}}^{\circ}$ (resp. $\Lambda_{i+1\boldsymbol{R}}^{\circ}$), then it is zero on $\Lambda_{i+1\boldsymbol{R}}^{\circ}$ (resp. $\Lambda_{i\boldsymbol{R}}^{\circ}$) as well.*

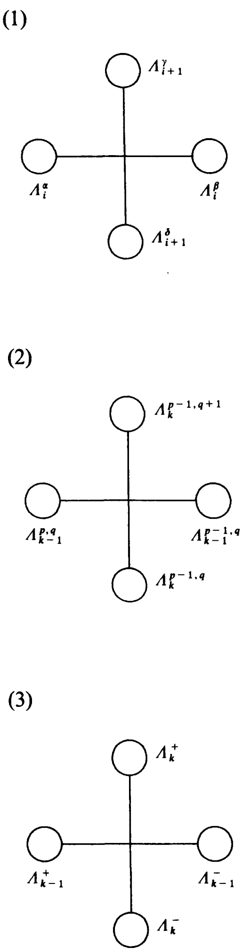

FIGURE 4.

PROOF. Let Λ_{i+1}^{γ} be a connected component in $\Lambda_{i+1\boldsymbol{R}}^{o}$. Then there exist two connected components Λ_i^{α}, Λ_i^{β} of $\Lambda_{i\boldsymbol{R}}^{o}$ and Λ_{i+1}^{δ} of $\Lambda_{i+1\boldsymbol{R}}^{o}$ which form the real holonomy diagram Figure 4, (1). This easily follows from the real holonomy diagrams calculated in Proposition 4.2, 4.4 and 4.6.

Let $v(x)$ be a hyperfunction solution to the holonomic system $\mathfrak{M}_{\lambda}$ such that $sp(v(x))$ is zero on $\Lambda_{i\boldsymbol{R}}^{o}$. Then $sp(v(x))$ is zero on Λ_i^{α} and Λ_i^{β}. Since $\mathfrak{M}_s$ is not critical at $s=\lambda$ from $\Lambda_{i\boldsymbol{C}}$ to $\Lambda_{i+1\boldsymbol{C}}$ by assumption, $sp(v(x))$ is zero on Λ_{i+1}^{γ} and Λ_{i+1}^{δ} in a neighborhood of the intersection of $\Lambda_{i\boldsymbol{R}}$ and $\Lambda_{i+1\boldsymbol{R}}$ by Proposition 3.3, (2). Moreover $sp(v(x))$ is zero on Λ_{i+1}^{γ} and Λ_{i+1}^{δ} globally. Thus $sp(v(x))$ is zero on Λ_{i+1}^{ρ} for every index ρ. Thus means that $sp(v(x))$ is zero on all the connected components of $\Lambda_{i+1\boldsymbol{R}}^{o}$. When $\mathfrak{M}_s$ is not critical at $s=\lambda$ from $\Lambda_{i+1\boldsymbol{C}}$ to $\Lambda_{i\boldsymbol{C}}$, we can show the converse in the same way and complete the proof of Lemma 5.2.1.

By Proposition 5.1, (1), if $\lambda\notin \mathrm{Crit}(P(x)^s)$, then $\mathfrak{M}_s$ is not critical at $s=\lambda$ from $\Lambda_{i\boldsymbol{C}}$ to $\Lambda_{i+1\boldsymbol{C}}$ for all $i=0, 1, \cdots, m-1$. Therefore, by induction on i, if $sp(v(x))|_{\Lambda_{0\boldsymbol{R}}^{o}}=0$, then $sp(v(x))|_{\Lambda_{i\boldsymbol{R}}^{o}}=0$ for all $i=0, 1, \cdots, m$.

