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DECOMPOSITION THEOREM FOR PROPER KÄHLER MORPHISMS

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Introduction. In [S1], [S2] we introduced the notion of *polarizable Hodge Modules* on complex analytic spaces, which corresponds philosophically to that of pure perverse sheaves in characteristic p [BBD]. If X is smooth, $\mathrm{MH}(X, n)^p$ the category of polarizable Hodge Modules of weight n (and with $\boldsymbol{k}$-structure) is a full subcategory of the category of filtered holonomic $\mathscr{D}_X$-Modules (M, F) with $\boldsymbol{k}$-structure by a given isomorphism $\alpha: \mathrm{DR}(M) \simeq \boldsymbol{C} \otimes_{\boldsymbol{k}} K$ for a perverse sheaf K (defined over $\boldsymbol{k}$). Here $\boldsymbol{k}$ is a subfield of $\boldsymbol{R}$, and we assume for simplicity $\boldsymbol{k} = \boldsymbol{R}$ in this note. In general $\mathrm{MH}(X, n)^p$ is defined using local embeddings into smooth varieties, and the underlying perverse sheaves K are globally well-defined. We can show that the category $\mathrm{MH}(X, n)^p$ is a *semi-simple abelian* category, and admits the *strict support decomposition*:

$$\mathrm{MH}(X, n)^p = \bigoplus_Z \mathrm{MH}_Z(X, n)^p \qquad \text{locally finite on} \quad X, \tag{0.1}$$

where Z are closed irreducible subspaces of X, and $\mathrm{MH}_Z(X, n)^p$ is the full subcategory of $\mathrm{MH}(X, n)^p$ with *strict support* Z, i.e. the underlying perverse sheaves of its objects are intersection complexes with local system coefficients, and supported on Z (or $\varnothing$). This decomposition is *unique*, because there is no nontrivial morphism between the Hodge Modules with different strict supports. The category $\mathrm{MH}_Z(X, n)^p$ depends only on Z and n (independent of X), and we have the equivalence of categories [S5]:

$$\mathrm{MH}_Z(X, n)^p \simeq \mathrm{VSH}(Z, n - \dim Z)^p_{\mathrm{gen}} \tag{0.2}$$

where the right hand side is the category of polarizable variations of $\boldsymbol{R}$-Hodge structures of weight $n - \dim Z$ defined on Zariski-open dense smooth subsets of Z, and the polarizations on Hodge Modules correspond bijectively to those of variations of Hodge structures. The main result of [S1], [S2] was the relative version of the Kähler package:

(0.3) THEOREM. *Let $f: X \to Y$ be a cohomologically projective morphism of complex analytic spaces, i.e. there is $l \in H^2(X, \boldsymbol{R}(1))$ which is locally on Y the pull-back of a multiple of the hyperplane section class by $X \subsetneq Y \times \boldsymbol{P}^m$. Then we have the natural functors*:

$$\mathscr{H}^j f_*: \mathrm{MH}(X, n)^p \to \mathrm{MH}(Y, n+j)^p \tag{0.3.1}$$

compatible with the corresponding functors ${}^p\mathscr{H}^j f_*$ on the underlying perverse sheaves [BBD], *and the relative hard Lefschetz*:

$$(0.3.2)\qquad l^j\colon \mathscr{H}^{-j}f_*\mathscr{M} \xrightarrow{\sim} \mathscr{H}^j f_*\mathscr{M}(j) \quad \text{for} \quad \mathscr{M}\in \mathrm{MH}(X,n)^p \quad \text{and} \quad j\geq 0\,,$$

with the induced polarization on the relative primitive part $P_l\mathscr{H}^{-j}f_*\mathscr{M}$ $(:=\mathrm{Ker}\, l^{j+1})$ *by* $(-1)^{j(j-1)/2}f_*\boldsymbol{S}\circ(\mathrm{id}\otimes l^j)$ *for* $j\geq 0$.

Then we have naturally:

(0.4) CONJECTURE. The Theorem (0.3) is valid with the assumption f projective replaced by f proper and X smooth Kähler.

In fact, it is not so difficult to show (0.3.1) under the assumption of (0.4), using a recent result of Kashiwara-Kawai [KK2], cf. the remark after 3.21 in [S5], and we can get the natural pure Hodge structure on the intersection cohomologies of a compact analytic space in the class C in the sense of Fujiki, associated to a polarizable variation of Hodge structures on a nonsingular Zariski-open subset [loc. cit.]. In this note we prove:

(0.5) THEOREM. *The conjecture* (0.4) *is true for the direct image of the constant sheaf* (*i.e.* $(M, F, K)=(\mathscr{O}_X, F, \boldsymbol{R}_X[d_X])$ *with* $\mathrm{Gr}_i^F\mathscr{O}_X=0$ *for* $i\neq 0$).

Combining with the decomposition (0.1) and Deligne's decomposition [D3], we get as a corollary (cf. [BBD] in the algebraic case):

(0.6) THEOREM. *Let* $f\colon X\to Y$ *be a proper morphism of irreducible analytic spaces. Assume that there is a proper surjective morphism* $\pi\colon \tilde{X}\to X$ *with* $\tilde{X}$ *smooth Kähler. Then we have the decomposition theorem for the direct image of the intersection complex, i.e.* $f_*\mathrm{IC}_X\boldsymbol{R}$ *is a direct sum of intersection complexes with local system coefficients and with some shift of complex.*

Here the assertion is valid also for $f_*\mathrm{IC}_XL$, if L is "geometric" in the following sense: L is a direct factor of the restriction of $R^j\pi_*\boldsymbol{R}_{\tilde{X}}$ to a smooth Zariski open subset for some π as above. In fact we have the decomposition by (0.1) and [D3] in the case of (0.5), and IC_XL is a direct factor of $\pi_*(\boldsymbol{R}_{\tilde{X}}[d_{\tilde{X}}])$ up to a shift of complex. Therefore the assertion is reduced to that for $(f\pi)_*(\boldsymbol{R}_{\tilde{X}}[d_{\tilde{X}}])$ by [D3] and (1.5) for the perverse sheaves, and we can apply (0.5) to $f\pi$. Note that the decomposition theorem can be divided into the two assertions:

$$(0.7)\qquad f_*\mathrm{IC}_XL\simeq \bigoplus_j ({}^p\mathscr{H}^jf_*\mathrm{IC}_XL)[-j] \qquad \text{(non-canonically)}\,,$$

$$(0.8)\qquad {}^p\mathscr{H}^jf_*\mathrm{IC}_XL=\bigoplus_Z \mathrm{IC}_ZL_Z^j \qquad \text{(canonically)}$$

with L_Z^j local systems on smooth Zariski open subsets of Z, and (0.7) follows from the hard Lefschetz by [D3], and (0.8) from (0.1).

The decomposition (0.7) implies the E_2-degeneration of the perverse *Leray spectral sequence*:

(0.9) $$E_2^{ij}=H^i(Y,{}^p\mathscr{H}^jf_*\mathrm{IC}_XL)\Rightarrow IH^{i+j+d_X}(X,L)\,(=H^{i+j}(X,\mathrm{IC}_XL))\,.$$

Applying it to $\pi_*(\boldsymbol{R}_{\tilde{X}}[d_{\tilde{X}}])$ for π as above, we see that the intersection cohomology $IH^{\cdot}(X)$ is a *canonical subquotient* of $H^{\cdot}(\tilde{X})$, more precisely, $IH^{\cdot}(X)$ is a *canonical direct factor* of $\mathrm{Gr}^G H^{\cdot}(\tilde{X})$ by the uniqueness of the decomposition (0.8), where G is the filtration associated to the Leray spectral sequence. Therefore in the compact case, we get a *canonical Hodge structure* on the intersection cohomologies $IH^{\cdot}(X)$, if the Leray spectral sequence and the decomposition of $\mathrm{Gr}^G H^{\cdot}(\tilde{X})$ are compatible with the Hodge structure of $H^{\cdot}(\tilde{X})$. Actually we can prove these using the theory of Hodge Modules. This argument can be generalized to the case of variation of Hodge structure, using (0.3) and [KK2], if π can be taken to be projective (this condition is satisfied in the case X in class C by a recent result of Varouchas). Moreover (0.3.1) can be generalized to:

(0.10) $$\mathscr{H}^jf_*:\ \mathrm{MH}_X(X,n)^p\to\mathrm{MH}(Y,n+j)^p$$

for f as in (0.6) with π projective. In the case of $Y=\mathrm{pt}$ and X in class C, the Hodge structure on $IH^{\cdot}(X)$ obtained by (1.10) coincides with the one by the Leray spectral sequence, etc. for any π.

In §1 we review the definition and some properties of polarizable Hodge Modules. In §2 we prove (0.10) using [KK2]. In §3 we prove (0.5) using essentially Hironaka's resolution.

1. **Polarizable Hodge Modules** (cf. [S1]$\sim$[S2]).

(1.1) Let X be a complex manifold of dimension d_X, and $\mathscr{D}_X$ the sheaf of holomorphic differential operators with the filtration F by the degree of operators. In this note we use the (filtered) *left* $\mathscr{D}_X$-Modules. For the correspondence to the right Modules in [S1]$\sim$[S5] we use the functor $\otimes_{\mathscr{O}_X}(\Omega_X^{d_X},F)$ with $\mathrm{Gr}_i^F\Omega_X^{d_X}=0$ for $i\neq -d_X$, so that the filtration is shifted by $-d_X$.

(1.2) Let $\mathrm{MF}_h(\mathscr{D}_X,\boldsymbol{R})$ be the category of filtered holonomic $\mathscr{D}_X$-Modules (M,F) with $\boldsymbol{R}$-structure given by $\alpha:\ \mathrm{DR}(M)\cong\boldsymbol{C}\otimes_{\boldsymbol{R}}K$ for a perverse sheaf K defined over $\boldsymbol{R}$, where the morphisms are the pairs of morphisms of (M,F) and K compatible with α. The functor $(M,F,K)\mapsto K$ is exact and faithful, because $M=0$ if $\mathrm{DR}(M)=0$.

(1.3) Let $i:\ X\to Y$ be a closed embedding locally defined by $X=\{x_1=\cdots=x_k=0\}$ with $(x_1,\cdots,x_m)$ local coordinates of Y. Then for a filtered holonomic $\mathscr{D}_X$-Module (M,F), the direct image $(\tilde{M},F)=i_*(M,F)$ is defined locally by:

$$F_p\tilde{M}=\bigoplus_{q+|\nu|\le p-k}F_qM\otimes\partial^\nu$$

where $\partial^\nu=\prod_{1\le i\le k}\partial_i^{\nu_i}$, $|\nu|=\sum\nu_i$, $\partial_i=\partial/\partial x_i$. Then we have $\mathrm{DR}\circ i_*=i_*\circ\mathrm{DR}$ and we get the functor

(1.3.1) $$i_*:\ \mathrm{MF}_h(\mathscr{D}_X,\boldsymbol{R})\to\mathrm{MF}_h(\mathscr{D}_Y,\boldsymbol{R})\,.$$

(1.4) Let g be a holomorphic function on X, and $i_g: X \to X \times \boldsymbol{C}$ the embedding by graph. Put $(\tilde{M}, F) = (i_g)_*(M, F)$ and consider the conditions:

(1.4.1) $\tilde{M}$ has the filtration V of Malgrange-Kashiwara [K2] indexed by $\boldsymbol{Q}$,

$$t: F_pV^\alpha\tilde{M} \xrightarrow{\sim} F_pV^{\alpha+1}\tilde{M} \quad \text{for} \quad \alpha > -1, \tag{1.4.2}$$

$$\partial_t: F_p\mathrm{Gr}_V^\alpha\tilde{M} \xrightarrow{\sim} F_{p+1}\mathrm{Gr}_V^{\alpha-1}\tilde{M} \quad \text{for} \quad \alpha < 0, \tag{1.4.3}$$

where V is indexed decreasingly so that $t\partial_t - \alpha$ on $\mathrm{Gr}_V^\alpha\tilde{M}$ is nilpotent. A filtered holonomic $\mathscr{D}_X$-Module (M, F) is said to be *regular and quasi-unipotent along g*, if the conditions (1.4.1–3) are satisfied. Sometimes it is more convenient to replace the condition (1.4.1) by

(1.4.4) $\tilde{M}$ has the filtration V indexed by $\boldsymbol{R}$,

because it is always satisfied in the case of polarizable variation of Hodge structure defined over $\boldsymbol{R}$. Here the filtration is assumed to be indexed discretely. If (M, F) satisfies (1.4.2–4), we define

$$\psi_g(M, F) = \bigoplus_{-1<\alpha\leq 0} \mathrm{Gr}_V^\alpha(\tilde{M}, F), \qquad \phi_{g,1}(M, F) = Gr_V^{-1}(\tilde{M}, F[-1]). \tag{1.4.5}$$

Then $\psi_g\mathrm{DR}[-1] = \mathrm{DR}\psi_g$ (same for $\phi_{g,1}$) and $-\partial_t$, t correspond to can, Var (cf. [S2, 3.4.12]). If (M, F) has a real structure K, we put

$$\psi_g(M, F, K) = (\psi_g(M, F), \psi_gK[-1]) \quad \text{(same for } \phi_{g,1}) \tag{1.4.6}$$

and we get the morphisms

(1.4.7) $\mathrm{can}: \psi_{g,1}(M, F, K) \to \phi_{g,1}(M, F, K), \quad \mathrm{Var}: \phi_{g,1}(M, F, K) \to \psi_{g,1}(M, F, K)(-1)$

induced by $-\partial_t$, t. Here $\psi_{g,1}$ is the unipotent monodromy part of ψ_g (same for $\phi_{g,1}$), cf. [D4] for the definition of ψ_g, ϕ_g. We have

$$\psi_g(M, F) = 0, \quad \phi_{g,1}(M, F) = (M, F), \quad \text{if} \quad \mathrm{supp}\, M \subset g^{-1}(0), \tag{1.4.8}$$

because (1.4.2–4) are equivalent to $g(F_pM) \subset F_{p-1}M$ in this case, cf. [S2, 3.2.6].

(1.5) PROPOSITION (cf. [S2, 5.1.4]). *If* (M, F) *satisfies the conditions* (1.4.2–4) *for any g locally defined on X, the following conditions are equivalent*:

(1.5.1) $\phi_{g,1}(M, F) = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{Var}$ *for any locally defined g*,

(1.5.2) *for any open set U of X*, $(M, F)|_U$ *has the canonical decomposition* $\bigoplus_Z(M_Z, F)$ *for Z closed irreducible subspaces of U, such that* M_Z *has strict support Z, i.e.* $\mathrm{supp}\, M_Z = Z$ *(or* $\emptyset$*) and* M_Z *has no nontrivial sub or quotient supported in a proper subspace of Z.*

Moreover M has strict support Z, if and only if $\mathrm{supp}\, M = Z$ *and* can *is surjective*, Var *is injective for any locally defined g such that* $\dim g^{-1}(0) \cap Z < \dim Z$.

(1.6) The proposition (1.5) holds with (M, F) replaced by K or (M, F, K), and the decomposition in (1.5.2) is called the *strict support decomposition*. In the case of perverse sheaves, no assumption is necessary (i.e. K may be non quasi-unipotent), and (1.5.1–2) are equivalent to

(1.6.1) K is a direct sum of intersection complexes with local system coefficients.

(1.7) Let $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_{\mathrm{dec}}$ be the full subcategory of $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})$ satisfying (1.4.2–4) and (1.5.1–2). Let $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_Z$ be the full subcategory of $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_{\mathrm{dec}}$ with strict support Z, i.e. the underlying perverse sheaves are intersection complexes with support Z, cf. (1.6). Then we have the canonical decomposition (locally finite on X):

(1.7.1) $$\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_{\mathrm{dec}} = \bigoplus_Z \mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_Z ,$$

where Z are closed irreducible subspaces of X.

Let (M, F, K) be an object of $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_Z$, and g a holomorphic function on X such that $g^{-1}(0) \not\supset Z$ and $\mathrm{can}: \psi_{g,1}(M, F) \to \phi_{g,1}(M, F)$ is *strictly* surjective. Then we have

(1.7.2) $$F_p\tilde{M} = \sum_i \partial_t^i(V^{>-1}\tilde{M} \cap j_*j^{-1}F_{p-i}\tilde{M})$$

with $j: X \times \boldsymbol{C}^* \to X \times \boldsymbol{C}$ and $(\tilde{M}, F) = (i_g)_*(M, F)$ as above. In this case the filtration F on M is uniquely determined by its restriction to the complement of $g^{-1}(0)$.

(1.8) Definition. The category $\mathrm{MH}(X, n)$ of *Hodge Modules of weight n* is the largest full subcategory of $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_{\mathrm{dec}}$ such that the objects (M, F, K) satisfy the following conditions:

(1.8.1) If $\mathrm{supp}\, M = \{x\}$, there is an $\boldsymbol{R}$-Hodge structure $(H_{\boldsymbol{C}}, F, H_{\boldsymbol{R}})$ of weight n (cf. [D1]) such that $(M, F, K) = (i_x)_*(H_{\boldsymbol{C}}, F, H_{\boldsymbol{R}})$, cf. (1.3.1), where $i_x: \{x\} \to X$ and $F_p = F^{-p}$.

(1.8.2) For any open subset U of X, any closed irreducible subspace Z of U, any holomorphic function g on U such that $g^{-1}(0) \not\supset Z$, we have

$$\mathrm{Gr}_i^W \psi_g(M_Z, F, K_Z), \quad \mathrm{Gr}_i^W \phi_{g,1}(M_Z, F, K_Z) \in \mathrm{MH}(U, i)$$

where (M_Z, F, K_Z) is the direct factor of $(M, F, K)|_U$ with strict support Z, cf. (1.5.2), and W is the monodromy filtration shifted by $n-1$ and n (i.e. the center is $n-1$ and n).