LEMMA 5.2.2. *For an arbitrary complex number λ, let $v(x)$ be a hyperfunction solution to the holonomic system $\mathfrak{M}_{\lambda}$. If the microfunction $sp(v(x))$ is zero on $\Lambda_{i\boldsymbol{R}}^{o}$ for all $i=0, \cdots, m$, then $v(x)=0$ as a hyperfunction on V.*

A theorem more general than Lemma 5.2.2 was proved in [Mr3], which would be an interesting result in itself. We omit the proof.

Consider the hyperfunction solution $v(x)$ in the form $u(x)-\sum_{i=1}^{l} a_i\cdot|P(x)|_i^{\lambda}$ again. Since $sp(v(x))$ is zero on $\Lambda_{0\boldsymbol{R}}^{o}$, it is zero on $\Lambda_{1\boldsymbol{R}}^{o}$, $\Lambda_{2\boldsymbol{R}}^{o}$, $\cdots$, $\Lambda_{m\boldsymbol{R}}^{o}$ by induction from Lemma 5.2.1, and hence it is zero on $\bigcup_{i=1}^{m}\Lambda_{i\boldsymbol{R}}^{o}$. By Lemma 5.2.2, we have $v(x)=0$, which means $u(x)=\sum_{i=1}^{l} a_i\cdot|P(x)|_i^{\lambda}$. Thus we see that any χ^{λ}-invariant hyperfunction $u(x)$ is expressed as a linear combination of $|P(x)|_i^{\lambda}$ $(i=1, \cdots, l)$ if $\lambda\notin\mathrm{Crit}(P(x)^s)$. Hence the dimension of the space of χ^{λ}-invariant hyperfunctions is at most l. Thus we have the desired result. q.e.d.

COROLLARY 5.3. *Let $\lambda\notin\mathrm{Crit}(P(x)^s)$. Then any χ^{λ}-invariant hyperfunction is written as a linear combination of $|P(x)|_i^{s}$ at $s=\lambda$ in the sense of* (1.3).

PROOF. We have seen in Proposition 5.2 that the space of linear combinations of $|P(x)|_i^{s}$ at $s=\lambda$ coincides with the space of χ^{λ}-invariant hyperfunctions. Thus we have the desired results. q.e.d.

5.3. Proof of the main theorem at critical points.

PROPOSITION 5.4. *Let $\lambda\in\mathrm{Crit}(P(x)^s)$. Then the dimension of the space of χ^{λ}-invariant hyperfunctions is l, the number of the connected components of $V-S_{\boldsymbol{R}}$.*

PROOF. It suffices to prove that the dimension of the space of χ^{λ}-invariant hyperfunctions is at most l. When $\lambda\in\mathrm{Crit}(P(x)^s)$, we may suppose that $\lambda\leq\lambda_1$ by

Proposition 5.1, (2). First we prove Proposition 5.4 when $\lambda \leq \lambda_m$.

LEMMA 5.4.1. *Suppose $\lambda \leq \lambda_m$. Then the dimension of the space of χ^λ-invariant hyperfunctions is at most l.*

PROOF. Let $\mathcal{S}ol(\mathfrak{M}_\lambda)$ be the space of hyperfunction solutions of $\mathfrak{M}_\lambda$ on T^*V. We denote by $\mathcal{S}ol(\mathfrak{M}_\lambda)|_{\Lambda_{iR}^\circ}$ the space of the restrictions to Λ_{iR}° of the sp-image of elements of $\mathcal{S}ol(\mathfrak{M}_\lambda)$. Recall that Λ_{mR}° decomposes into l connected components by Proposition 4.1, 4.3 and 4.5. Then $\mathcal{S}ol(\mathfrak{M}_\lambda)|_{\Lambda_{mR}^\circ}$ is at most l-dimensional because $\mathcal{S}ol(\mathfrak{M}_\lambda)$ is one dimensional on each connected component of Λ_{iR}° by Proposition 3.2.