The condition (1.8.2) is well-defined by induction on $\dim \mathrm{supp}\, M$. Put

(1.8.3) $$\mathrm{MH}_Z(X, n) = \mathrm{MH}(X, n) \cap \mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})_Z$$

so that $\mathrm{MH}(X, n) = \bigoplus_Z \mathrm{MH}_Z(X, n)$ with Z closed irreducible subspaces of X.

(1.9) LEMMA [S2, 5.1.9–10]. *The category* $\mathrm{MH}_Z(X, n)$ *depends only on* Z *and* n, *i.e. independent of* X *via* (1.3.1), *and* $(M, F, K) \in \mathrm{MH}_Z(X, n)$ *is generically a variation of Hodge structure, i.e. if* $Z=X$ *and* $K[-d_X]$ *is a local system* L, $(M=\mathcal{O}_X \otimes L, F, L)$ *is a variation of* $\boldsymbol{R}$*-Hodge structure of weight* $n-d_X$.

(1.10) PROPOSITION [S2, 5.1.14]. *The categories* $\mathrm{MH}(X, n)$ *and* $\mathrm{MH}_Z(X, n)$ *are abelian categories such that any morphisms are strictly compatible with the Hodge filtration* F.

(1.11) REMARK. In the definition (1.8), K is not supposed quasi-unipotent, because (1.4.1) is replaced by (1.4.4). But the same argument works.

For a Hodge Module (M, F, K) with strict support Z, M is *regular* holonomic. In fact $\mathcal{H}^j f^! M$ are regular for any $f: S \to X$ with $\dim S = 1$. Therefore $\mathcal{H}^0 \pi^! M$ is regular for a resolution $\pi: \tilde{Z} \to Z$, and M is a subquotient of $\mathcal{H}^0 \pi_* \tilde{M}$ with $\tilde{M}$ a (minimal) subquotient of $\mathcal{H}^0 \pi^! M$ by the adjunction of π. We can also show that (M, F) is *Cohen-Macaulay*, i.e. $\mathrm{Gr}^F M$ is a Cohen-Macaulay $\mathrm{Gr}^F \mathcal{D}_X$-Module, so that the dual $\boldsymbol{D}(M, F, K) = (\boldsymbol{D}(M, F), \boldsymbol{D}K)$ is well-defined, cf. [S2, 5.1.13].

(1.12) DEFINITION. A *polarization* of a Hodge Module (M, F, K) of weight n is a pairing $\boldsymbol{S}: K \otimes K \to a_X^! \boldsymbol{R}(-n)$ with $a_X: X \to \mathrm{pt}$, and satisfies the following conditions by induction on dim supp M:

(1.12.1) $\boldsymbol{S}$ is compatible with the Hodge filtration F, i.e. the corresponding morphism $K \to (\boldsymbol{D}K)(-n)$ can be extended to

$$(M, F, K) \to \boldsymbol{D}(M, F, K)(-n).$$

(1.12.2) If $\operatorname{supp} M = \{x\}$, $\boldsymbol{S}$ is a polarization of $(H_{\boldsymbol{C}}, F, H_{\boldsymbol{R}})$ in the sense of [D1] for $(H_{\boldsymbol{C}}, F, H_{\boldsymbol{R}})$ as in (1.8.1).

(1.12.3) For U, Z, g as in (1.8.2), the restriction of

$${}^p\psi_g \boldsymbol{S} \circ (\mathrm{id} \otimes N^i): \mathrm{Gr}^W_{n-1+i} {}^p\psi_g K_Z \otimes \mathrm{Gr}^W_{n-1+i} {}^p\psi_g K_Z \to a_U^! \boldsymbol{R}(1-n-i)$$

to the primitive part ($=\operatorname{Ker} N^{i+1}$) is a polarization of

$$P\mathrm{Gr}^W_{n-1+i} \psi_g(M_Z, F, K_Z) \qquad \text{for} \quad i \geq 0\,.$$

(See [S2], [S7] for the definition of $\boldsymbol{D}(M, F, K)$, ${}^p\psi$, etc.) Here the *Tate twist* $(\tilde{M}, F, \tilde{K}) = (M, F, K)(m)$ is defined by $(\tilde{M}, F) = (M, F[m])$, $\tilde{K} = K \otimes_{\boldsymbol{R}} (2\pi i)^m \boldsymbol{R}$. We say that a Hodge Module is *polarizable*, if it has a polarization. By definition, the polarizations (and the polarizability) are compatible with the strict support decomposition. We denote by $\mathrm{MH}(X, n)^p$, $\mathrm{MH}_Z(X, n)^p$ the full subcategories of the polarizable Hodge Modules (with strict support Z). Note that (1.12.3) implies

(1.12.4) ${}^p\phi_{g,1} \boldsymbol{S} \circ (\mathrm{id} \otimes N^i)$ is a polarization of $P\mathrm{Gr}^W_{n+i} \phi_{g,1}(M_Z, F, K_Z)$, cf. [S2, 5.2].

(1.13) LEMMA [S2, 5.2.11–12]. *A polarization of a Hodge Module is non-degenerate* (*i.e. induces* $K \xrightarrow{\sim} \boldsymbol{D}K(-n)$), *is independent of* X, *i.e.*

$$(1.13.1) \qquad i_*: \mathrm{MH}_Z(X, n)^p \xrightarrow{\sim} \mathrm{MH}_Z(Y, n)^p \qquad \textit{for a closed immersion} \quad i: X \to Y,$$

and gives a polarization of the generic variation of Hodge structure in (1.9).

(1.14) PROPOSITION. *The full subcategories* $\mathrm{MH}(X, n)^p$, $\mathrm{MH}_Z(X, n)^p$ *are abelian* (*i.e. stable by* Ker, Coker) *and semi-simple.*

In fact this follows from (1.7.2) (with (1.10)) and (1.13). We have also

(1.15) LEMMA. *The categories* $\mathrm{MH}(X, n)$, $\mathrm{MH}_Z(X, n)$, $\mathrm{MH}(X, n)^p$, *etc. are stable by direct factors in* $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})$.

(1.16) For an analytic space X, the categories $\mathrm{MH}(X, n)^p$ and $\mathrm{MH}_Z(X, n)^p$ are defined using local closed embeddings into complex manifolds, cf. [S2, 5.3.12]. This is well-defined by (1.13.1) where i_* depends only on the restriction of i to Z, by (1.4.8). One of the main results of [S5] is the equivalence of categories (0.2), i.e. the converse of (1.9), (1.13) holds. The functor given in (1.9) is fully faithful by (1.7.2), (1.11), and the essential surjectivity was shown using [S7, § 3] and Kashiwara's lemma on nilpotent orbit, cf. [S5, 3.21].

(1.17) For the proof of (0.10) we have to treat the mixed case a little bit, because the vanishing cycles of Hodge Modules are *mixed*. We denote by $\mathrm{MHW}(X)$ (resp. $\mathrm{MHW}(X)^p$) the category whose objects are obtained by extensions of (polarizable) Hodge Modules. If X is smooth, it is the category of $(M, F, K; W)$ such that $\mathrm{Gr}_n^W(M, F, K) \in \mathrm{MH}(X, n)$ (resp. $\mathrm{MH}(X, n)^p$) where $(M, F, K) \in \mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})$ with W a pair of filtrations of M, K compatible via α. We also assume that $gF_pM \subset M_{p-1}M$ if $g^{-1}(0) \supset \mathrm{supp}\, M$ so that $\mathrm{MHW}(X)^p$ is well-defined also for X singular, using local closed embeddings as in (1.16). Here (K, W) are globally well-defined.

Let $N: (M, F, K; W) \to (M, F, K; W)(-1)$ $(:= (M, F[-1], K(-1); W[2])$ be a morphism of $\mathrm{MHW}(X)$ and $\boldsymbol{S}: K \otimes K \to a_X^! \boldsymbol{R}(-n)$ a morphism of $D_c^b(\boldsymbol{R}_X)$. We say that $(M, F, K; W)$ is *strongly polarized by* $(\boldsymbol{S}, N)$ *with weight* n, if the following conditions are satisfied:

$$(1.17.1) \qquad N^i: \mathrm{Gr}_{n+i}^W(M, F, K) \xrightarrow{\sim} \mathrm{Gr}_{n-i}^W(M, F, K)(-i) \qquad \text{for} \quad i > 0,$$

$$(1.17.2) \qquad \boldsymbol{S} \circ (\mathrm{id} \otimes N) + \boldsymbol{S} \circ (N \otimes \mathrm{id}) = 0,$$

(1.17.3) the restriction of $\boldsymbol{S} \circ (\mathrm{id} \otimes N^i): \mathrm{Gr}_{n+i}^W K \otimes \mathrm{Gr}_{n+i}^W K \to a_X^! \boldsymbol{R}(-n-\mathrm{i})$ to the N-primitive part $(:= \mathrm{Ker}\, N^{i+1})$ is a polarization of $P_N \mathrm{Gr}_{n+i}^W(M, F, K) \in \mathrm{MH}(X, n+i)$ for $i \geq 0$.

In this case $\mathrm{Gr}_k^W(M, F, K)$ are polarizable, because (1.17.1) implies the *primitive decomposition*:

$$(1.17.4)\qquad \mathrm{Gr}_k^W(M,F,K)=\bigoplus_{i\geq\max(0,n-k)} N^iP_N\mathrm{Gr}_{k+2i}^W(M,F,K)(i)\,.$$

These are generalized to the singular case as above.

One of the key points in the proof of (0.3) (cf. [S2], [S5]) is:

(1.8) PROPSOITION. *Let $f: X\to Y$ be a proper morphism of complex analytic spaces, and $(\mathcal{M},W)\in\mathrm{MHW}(X)$. If the Hodge filtration F of $f_*\mathrm{Gr}_i^W\mathcal{M}$ are strict and $\mathcal{H}^jf_*\mathrm{Gr}_i^W\mathcal{M}\in\mathrm{MH}(Y,i+j)$, then F of $f_*\mathcal{M}$ is strict and $(\mathcal{H}^jf_*\mathcal{M}, W[j])\in\mathrm{MHW}(Y)$ with W the filtration induced by $\mathcal{H}^jf_*$, i.e. W is associated to the weight spectral sequence*:

$$(1.18.1)\qquad E_1^{-i,i+j}=\mathcal{H}^jf_*\mathrm{Gr}_i^W\mathcal{M}\Rightarrow\mathcal{H}^jf_*\mathcal{M}\qquad in\quad \mathrm{MHW}(Y)\,,$$

which degenerates at E_2. In particular $(\mathcal{H}^jf_*\mathcal{M}, W[j])$ are polarized if so are $\mathcal{H}^jf_*\mathrm{Gr}_i^W\mathcal{M}$.

(1.19) PROPOSITION. *Let f be as above, $l\in H^2(X,\boldsymbol{R}(1))$, and $(\mathcal{M},W)\in\mathrm{MHW}(X)$ strongly polarized by (S,N) with weight n. Assume F of $f_*P_N\mathrm{Gr}_{n+i}^W\mathcal{M}$ is strict and $\mathcal{H}^jf_*P_N\mathrm{Gr}_i^W\mathcal{M}$ satisfies* (0.3.1–2) (*with the induced polarization on the l-primitive parts*). *Then the hard Lefschetz* (0.3.2) *holds for* $(\mathcal{H}^jf_*\mathcal{M}, W[j])\in\mathrm{MHW}(Y)$ (*cf.* (1.18)), *the weight filtration $W[j]$ of $\mathcal{H}^jf_*M$ is the monodromy filtration shifted by $n+j$, and the l-primitive part $P_l(\mathcal{H}^{-j}f_*\mathcal{M}, W[-j])$ is strongly polarized by* $((-1)^{j(j-1)/2}f_*\boldsymbol{S}\circ(\mathrm{id}\otimes l^j), N)$ $(j\geq 0)$.

(1.20) In [S3]~[S5], the notion of *mixed Hodge Module* is defined for complex analytic spaces. The category $\mathrm{MHM}(X)$ of mixed Hodge Modules is the largest full subcategory of $\mathrm{MHW}(X)$ stable by the (exact) functors: ψ_g, $\phi_{g,1}$, $j_!j^{-1}$, j_*j^{-1}, $\mathcal{H}^dp^*$, where g is a locally defined holomorphic function, j is an open immersion whose complement is a locally principal divisor, and p is a smooth morphism with d the relative dimension. Put

$$\mathrm{MHM}(X)^p=\mathrm{MHM}(X)\cap\mathrm{MHW}(X)^p\,.$$

Then it is stable by the above exact functors and also by $\mathcal{H}^jf_*$ for f projective and $\mathcal{H}^ji^*$, $\mathcal{H}^ji^!$ for i a closed embedding [S5]. We have

$$(1.20.1)\qquad \mathrm{MH}(X,n)^p=\{(\mathcal{M},W)\in\mathrm{MHM}(X)^p\colon \mathrm{Gr}_i^W\mathcal{M}=0 \text{ for } i\neq n\}$$

using [S5, 3.27] and the intermediate direct image $j_{!*}$.

The following proposition will be used in the proof of the global polarizability of $\mathcal{H}^jf_*\mathcal{M}$ in (0.10).

(1.21) PROPOSITION. *Let $f: X\to Y$ be a proper surjective morphism of complex manifolds, $\mathcal{M}\in\mathrm{MH}_X(X,n)^p$, and $g_1,\cdots,g_k$ holomorphic functions on Y. Put $h_i=f^*g_i$, $Y_0=\bigcap g_i^{-1}(0)$ and $X_0=f^{-1}(Y_0)$ with $i\colon X_0\to X$, $i\colon Y_0\to Y$ the natural inclusions. Assume that X_0 is a locally principal divisor on X, and for $\mathcal{M}'$ any iterations of $P_N\mathrm{Gr}^W\psi_{h_i}$ or $P_N\mathrm{Gr}^W\phi_{h_i,1},\cdots,P_N\mathrm{Gr}^W\psi_{h_1}$ or $P_N\mathrm{Gr}^W\phi_{h_1,1}$ of $\mathcal{M}(0\leq i\leq k)$, the assumption of* (1.18) *is* *satisfied, and the weight filtration of* $\mathscr{H}^j f_* \psi_{h_{i+1}} \mathscr{M}'$, $\mathscr{H}^j f_* \phi_{h_{i+1},1} \mathscr{M}'$ *is the monodromy filtration shifted by* $j+n'-1$, $j+n'$ *respectively, where* n' *is the weight of* $\mathscr{M}'$. *Then the iterations of* ψ_{g_i} *or* $\phi_{g_i,1}, \cdots, \psi_{g_1}$ *or* $\phi_{g_1,1}$ *on* $\mathscr{H}^j f_* \mathscr{M}$ *are inductively well-defined and we have the spectral sequence:*

$$E_2^{pq} = \mathscr{H}^p i^* \mathscr{H}^q f_* \mathscr{M} \Rightarrow H^{p+q+1} f_* \mathscr{H}^{-1} i^* \mathscr{M} \quad \text{in } \mathrm{MHW}(Y) \tag{1.21.1}$$

compatible with the natural spectral sequence on the underlying perverse sheaves. Moreover $i^*\mathscr{M}$ *is defined in* $D^b\mathrm{MHM}(X)^p$ *so that* $\mathscr{H}^j f_*(i^*\mathscr{M}) = \mathscr{H}^{j+1}\mathscr{H}^{-1} i^* \mathscr{M}$.

PROOF. By (1.20.1) the iterations of vanishing cycle functors $A_j \cdots A_1 \mathscr{M}$ with $A_j = \psi_{h_j}$ or $\phi_{h_j,1}$ $(1 \le j \le k)$ are inductively well-defined. We check inductively that the weight filtrations $W^{(j)}$ associated to A_j (i.e. the monodromy filtration relative to $W^{(j-1)}$ up to shift) induce compatible filtrations on $A_k \cdots A_1 \mathscr{M}$. By the canonical splitting of Kashiwara, $\mathrm{Gr}^{W^{(j)}}(A_j \cdots A_1 \mathscr{M})$ is the direct sum of $(\mathrm{Gr}^{W^{(j)}} A_j) \cdots (\mathrm{Gr}^{W^{(1)}} A_1) \mathscr{M}$, where A_j are exact and commute with Gr^W. Therefore the assumption of [S5, 2.16] is satisfied, and we can apply it inductively so that the (iterations of) vanishing cycle functors commute with $\mathscr{H}^j f_*$ (e.g. the direct image of V is strict, the (decalage of) direct image of the relative monodromy filtration induces the relative monodromy filtration, i.e. the weight filtration, cf. [loc. cit.], etc.)

Put $i_j : h_j^{-1}(0) \to X$ (or $g_j^{-1}(0) \to Y$) so that $i_* i^* = (i_k)_* i_k^* \cdots (i_1)_* i_1^*$ and

$$(i_j)_* i_j^* = C(\mathrm{can} : \psi_{h_j,1} \to \phi_{h_j,1}) \quad \text{(same for } g_j\text{), cf. [S5, 2.24]}. \tag{1.21.2}$$

This implies the last assertion using (1.5), because it is equivalent to $\mathscr{H}^j i^* \mathscr{M} = 0$ $(i \neq -1)$ and $\mathscr{H}^j i^*$ is independent of the equations [S5, 2.20]. The spectral sequence is then induced by the pair of canonical filtrations τ of $f_* A_k \cdots A_1 M$ and ${}^p\tau$ of $f_* K$, where (M, F) and K are the underlying filtered $\mathscr{D}$-Module (cf. [S2, 2.1.20]) and perverse sheaf of $\mathscr{M}$, and the filtrations F, $\mathrm{Dec}\, W^{(k)}$ on $f_* A_k \cdots A_1 M$ are bistrict by [S5, 2.15]. Here $i_* i^* \mathscr{M}$ is represented as above, and $\mathscr{H}^j f_*(i^*\mathscr{M})$ can be defined using the shifted weight filtration on $i_* i^* \mathscr{M}$.