Let $v(x)$ be a hyperfunction solution of $\mathfrak{M}_\lambda$ on V such that $sp(v(x))|_{\Lambda_{mR}^\circ}=0$. Then

$$sp(v(x))|_{\Lambda_{iR}^\circ}=0 \qquad \text{for all} \quad i=0, 1, \cdots, m. \tag{5.2}$$

Indeed, since $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{i+1C} to Λ_{iC} for all $i=m-1, m-2, \cdots, 0$, by Proposition 5.1 (3), $sp(v(x))|_{\Lambda_{i+1R}^\circ}=0$ implies that $sp(v(x))|_{\Lambda_{iR}^\circ}=0$ for $i=m-1$, $m-2, \cdots, 0$ by Lemma 5.2.1. Thus, by induction on i, we have (5.2). Moreover, (5.2) means that $v(x)=0$ as a hyperfunction on V by Lemma 5.2.2. Thus, for two solutions $v_1(x)$, $v_2(x) \in \mathcal{S}ol(\mathfrak{M}_\lambda)$, if $sp(v(x))|_{\Lambda_{mR}^\circ}=sp(v_2(x))|_{\Lambda_{mR}^\circ}$, then $v_1(x)=v_2(x)$. Therefore any hyperfunction solution $v(x)$ of $\mathfrak{M}_\lambda$ is uniquely determined by the data $sp(v(x))|_{\Lambda_{mR}^\circ}$. Hence the dimension of the hyperfunction solutions of $\mathfrak{M}_\lambda$ is at most l.

LEMMA 5.4.2. *Let $\lambda \in \mathrm{Crit}(P(x)^s)$ and suppose that $\lambda_m < \lambda \leq \lambda_1$. Then the dimension of the space of χ^λ-invariant hyperfunctions is at most l-dimensional.*

PROOF. We show Lemma 5.4.2 by reducing it to the following sublemma.

SUBLEMMA 5.4.2.1. *Let $\lambda \in \mathrm{Crit}(P(x)^s)$ and suppose that $\lambda_{k+1} < \lambda \leq \lambda_k$. Then $\mathcal{S}ol(\mathfrak{M}_\lambda)|_{\Lambda_{k-1R}^\circ \cup \Lambda_{kR}^\circ}$ is at most l-dimensional.*

Sublemma 5.4.2.1 implies Lemma 5.4.2. Indeed, let $v(x)$ be a hyperfunction solution of $\mathfrak{M}_\lambda$ on V such that $sp(v(x))|_{\Lambda_{k-1R}^\circ \cup \Lambda_{kR}^\circ}=0$. Then

$$sp(v(x))|_{\Lambda_{iR}^\circ}=0, \qquad \text{for all} \quad i=0, 1, \cdots, m. \tag{5.3}$$

Since $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{iC} to Λ_{i+1C} for all $i=k, k+1, \cdots, m-1$, by Proposition 5.1 (4), $sp(v(x))|_{\Lambda_{iR}^\circ}=0$ implies that $sp(v(x))|_{\Lambda_{i+1R}^\circ}=0$ for $i=k+1, k+2, \cdots, m-1$ by Lemma 5.2.1. Similarly, since $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{i+1C} to Λ_{iC} for all $i=k-2, k-3, \cdots, 0$, by Proposition 5.1 (4), $sp(v(x))|_{\Lambda_{i+1R}^\circ}=0$ implies that $sp(v(x))|_{\Lambda_{iR}^\circ}=0$ for $i=k-2, k-3, \cdots, 0$ by Lemma 5.2.1. Thus, by induction on i, we have (5.3). Moreover, (5.3) means that $v(x)=0$ as a hyperfunction on V by Lemma 5.2.2. Therefore any hyperfunction solution $v(x)$ of $\mathfrak{M}_\lambda$ is uniquely determined by the data $sp(v(x))|_{\Lambda_{k-1R}^\circ \cup \Lambda_{kR}^\circ}$. This implies that the dimension of the hyperfunction solutions of $\mathfrak{M}_\lambda$ is at most l if $\lambda_m < \lambda \leq \lambda_1$. Thus we complete the proof of Lemma 5.4.2 if Sublemma 5.4.2.1 is proved.