(1.22) PROPOSITION. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be proper morphisms of complex analytic spaces, and* $(\mathscr{M}, W) \in \mathrm{MHM}(X)^p$. *Assume the hypothesis of* (1.18) *is satisfied for the direct image of* $\mathrm{Gr}_i^W \mathscr{M}$ *by* f, $h := gf$, *and that of* $\mathscr{H}^j f_* \mathrm{Gr}_i^W \mathscr{M}$ by g, $\mathscr{H}^j f_* \mathrm{Gr}_i^W \mathscr{M} \in \mathrm{MH}(Y, j+i)^p$, *and* $f_* \mathrm{Gr}_i^W (M, F) \simeq \bigoplus \mathscr{H}^j f_* \mathrm{Gr}_i^W (M, F)[-j]$ *in* $DF(\mathscr{D}_Y)$, *where* $(M; F, W)$ *denotes the underlying filtered* $\mathscr{D}_X$*-Module of* $\mathscr{M}$, *cf.* [S2, 2.1.20] [S5, 2.13]. *Then we have the Leray spectral sequence in* $\mathrm{MHM}(Z)$:

$$E_2^{ij} = \mathscr{H}^i g_* \mathscr{H}^j f_* \mathscr{M} \Rightarrow \mathscr{H}^{i+j} h_* \mathscr{M} \tag{1.22.1}$$

compatible with the (perverse) Leray spectral sequence of the underlying perverse sheaves.

PROOF. Let $(M_Y; F, W)$, $(M_Z; F, W)$ denote the underlying filtered complexes of $\mathscr{D}$-Modules of $f_* \mathscr{M}$, $g_* f_* \mathscr{M}$, and L, L^* the filtration on M_Y defined by the canonical and cocanonical filtration, i.e. $L_iM_Y^j=M_Y^j$ $(j<i)$, $\mathrm{Ker}\, d$ $(j=i)$ and 0 $(j>i)$ and $L_i^*M_Y^j=M_Y^j$ $(j\le i)$, $\mathrm{Im}\, d$ $(j=i+1)$ and 0 $(j>i+1)$. Put $\tilde{L}_{2i}=L_i$, $\tilde{L}_{2i+1}=L_i^*$. We denote by the same symbol the filtration on M_Z induced by L, L^*, $\tilde{L}$. By E_2-degeneration of the weight spectral sequence (1.18.1) the filtration $\tilde{L}$ on $\mathrm{Gr}_i^W(M_Y, F)$ corresponds to that on $\mathscr{H}^j f_*\mathrm{Gr}_i^W\mathscr{M}$ defined by $\mathrm{Ker}\, d_1$, $\mathrm{Im}\, d_1$, and splits by hypothesis and the semisimplicity of $\mathrm{MH}(Y, i)^p$. Therefore the hypothesis of [S2, 1.3.8] is satisfied and Dec of W on (M_Z, F) commutes with $\mathrm{Gr}_i^{\tilde{L}}$. We check that W on $\mathrm{Gr}_{2i}^{\tilde{L}}M_Y=\mathscr{H}^i f_*M[-i]$ is the weight filtration up to shift by $-i$ so that Dec W on $\mathrm{Gr}_{2i}^{\tilde{L}}M_Z=g_*\mathscr{H}^i f_*M[-i]$ gives the weight filtration on $\mathscr{H}^j g_*\mathscr{H}^i f_*\mathscr{M}$ by [S5, 2.15], and $\mathrm{Gr}_{2i+1}^{\tilde{L}}(M_Z; F, \mathrm{Dec}\, W)$ is acyclic by the spectral sequence by W, because its E_0-complex $\mathrm{Gr}_{2i+1}^{\tilde{L}}\mathrm{Gr}_k^W(M_Z, F)$ is isomorphic to the direct sum of $g_*\mathrm{Coim}\, d_1[-i]$ and $g_*\mathrm{Im}\, d_1[-i-1]$ by the above decomposition and its E_1-complex is filtered acyclic, where d_1 is the differential of the above weight spectral sequence. Then we get (1.22.1) by L or L^*, using the same argument as in [S5, 2.16]. In fact (1.22.1) is well-defined for the underlying $\mathscr{D}$-Module with filtration F, Dec W and for the underlying perverse sheaf, and we check inductively that E_r-terms are mixed Hodge Modules and d_r are morphisms of mixed Hodge Modules, and finally the converging filtration is a filtration of mixed Hodge Modules.

(1.23) COROLLARY. *With the above notation and assumption, let K denote the underlying perverse sheaf of $\mathscr{M}$, and assume K has an endomorphism $N\colon K\to K(-1)$. If we have a decomposition $f_*K=\bigoplus({}^p\mathscr{H}^j f_*K)[-j]$ compatible with the action of N, and the weight filtration of ${}^p\mathscr{H}^j h_*K$ is the monodromy filtration by N shifted by $j+w$, then the weight filtration of ${}^p\mathscr{H}^i g_*{}^p\mathscr{H}^j f_*K$ is the monodromy filtration shifted by $i+j+w$.*

PROOF. This is clear by (1.22), because the spectral sequence (1.22.1) degenerates at E_2 and the converging filtration L has a splitting compatible with N and hence with W.

REMARK. If $\mathscr{M}=\psi_k\mathscr{M}'$ or $\phi_{k,1}\mathscr{M}'$ for $\mathscr{M}'\in\mathrm{MH}(X, n)^p$ with $n=w+1$ or w and for $k=k'h$ with k' a holomorphic function on Z, the assumption on the decomposition of f_*K follows from that of f_*K' in the case f projective by the commutativity of the vanishing cycles with the direct images by proper images, where K' is the underlying perverse sheaf of $\mathscr{M}'$.

2. Stability by proper Kähler morphisms.

(2.1) Let $f\colon X\to Y$ be a proper morphism of complex analytic spaces such that X is smooth. We say that f is *cohomologically Kähler* with Kähler class $l\in H^2(X, \boldsymbol{R}(1))$, if l is represented by a Kähler form locally on Y. Let f be cohomologically Kähler with $l\in H^2(X, \boldsymbol{R}(1))$, and $\pi\colon \tilde{X}\to X$ a projective morphism of complex manifolds with l' the first Chern class of a π-ample line bundle L. Then the restriction of $f\pi$ to any relatively compact open subset U of Y is also cohomologically Kähler with Kähler class π^*l+cl' for $0<c\ll 1$, where the range of c depends on U and perhaps it does not exist globally. For the proof, we use the representative of l' as $\partial\bar{\partial}\log u$ with u a metric of

L and a partition of unity on X associated to local embeddings $\tilde{X}_i \to \boldsymbol{P}^{n_i-1} \times X_i$ induced by $\mathcal{O}_{\tilde{X}_i}^{n_i} \twoheadrightarrow \pi_* L^m|_{X_i}$ $(i>0)$, where $\{X_i\}_{i\geq 0}$ is an open covering of X such that $X_0 = X \setminus U$, $\tilde{X}_i = \pi^{-1} X_i$ and m is highly divisible and independent of $i>0$. Note that at each $x \in X$, we have the minimal embedding $\tilde{X} \to \boldsymbol{P}^{n-1} \times X$ with $n = \dim \pi_* L^m \otimes_{\mathcal{O}} (\mathcal{O}_X/\mathfrak{m}_x)$ on a neighborhood of x which factorize the other embeddings. This is an analogue of the fact that the compsition of projective morphisms is projective, if we restrict it to a relatively compact open subset of the image.

In this section we prove the following theorems by induction on dim Y:

(2.2) THEOREM. *Let $f: X \to Y$ be a proper morphism of complex analytic spaces (assumed always reduced and separated). We assume that there is a proper surjective morphism $\pi: \tilde{X} \to X$ such that $\tilde{X}$ is smooth, π is a composition of projective morphisms and the restriction of $\tilde{f} := f\pi$ to any relatively compact open subset U of Y is cohomologically Kähler with Kähler class l in* $\mathrm{Im}(H^2(\tilde{X}, \boldsymbol{R}(1)) \to H^2(\tilde{f}^{-1}(U), \boldsymbol{R}(1)))$. *Then for $\mathcal{M} \in$* $\mathrm{MH}(X, n)^p$ *the Hodge filtration F on $f_* \mathcal{M}$ is strict, and we have the natural functor*

$$\text{(2.2.1)} \qquad \mathcal{H}^j f_*: \mathrm{MH}(X, n)^p \to \mathrm{MH}(Y, n+j)^p$$

compatible with the corresponding functor ${}^p\mathcal{H}^j f_*$ on the underlying perverse sheaves.

REMARK. By Hironaka, the assumption is satisfied if X is Kähler (X may be singular). By (2.1) and Hironaka, the assumption is stable by restriction of X to closed subspaces; in particular, we may assume X, Y irreducible and restrict to $\mathrm{MH}_X(X)^p$. By (0.3) and Deligne's uniqueness of decomposition, we may also assume that X is smooth and f satisfies the assumption on $\tilde{f}$, i.e. replace X by $\tilde{X}$, cf. (2.4–5).

(2.3) THEOREM. *Let $f: X \to Y$ be a proper surjective morphism of irreducible analytic spaces, D a divisor on X, and $g_1, \cdots, g_k$ holomorphic functions on Y such that $\bigcap g_i^{-1}(0) = \{y\}$ and $f^{-1} g_i^{-1}(0) \subset D$. Assume X is smooth Kähler with Kähler class $l \in H^2(X, \boldsymbol{R}(1))$ and D is a normal crossing divisor with smooth irreducible components. Then for $(M, F, K) \in \mathrm{MH}_X(X, n)^p$ such that $K[-d_X]$ is a local system on $X \setminus D$, we have the following on a neighborhood of y:*

(2.3.1) *$f_*(M, F)$ is strict and $\mathcal{H}^j f_*(M, F, K) \in \mathrm{MH}(Y, n+j)^p$*

(2.3.2) *the hard Lefschetz* (0.3.2) *with the induced polarization on the primitive part holds for $\mathcal{H}^j f_* \mathcal{M}$.*

REMARK. If $Y = \mathrm{pt}$, the assertions (2.3.1–2) were proved in [KK1], [KK2]. In fact the Poincaré lemma for the L^2-complex $\mathcal{L}_2$ was shown in [KK1], [CKS], and the filtered L^2-complex $(\mathcal{L}_2, F)$ underlies a cohomological Hodge complex inducing the Hodge structure on the intersection cohomologies [KK1]. Then (2.3.1) follows from the isomorphism $f_*(M, F) \cong \boldsymbol{R}\Gamma(X, (\mathcal{L}_2, F))[d_X]$ constructed in [KK2]. Here it is enough to construct a morphism $\mathrm{DR}(M, F) \to (\mathcal{L}_2, F)[d_X]$ in the derived category of filtered differential complex [S2, §2] by the self duality of (M, F) [S5, 3.15], and we use the filtration V in the non-unipotent case, i.e. replace $\otimes_{\mathcal{O}_X}\mathcal{O}_{D_i}$ by Gr_V. The hard Lefschetz was proved for the action of $l+cl'$ on $H^{\cdot}(X, \mathscr{L}_2)$ for $0<c\ll 1$ with $l'=\sum\partial\bar\partial\log\log h_i$, where $D=\bigcup D_i$ and h_i is the norm of $1\in\Gamma(X, \mathcal{O}_X(D_i))$ by a Hermitian metric of the line bundle $\mathcal{O}_X(D_i)$, i.e. locally of the form $u_iz_i\bar z_i$ with u_i a nowhere vanishing C^∞-function and z_i a local equation of D_i. But the action of l' is zero, because $\bar\partial\log\log h_i\wedge v\in\mathscr{L}_2$ if $v\in\mathscr{L}_2$. For the polarization we use the natural pairing

$$\mathscr{L}_2[d_X]\otimes\mathscr{L}_2[d_X]\to\mathscr{L}_1(\boldsymbol{C})[2d_X]\to\mathscr{D}\mathscr{E}[2d_X]\,,\quad \text{cf. [KK1], [S5, 3.15]}\,,$$

which represents $K_{\boldsymbol{C}}\otimes K_{\boldsymbol{C}}\to a_X^!\boldsymbol{C}$ by $\mathrm{Hom}(K_{\boldsymbol{C}}\otimes K_{\boldsymbol{C}}, a_X^!\boldsymbol{C})=\mathrm{Hom}(K_{\boldsymbol{C}}|_U\otimes K_{\boldsymbol{C}}|_U, a_U^!\boldsymbol{C})$. Then the assertion follows from the harmonic theory [KK1].

Note that (2.3) implies (2.2) locally on Y, using Deligne's uniqueness of decomposition (cf. also (2.11)):

(2.4) PROPOSITION (Deligne). *Let $\mathscr{D}$ be a triangulated category with t-structure given by τ, where the associated cohomological functor is denoted by H^j. Let M be an object of $\mathscr{D}$ with a morphism $\eta: M\to M[2]$ such that $\eta^i: H^{-i}M\xrightarrow{\sim} H^iM$ $(i>0)$ and $H^jM=0$ $(j\gg 0)$. Then we have a non-canonical decomposition* [D3]:

$$M\simeq\bigoplus\,(H^jM)[-j]\,. \tag{2.4.1}$$

Moreover, if $\mathrm{Ext}^k(H^iM, H^jM)$ *are $\boldsymbol{Q}$-modules, we have a canonical choice of the isomorphism* (2.4.1) *uniquely characterized by*:

$$(\mathrm{ad}\,\eta_0)^{i-1}\eta_i=0\qquad \textit{for}\quad i>0\ (\textit{in particular } \eta_1=0)\,, \tag{2.4.2}$$

where $\eta=\sum\eta_i$ is the decomposition of

$$\eta:\ \bigoplus(H^jM)[-j]\to\bigoplus(H^jM)[2-j]\quad (\textit{via } (2.4.1))$$

such that $\eta_i\in\bigoplus_j\mathrm{Ext}^i(H^jM, H^{j+2-i}M)$.

In fact, combining with (0.3) and the uniqueness of decomposition of (0.1), this implies:

(2.5) PROPOSITION. *Let $\pi:\tilde X\to X$ be a composition of surjective projective morphisms of irreducible analytic spaces with $d=\dim\tilde X-\dim X$. Let $\tilde{\mathscr{M}}\in\mathrm{MH}_{\tilde X}(\tilde X, n+d)^p$ be the generic pull-back of $\mathscr{M}\in\mathrm{MH}_X(X, n)^p$, i.e. the generic variation of Hodge structure is the pull-back of that of $\mathscr{M}$. Then $\mathscr{M}$ is a direct factor of $\pi_*\tilde{\mathscr{M}}[-d]$, i.e. the direct factor for the underlying complexes of filtered $\mathscr{D}$-modules* (*in the sense of* [S2, 2.1.20], [S5, 2.13]) *and for the underlying $\boldsymbol{R}$-complexes is compatible via α.*

REMARK. X is a projective limit and $\pi_*\tilde{\mathscr{M}}$ is an inductive limit, where the projective system and the inductive system are locally constant on X so that they are well-defined.

(2.6) For the proof of (2.2), we apply (2.5) to $\pi:\tilde X\to X$ such that $f\pi$ satisfies the assumption of (2.3) for some $g_1, \cdots, g_k\in\mathcal{O}_{Y,y}$, where π exists locally on Y by Hironaka and the assumption of (2.2), because we may assume X, Y irreducible and restrict to $\mathrm{MH}_X(X, n)^p$, cf. the remark after (2.2). Then $f_*\mathcal{M}$ is a direct factor of $(f\pi)_*\tilde{\mathcal{M}}[-d]$ so that the Hodge filtration is strict, and $\mathcal{H}^j f_*\mathcal{M}$ is a direct factor of $\mathcal{H}^{j-d}(f\pi)_*\tilde{\mathcal{M}}$ so that $\mathcal{H}^j f_*\mathcal{M} \in \mathrm{MH}(Y, n+j)^p$ by (1.15) locally on Y.

For the global polarizability of $\mathcal{H}^j f_*\mathcal{M}$, we may replace X by $\tilde{X}$, restrict to $\mathcal{M} \in \mathrm{MH}_X(X, n)^p$ and assume that X, Y irreducible, f is surjective and $\mathcal{M}$ is a variation of Hodge structure on the complement of a normal crossing divisor D on X by the same argument as above. Let Y_0 be a closed proper subspace of Y such that any local intersection of irreducible components of D (in particular X) is smooth over $U := Y \setminus Y_0$. We may further assume that $X_0 := f^{-1}Y_0$ is a divisor. Then we have the spectral sequence (1.21.1) by (1.21–23) and (2.3), because the assumption of (1.21) is local on Y and we can apply (1.23) to a projective morphism $\pi: \tilde{X} \to X$ as above (restricting Y). Then (1.21.1) degenerates at E_2 by the decomposition (0.7) (locally on Y) so that $\mathcal{H}^0 i^* \mathcal{H}^j f_*\mathcal{M}$ is a subobject of $\mathcal{H}^{j+1} f_* \mathcal{H}^{-1} i^* \mathcal{M} \in \mathrm{MHW}(Y)$, and polarizable by (1.18) (with the inductive assumption). As $\mathcal{H}^0 i^* i_* = \mathrm{id}$, it remains to show the polarizability of the variation of Hodge structure $\mathcal{H}^j f_*\mathcal{M}|_U$ by (0.2). Replacing Y by U, we may assume X, Y smooth connected. By assumption there is a nonempty open subset U' of Y such that f is cohomologically Kähler with l coming from $H^2(X, \boldsymbol{R}(1))$. Then by [KK1] we have the hard Lefschetz by l^j and the polarization by $(-1)^{j(j-1)/2} f_*\boldsymbol{S} \circ (\mathrm{id} \otimes l^j)$ on the primitive part, because it holds on U' and Y is connected. This completes the proof of (2.2) (assuming (2.3)).