PROOF OF SUBLEMMA 5.4.2.1. We consider the cases of i) in (4.1)–(4.4). The number l coincides with $m+1$ in these cases. The connected component decompositions of $\Lambda^o_{k-1\mathbf{R}}$ and $\Lambda^o_{k\mathbf{R}}$ were given in Proposition 4.1. The real holonomy diagrams of the $(n-1)$-dimensional intersections between $\Lambda_{k-1\mathbf{R}}$ and $\Lambda_{k\mathbf{R}}$ is given by Figure 4, (2) with $1 \leq p \leq n-k+1$ and $0 \leq q \leq k-1$ (see Proposition 4.2).

We set $W := \mathscr{S}ol(\mathfrak{M}_\lambda)|_{\Lambda^o_{k-1\mathbf{R}} \cup \Lambda^o_{k\mathbf{R}}}$, $W_1 := \{v(x) \in W; v(x)|_{\Lambda^o_{k-1\mathbf{R}}} = 0\}$. We would like to prove

$$(5.4) \qquad \begin{aligned} &1)\ \dim W_1 \leq m-k+1\,, \\ &2)\ \dim (W/W_1) \leq k\,, \end{aligned}$$

which means that $\dim W = m+1 = l$.

As for (5.4), 1), let $v(x)$ be an element of W_1. Thus $v(x)$ is zero on $\Lambda^{p,q}_{k-1}$ and $\Lambda^{p-1,q}_{k-1}$ in the real holonomy diagram Figure 4 (2). Since $\mathfrak{M}_s$ is critical at $s=\lambda$ from Λ_{k-1C} to Λ_{kC}, the value of $v(x)$ on $\Lambda^{p-1,q+1}_k$ is determined by the value of $v(x)$ on $\Lambda^{p-1,q}_k$ by Proposition 3.3, (3). Therefore, by induction on q, the values of $v(x)$ on $\Lambda^{p-1,q}_k$ $(q=0, 1, \cdots, k)$ are determined by the value of $v(x)$ on $\Lambda^{p-1,0}_k$. Hence the values $v(x)|_{\Lambda^o_{k\mathbf{R}}}$ is completely determined by the data $v(x)|_{\bigcup_{1 \leq p \leq m-k+1} \Lambda^{p-1,0}_k}$ because $\Lambda^o_{k\mathbf{R}}$ consists of the connected components in $\bigcup_{\substack{0 \leq p \leq m-k \\ 0 \leq q \leq k}} \Lambda^{p,q}_k$ (see (4.10)). Since the dimension of the solution space on the connected component $\Lambda^{p-1,0}_k$ is one for each $p=1, 2, \cdots, m-k+1$, we have (5.4), 1).

To show (5.4), 2), let $v_1(x)$ and $v_2(x)$ be elements of W. If $v_1(x) - v_2(x) \in W_1$, then $v_1(x)$ and $v_2(x)$ coincide with each other in W/W_1 and vice versa. Namely, the representative of $v_1(x)$ in W/W_1 coincides with that of $v_2(x)$ if and only if $v_1(X)|_{\Lambda^o_{k-1\mathbf{R}}} = v_2(x)|_{\Lambda^o_{k-1\mathbf{R}}}$. Therefore the dimension of the space $\mathscr{S}ol(\mathfrak{M}_\lambda)|_{\Lambda^o_{k-1\mathbf{R}}}$ is the dimension of (W/W_1).

Let $v(x)$ be an element of W. In the real holonomy diagram Figure 4 (2), the value of $v(x)$ on $\Lambda^{p,q}_{k-1}$ is determined by the value of $v(x)$ on $\Lambda^{p-1,q}_{k-1}$ by Proposition 3.3 (3), because $\mathfrak{M}_s$ is critical at $s=\lambda$ from Λ_{k-1C} to Λ_{kC}. Therefore, by induction on p, the values of $v(x)$ on $\Lambda^{p,q}_{k-1}$ $(p=0, 1, \cdots, m-k+1)$ are determined by the value of $v(x)$ on $\Lambda^{0,q}_{k-1}$. This means that the values $v(x)|_{\Lambda^o_{k-1\mathbf{R}}}$ are completely determined by the data $v(x)|_{\bigcup_{0 \leq q \leq k-1} \Lambda^{p,q}_k}$. The dimension of the solution space on the connected component $\Lambda^{p,q}_k$ is one for each $q=0, \cdots, k-1$ and hence we have (5.4), 2): $\dim(W/W_1) \leq k$.