(2.7) Proof of (2.3). We show the assertion by induction on $d = \dim Y$. If $d = 0$, it follows from [KK1], [KK2], cf. the remark after (2.3). Assume $d > 0$, and take g_1 such that $g_1^{-1}(0) \not\supset Y$. Then the assumption of (2.3) is satisfied also for the direct factors of $P_N \mathrm{Gr}^W \psi_{h_1}\mathcal{M}$, $P_N \mathrm{Gr}^W \phi_{h_1,1}\mathcal{M}$ with $h_i = f^* g_i$, where the support of the direct factors are the intersections of local irreducible components of D, cf. [S5, the proof of 3.20], and we may assume Y irreducible at y. Then by induction hypothesis and [S2, 3.3.17], (1.18–19), we get the following on a neighborhood of y:

(2.7.1) $f_*(M, F)$ is strict and $\psi_{g_1}\mathcal{H}^j f_*(M, F) = \mathcal{H}^j f_* \psi_{h_1}(M, F)$ (same for ϕ_1),

(2.7.2) $(\psi_{g_1}\mathcal{H}^j f_*(M, F, K), W) = (\mathcal{H}^j f_* \psi_{h_1}(M, F, K), W) \in \mathrm{MHW}(Y)^p$ with W the monodromy filtration up to shift (same for ϕ_1),

(2.7.3) the hard Lefschetz with the induced polarization on the l-primitive parts holds for ψ_{g_1} and $\phi_{g_1,1}$ of $\mathcal{H}^j f_*(M, F, K)$, cf. (1.17).

In particular we get the hard Lefschetz for $\mathcal{H}^j f_*\mathcal{M}$ on a neighborhood of y. We can apply the same argument for any y' and $g'_1, \cdots, g'_{k'}$ such that $\bigcap g_i'^{-1}(0) = \{y'\}$, replacing X by a resolution of $\bigcup f^{-1} g_i'^{-1}(0) \cup D$ in X. Then we get $\mathcal{H}^j f_*\mathcal{M} \in \mathrm{MH}(Y, n+j)$ by (2.5), where (1.4.2–3) and (1.5.1) are satisfied by [S2, 3.3.17 and 5.2.14]. Moreover the assertion on the induced polarization on $P_l \mathcal{H}^{-j} f_*\mathcal{M}$ follows from the lemma below, which we apply to $A_{k'}\cdots A_1\mathscr{H}^j(f\pi)_*\tilde{\mathscr{M}}$ with $A_j=P_N\mathrm{Gr}^W\psi_{g'_j}$ or $P_N\mathrm{Gr}^W\phi_{g'_j,1}$, where g'_j, π, $\tilde{\mathscr{M}}$ are defined on a neighborhood of y' as above. (We can apply it also to the generic variations of Hodge structures of the direct factors of $\mathscr{H}^j(f\pi)_*\tilde{\mathscr{M}}$, if we use (0.2).) In fact the functors A_j are exact so that the Leray spectral sequence induces

$$(2.7.4)\qquad E_2^{pq}=A_{k'}\cdots A_1\mathscr{H}^pf_*\mathscr{H}^q\pi_*\tilde{\mathscr{M}}\Rightarrow A_{k'}\cdots A_1\mathscr{H}^{p+q}(f\pi)_*\tilde{\mathscr{M}}$$

degenerating at E_2, and the restriction of $\pi_*\tilde{S}$ to $\mathscr{M}\subset\mathscr{H}^0\pi_*\tilde{\mathscr{M}}$ coincides with S, where $(\tilde{\mathscr{M}},\tilde{S})$ is the generic pull-back of $(\mathscr{M},S)$. Here G in the lemma is induced by τ on $\pi_*\tilde{\mathscr{M}}$, S by the iteration of $\mathrm{Gr}^W\psi_{g'_j}$ (or $\phi_{g'_j,1}$) and $(\mathrm{id}\otimes N^{m_j})$ on ${}^p\mathscr{H}^{\cdot}(f\pi)_*\tilde{S}$, and N_1, N_2 by π^*l, l' as in (2.1). Then the direct factor $\mathscr{H}^{-j}f_*\mathscr{M}$ of $\mathscr{H}^{-j}f_*\mathscr{H}^0\pi_*\tilde{\mathscr{M}}=\mathrm{Gr}_0^G\mathscr{H}^{-j}(f\pi)_*\tilde{\mathscr{M}}$ corresponds to a direct factor of $H_{j,0}=\mathrm{Gr}_G^0H_j$ contained in $P_{N_2}H_{j,0}=\mathrm{Ker}\,\mathrm{Gr}_G^0N_2$, so that $S\circ(\mathrm{id}\otimes N_1^j)$ induces a polarization of the N_1-primitive part, where the index of H_i is reversed (with Tate twist $(-i)$) as in [CKS, end of §3].

(2.8) LEMMA. *Let H_i be $\boldsymbol{R}$-Hodge structures of weight $n+i$ with a decreasing filtration G and morphisms*

$$N_1, N_2: H_i\to H_{i-2}(-1)$$

such that $H_i=0$ for $|i|\gg 0$, $N_1G^kH_i\subset G^kH_{i-2}(-1)$, $N_2G^kH_i\subset G^{k-2}H_{i-2}(-1)$. Put $H_{ij}=\mathrm{Gr}_G^jH_{i+j}$ so that N_1, N_2 induce

$$\mathrm{Gr}_GN_1: H_{ij}\to H_{i-2,j}(-1)\,,\qquad \mathrm{Gr}_GN_2: H_{ij}\to H_{i,j-2}(-1)\,.$$

Let $S: (H_i)_{\boldsymbol{R}}\otimes(H_{-i})_{\boldsymbol{R}}\to\boldsymbol{R}(-n)$ $(i\in\boldsymbol{Z})$ be nondegenerate pairings such that $S(u,v)=(-1)^nS(v,u)$, $S(G^k,G^{1-k})=0$, $S(N_a\otimes\mathrm{id})+S(\mathrm{id}\otimes N_a)=0$ $(a=1,2)$, and Gr_GS induces nondegenerate pairings on $(H_{ij})_{\boldsymbol{R}}\otimes(H_{-i,-j})_{\boldsymbol{R}}$. Put $N_c=N_1+cN_2$ for $0<c\ll 1$. Assume:

(2.8.1) $N_1^i: H_{ij}\xrightarrow{\sim}H_{-i,j}(-i)$ $(i>0)$,

(2.8.2) $N_c^i: H_i\xrightarrow{\sim}H_{-i}(-i)$ $(i>0)$, and $S\circ(\mathrm{id}\otimes N_c^i)$ *induces a polarization of Hodge structures on th primitive part $P_{N_c}H^i=\mathrm{Ker}\,N_c^{i+1}$ for $i\geq 0$ and $0<c\ll 1$.*

Then we have

(2.8.3) $N_2^j: H_{ij}\xrightarrow{\sim}H_{i,-j}(-j)$ $(j>0)$,

(2.8.4) $S\circ(\mathrm{id}\otimes N_1^iN_2^j)$ *induces a polarization on the biprimitive part $P_{N_1}P_{N_2}H_{ij}:=\mathrm{Ker}\,N_1^{i+1}\cap\mathrm{Ker}\,N_2^{j+1}$ for $i,j\geq 0$.*

PROOF. This follows from [CKS, (2.11)], [CK, proof of (3.3)], because the filtration G' defined by $G'_kH_i=G^{i-k}H_i$ is the monodromy filtration of N_1 on $H:=\bigoplus H_i$ by (2.8.1). In fact $(H, S; N_1, N_2)$ is a nilpotent orbit and $H=\bigoplus H_i$ gives a splitting of the weight filtration, i.e. the monodromy filtration of N_1+N_c $(0<c\ll 1)$ by (2.8.2). This completes the proof of (2.2–3).

REMARK. For the moment [KK1], [KK2] is proved in the quasiunipotent monodromy case, and so are (2.2–3) (i.e. (1.4.1) is assumed). For the applications (e.g. the proof of (0.5)) it is sufficient in most cases. Note that (0.3) is valid in the nonquasiunipotent case (assuming (1.4.4)), because [Z] is proved in this case.

(2.9) REMARK. (2.3) gives a generalization of Kollár's torsionfreeness of $R^i f_* \omega_X$, the higher direct images of the dualizing sheaf, to the case X smooth Kähler and f proper, cf. [S5, 2.34].

(2.10) THEOREM. *Let $f: X \to Y$ be as in* (2.2). *Then the functor $\mathscr{H}^j f_*$ in* (1.18) *induces the cohomological functor*

$$\mathscr{H}^j f_* : \mathrm{MHM}(X)^p \to \mathrm{MHM}(Y)^p$$

compatible with the corresponding functor ${}^p\mathscr{H}^j f_*$ on the underlying perverse sheaves.