By (5.4), 1) and 2), we have $\dim W = \dim(W/W_1) + \dim W_1 \leq m+1 = l$. Then we complete the proof of sublemma 5.4.2.1 in the cases of i) in (4.1)–(4.4).

Next we consider the cases ii) in (4.2)–(4.4). The number $l=2$ in these cases. The connected component decompositions of $\Lambda^o_{k-1\mathbf{R}}$ and $\Lambda^o_{k\mathbf{R}}$ were given in (4.12) as $i=k-1$ and k. The real holonomy diagrams of the $(n-1)$-dimensional intersections of $\Lambda_{k-1\mathbf{R}}$ and $\Lambda_{k\mathbf{R}}$ is given by Figure 4, (3) as proved in Proposition 4.3. The holonomic system $\mathfrak{M}_s$ is critical at $s=\lambda$ from Λ_{k-1C} to Λ_{kC}, hence $\mathfrak{M}_s$ is not critical at $s=\lambda$ from Λ_{kC} to Λ_{k-1C}. Therefore, $\mathscr{S}ol(\mathfrak{M}_\lambda)|_{\Lambda^o_{k-1\mathbf{R}}}$ is determined by the data $\mathscr{S}ol(\mathfrak{M}_\lambda)|_{\Lambda^o_{k\mathbf{R}}}$. Since $\Lambda^o_{k-1\mathbf{R}}$

has only two connected components, the dimension of $\mathcal{S}o\ell(\mathfrak{M}_\lambda)|_{A^\circ_{k-1R}}$ is two and so is the dimension of $\mathcal{S}o\ell(\mathfrak{M}_\lambda)|_{A^\circ_{k-1R}\cup A^\circ_{kR}}$.

Lastly, we consider the case (4.5). In Proposition 4.6, the real holonomy diagram of (4.5) was proved to have the same form as that in the first case (resp. second case) when $p=1$ (resp. $p>1$). Thus we can prove this sublemma for the third case in the same way as in the first case or the second case. Thus we complete the proof of Sublemma 5.4.2.1.

By Lemma 5.4.1 and Lemma 5.4.2, we obtain the result claimed in Proposition 5.4.

COROLLARY 5.5. *Let $\lambda\in \boldsymbol{C}$ be a critical point for $P(x)^s$. Then any χ^λ-invariant hyperfunction is written as a linear combination of $|P(x)|_i^s$ at $s=\lambda$ in the sense of* (1.3).

This corollary is proved in the same way if we use Proposition 5.4 instead of Proposition 5.2.

5.4. Conclusions and a remark.

THEOREM 5.6. *Let $(\boldsymbol{G}_{\boldsymbol{R}}^+, \rho, V)$ be a one of the real forms in* (4.1)–(4.5). *Let λ be an arbitrary comlex number. Then*:

1) *The dimension of the space of χ^λ-invariant hyperfunctions coincides with the number of the connected components of $V-\{x\in V;\ P(x)=0\}$.*

2) *Any χ^λ-invariant hyperfunction is a tempered distribution and is written as a linear combination of $|P(X)|_i^s$ defined in* (1.4) *at $s=\lambda$ in the sense of* (1.3).

The claim 1) is the direct consequence of Proposition 5.2 and 5.4. The claim 2) follows from Corollary 5.3 and 5.5.

As an application of Theorem 5.6, we have the following:

THEOREM 5.7. *Let $(\boldsymbol{G}_{\boldsymbol{R}}^+, \rho, V)$ be a real form in* (4.1)–(4.5). *We put $\boldsymbol{G}_{\boldsymbol{R}}^1:=\{g\in \boldsymbol{G}_{\boldsymbol{R}}^+;\ \chi(g)=1\}$. Then any $\boldsymbol{G}_{\boldsymbol{R}}^1$-invariant tempered distribution whose support is contained in the real singular set $\boldsymbol{S}_{\boldsymbol{R}}=\{x\in \boldsymbol{V}_{\boldsymbol{R}};\ P(x)=0\}$ is obtained as a linear combination of negative order Laurent coefficients of $|P(x)|_i^s$ $(i=1,\cdots,l)$ at poles.*

PROOF. [Mr2, Theorem 2.7] proved that the theorem is valid if the singular set $\boldsymbol{S}_{\boldsymbol{C}}=\{x\in \boldsymbol{V}_{\boldsymbol{C}};\ P(x)=0\}$ decomposes into a finite number of $\boldsymbol{G}_{\boldsymbol{C}}^1$-orbits. Here, $\boldsymbol{G}_{\boldsymbol{C}}^1:=\{g\in \boldsymbol{G}_{\boldsymbol{C}};\ \chi(g)=1\}$. It is easily checked by calculating the action of the Lie algebra $\mathcal{G}_{\boldsymbol{C}}$ on $\boldsymbol{V}_{\boldsymbol{C}}$ that any $\boldsymbol{G}_{\boldsymbol{C}}$-orbits in the singular set $\boldsymbol{S}_{\boldsymbol{C}}$ is actually a $\boldsymbol{G}_{\boldsymbol{C}}^1$-orbit. The finiteness of the $\boldsymbol{G}_{\boldsymbol{C}}$-orbit decomposition was proved in Proposition 2.1. q.e.d.

REFERENCES

[Da-Wr] B. DATSKOVSKY AND D. J. WRIGHT, The adelic zeta functions associated to the space of binary cubic forms, Part II: Local theory, J. Reine Angew. Math. 367 (1986), 27–75.

[Ka1] M. KASHIWARA, *B*-functions and holonomic systems, Invent. math. 38 (1976), 33–53.

[Ka2] M. KASHIWARA, Systems of microdifferential equations, Progress in Math. 34, Boston, Birkhäuser, 1983.

[Ka3] M. KASHIWARA, Microlocal calculus of simple microfunctions, in 'Analysis and Algebraic Geometry', Papers in honor of Professor K. Kodaira, W. L. Baily and T. Shioda eds., Iwanami Shoten, Tokyo, 1977, 369–374.

[Ka4] M. KASHIWARA, Micro-local Analysis, in Proc. Intern. Congress of Math., Helsinki, 1978, O. Lehto ed., Academia Scientiarum Fennica, Helsinki, Vol. 1, 1980, 139–150.

[Ka-Kw-Ki] M. KASHIWARA, T. KAWAI AND T. KIMURA, Foundations of Algebraic Analysis (in Japanese), Kinokuniya, Tokyo, 1980, English translation (Princeton Mathematical Series No. 37), Princeton U.P., Princeton, 1986.

[Ki] T. KIMURA, The b-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, Nagoya Math. J. 85 (1982), 1–80.

[Mr1] M. MURO, Microlocal analysis and calculations on some relatively invariant hyperfunctions related to zeta functions associated with the vector spaces of quadratic forms, Publ. Res. Inst. Math. Sci. 22 (1986), 395–463.

[Mr2] M. MURO, Singular invariant tempered distributions on regular prehomogeneous vector spaces, J. Funct. Anal. 76 (1988), 317–345.

[Mr3] M. MURO, On uniqueness of hyperfunction solutions of holonomic systems, Ark. Mat. 26 (1988), 305–314.

[Mr4] M. MURO, The dimension of the space of relatively invariant hyperfunctions on regular prehomogeneous vector spaces, Proc. Japan Acad. 63 (1987), 66–68.

[Mu] I. MULLER, Décomposition orbitale des espaces préhomogènes réguliers de type parabolique commutatif et application, C. R. Acad. Sc. 303 (1986), 495–498.

[Mu-Ru-Sc] I. MULLER, H. RUBENTHALER AND G. SCHIFFMANN, Structure des espaces préhomogènes associés à certaines algèbres de Lie graduées, Math. Ann. 274 (1986), 95–123.

[Os-Se] T. OSHIMA AND J. SEKIGUCHI, Eigenfunctions of invariant differential operators on an affine symmetric space, Invent. math 57 (1980), 1–81.

[Ra] M. RAIS, Distributions homogènes sur des espaces de matrices, Bull. Soc. Math. France, Memoire 30 (1970), 5–109.

[Ric-St] F. RICCI AND E. M. STEIN, Homogeneous distributions on spaces of Hermitian matrices, J. Reine. Angew. Math. 368 (1986), 142–164.

[Ru1] H. RUBENTHALER, Distributions bi-invariantes par $SL_n(k)$, in Analyse Harmonique sur les Grupes de Lie, Séminaire Nancy-Strasburg 1973–75, Springer Lecture Note in Math. 497 (1975), 383–493.

[Ru2] H. RUBENTHALER, Espaces préhomogènes de type parabolique, in Lectures on Harmonic Analysis on Lie groups and related topics, Lec. in Math., Kyoto Univ. No. 14 (1981), 189–221.

[Ru3] H. RUBENTHALER, Classification infinitésimale de forms réeles de certains espaces préhomogènes., C. R. Acad. Sc. 295 (1982), 55–57.

[Ru4] H. RUBENTHALER, Forms réeles infinitésimales préhomogènes de type parabolique, Ann. Inst. Fourier 36–1 (1986), 1–38.

[Sk-Fa] I. SATAKE AND J. FARAUT, The functional equations of zeta-distributions associated with formally real Jordan algebras, Tôhoku Math. J. 36 (1984), 469–482.

[Sm-Kw-Ka] M. SATO, T. KAWAI AND M. KASHIWARA, Microfunctions and pseudo-differential equations, in 'Hyperfunctions and pseudo-differential equations', Springer Lecture Note in Math. 287 (1973), 263–529.

[Sm-Ki] M. SATO AND T. KIMURA, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. 65 (1977), 1–155.

[Sm-Sh] M. SATO AND T. SHINTANI, On zeta functions associated with prehomogeneous vector spaces, Ann. of Math. 100 (1974), 131–170.

[Sm-Ka-Ki-Os] M. SATO, M. KASHIWARA, T. KIMURA AND T. OSHIMA, Micro-local analysis of prehomogeneous vector spaces, Invent. math. 62 (1980), 17–179.

[Sh] T. SHINTANI, On Dirichlet series whose coefficients are class-numbers of integral binary cubic forms, J. Math. Soc. Japan 24 (1972), 132–188.

[St] E. M. STEIN, Analysis in Matrix spaces and some new representations of $SL(n, C)$, Ann. of Math. 86 (1967), 461–490.

[Su] T. SUZUKI, Fourier transforms of relative invariants of prehomogeneous vector spaces, (in Japanese), Master Thesis, Nagoya University (1975).

[Vi] E. B. VINBERG, On the classification of the nilpotent elements of graded Lie algebras, Soviet Math. Dokl. 16 (1975), 1517–1520.

[We] A. WEIL, Fonction zêta et distributions, Séminaire Bourbaki, no. 312, juin (1966), also published in André Weil, Collected papers, Springer, Vol. 3, (1979), 158–163.

[Wr] D. J. WRIGHT, The adelic zeta function associated to the space of binary cubic forms, Part I: Global theory, Math. Ann. 270 (1985), 503–534.

FACULTY OF GENERAL EDUCATION
GIFU UNIVERSITY
YANAGITO 1-1, GIFU, 501-11
JAPAN