PROOF. The assertion follows from (2.2) and [S5, 2c], if X is smooth. In the singular case we can apply the same argument, if the bifiltered direct image $\tilde{f}_*(\tilde{M}; F, V)$ is defined so that $\mathrm{Gr}_\alpha^V \tilde{f}_*(\tilde{M}, F) = f_* \mathrm{Gr}_\alpha^V(\tilde{M}, F)$, and $\mathscr{H}^j(F_p V_\alpha \tilde{f}_* \tilde{M})$, $\mathscr{H}^j(V_\alpha \tilde{f}_* \tilde{M})$ are coherent over $\mathscr{O}_{\tilde{Y}}$, $V_0 \mathscr{D}_{\tilde{Y}}$, where $\tilde{f} = f \times \mathrm{id}: \tilde{X} = X \times \boldsymbol{C} \to \tilde{Y} = Y \times \boldsymbol{C}$ and $(\tilde{M}, F) = (i_h)_*(M, F)$. With the notation of [loc. cit.], we have a natural isomorphism

$$(f_I)_!(M_I, F) = (p_I)_! \widetilde{\mathrm{DR}}_{V_I}((i_{f_I})_*(M_I, F))$$

where $i_{f_I}: V_I \to V_I \times V_I'$ is the immersion by graph of f_I, $p_I: V_I \times V_I' \to V_I'$ is the natural projection, $(i_{f_I})_*$ denotes the direct image of filtered $\mathscr{D}$-Modules, and $(p_I)_!$ the topological direct image with proper supports. Taking the fiber product with $\boldsymbol{C}$, we define

$$(\tilde{f}_I)_!(\tilde{M}_I; F, V) = (\tilde{p}_I)_! \widetilde{\mathrm{DR}}_{V_I}((\tilde{i}_{f_I})_*(\tilde{M}_I; F, V))$$

where $\tilde{f}_I = f_I \times \mathrm{id}$ (same for $\tilde{i}_{f_I}, \tilde{p}_I$) and $(\tilde{M}_I, F) = (i_{h_I})_*(M, F)$ with h_I an extension of $h|_{U_I}$ to V_I. Then $\tilde{f}_*(\tilde{M}; F, V)$ is defined as in [loc. cit.]. The assertion on the coherence is reduced to the pure case using the weight filtration, and then to the case X smooth using a resolution $\pi: X' \to X$ and the filtration τ on $\tilde{\pi}_*(\tilde{M}'; F, V)$ (by induction on d_X), where $(M', F, K) \in \mathrm{MH}_{X'}(X', n)^p$ is the generic pull-back of $(M, F, K) \in \mathrm{MH}_X(X, n)^p$ (X may be assumed irreducible), and τ exists by the strictness of $\tilde{\pi}_*(\tilde{M}'; F, V)$, cf. [S2, 1.2.3. iii, 3.3.17].

(2.11) REMARK. We can prove (2.2–3) by induction on $\dim \operatorname{supp} \mathscr{M}$ without using the uniqueness of the decomposition (2.4) as follows. It is enough to show the following assertion: Let $\pi: \tilde{X} \to X$, $f: X \to Y$ be proper morphisms, and $\mathscr{M} \in \mathrm{MH}(\tilde{X}, n)$. Assume the assertion (2.3.1) is satisfied for $\pi_* \mathscr{M}$, $(f\pi)_* \mathscr{M}$ and $f_*(\mathscr{H}^j \pi_* \mathscr{M})$ for $j \neq 0$, and the decomposition theorem holds for the underlying complexes of filtered $\mathscr{D}$-Modules (cf. [S2, 2.1.20]) and $\boldsymbol{R}$-Modules of $\pi_* \mathscr{M}$. Then (2.3.1) holds also for $f_*(\mathscr{H}^0 \pi_* \mathscr{M})$. In fact the filtration τ on $\pi_* \mathscr{M}$ induces a filtration G of $\mathscr{H}^j (f\pi)_* \mathscr{M}$ in $\mathrm{MF}_h(\mathscr{D}_X, \boldsymbol{R})$ so that

$$\mathrm{Gr}_i^G \mathscr{H}^j \tilde{f}_* \mathscr{M} \simeq \mathscr{H}^{j-i} f_* \mathscr{H}^i \pi_* \mathscr{M} \qquad \text{with } \tilde{f} := f\pi$$

by assumption. Then $\mathrm{Gr}_i^G \mathscr{H}^j \tilde{f}_* \mathscr{M} \in \mathrm{MH}(Y, n+j)^p$ for $i \neq 0$, and this implies $\mathrm{Gr}_0^G \mathscr{H}^j \tilde{f}_* \mathscr{M} \in \mathrm{MH}(Y, n+j)^p$ and G is a filtration of $\mathscr{H}^j \tilde{f}_* \mathscr{M}$ in $\mathrm{MH}(Y, n+j)^p$. Here it is enough to assume the decomposition theorem for the underlying $\boldsymbol{R}$-complex of $\pi_* M$, because we can apply the following to $f_*(\tau_{\leq i}\pi_{\geq j}\pi_*(M, F))$ $(i \geq 0, j \leq 0)$ inductively: For a short exact sequence of filtered complexes

$$0 \to (K', F) \to (K, F) \to (K'', F) \to 0$$

the following two assertions are equivalent:

(2.11.1) (K', F), (K, F) are strict and $H^j(K', F) \to H^j(K, F)$ are strictly injective.

(2.11.2) (K, F), (K'', F) are strict and $H^j(K, F) \to H^j(K'', F)$ are strictly surjective.

(This equivalence can be easily checked using the long exact sequence in the abelian category containing the exact category of filtered objects.) This argument shows also the compatibility of the two Hodge structures mentioned in the introduction.

3. Decomposition theorem for the proper Kähler direct image of constant sheaf. In this section we prove (0.5):

(3.1) THEOREM. *Let $f: X \to Y$ be a proper morphism of complex analytic spaces. Assume X is smooth Kähler with Kähler class l. Then*

(3.1.1) *$f_*(\mathscr{O}_X, F)$ is strict and $\mathscr{H}^j f_*(\mathscr{O}_X, F, \boldsymbol{R}_X[d_X]) \in \mathrm{MH}(Y, d_X + j)^p$,*

(3.1.2) *the hard Lefschetz* (0.3.2) *with the induced polarization on the relative primitive parts holds for $\mathscr{H}^j f_*(\mathscr{O}_X, F, \boldsymbol{R}_X[d_X])$,*
where $d_X = \dim X$.

PROOF. The assertion is *local* by definition. By (0.2) (or [S2, 5.4.3]) we have $(\mathscr{O}_X, F, \boldsymbol{R}_X[d_X]) \in \mathrm{MH}(X, d_X)^p$, and by (2.2) (or (2.3)(2.5)) it remains to show (3.1.2) locally on Y. By (2.3) there is a bimeromorphic projective morphism $\pi: \tilde{X} \to X$ such that (3.1.2) is satisfied for $\mathscr{H}^j(f\pi)_*(\mathscr{O}_{\tilde{X}}, F, \boldsymbol{R}_{\tilde{X}}[d_X])$, $\pi^* l + cl'$ $(0 < c \ll 1)$. By Hironaka π is a composition of blowing-ups along nonsingular centers, and we may assume that π itself is such a blow-up. Then the assertion follows from the next two lemmas by induction on d_X, because $\pi_*(\mathscr{O}_{\tilde{X}}, F, \boldsymbol{R}_{\tilde{X}}[d_X])$ is the direct sum of $(\mathscr{O}_X, F, \boldsymbol{R}_X[d_X])$ and

$$(i_Z)_*(\mathscr{O}_Z, F, \boldsymbol{R}_Z[d_Z])(-j-1)[d-2-2j] \qquad (0 \leq j \leq d-2)$$

with Z the center of the blow-up, $i_Z: Z \to X$ and $d = d_X - d_Z$. Here we apply the next lemma to $A_{k'} \cdots A_1$ of $\mathscr{H}^j(f\pi)_*(\mathscr{O}_{\tilde{X}}, F, \boldsymbol{R}_{\tilde{X}}[d_X])$ as in the proof of (2.3), cf. (2.7) (or to the generic variation of Hodge structure of the direct factors, if we use (0.2)).

(3.2) LEMMA. *Let $H = \bigoplus H_i$, $\boldsymbol{S}$, G, H_{ij}, N_1, N_2 and N_c be as in* (2.8). *Assume H_i has a decomposition $H'_i \oplus H''_i$ compatible with G, N_1, and satisfies*

(3.2.1) $$\mathrm{Gr}_G^k H'_i = 0 \qquad \text{for} \quad k \neq 0\,,$$

(3.2.2) *the decomposition* $\mathrm{Gr}_G H = \mathrm{Gr}_G H' \oplus \mathrm{Gr}_G H''$ *is compatible with* N_2 *and* $\boldsymbol{S}$,

(3.2.3) *the conditions* (2.8.1), (2.8.3–4) *for* $\mathrm{Gr}_G H''$ *and* (2.8.2) *for* H *hold.*

Then (2.8.1), (2.8.3–4) *hold for* H', *if the following condition is satisfied*:

$$N_2 H'_i \subset G^0 H''_{i-2}(-1)\,. \tag{3.2.4}$$

PROOF. Let $W^{(1)}$ be the monodromy filtration of N_1 on H. Then it is compatible with the decompositions $H = \bigoplus H_i$ and $H_i = H'_i \oplus H''_i$, and $W^{(1)} = G'$ on H'', where $G'_k H_i = G^{i-k} H_i$. By (3.2.1) and (2.8), the assertion is equivalent to $W^{(1)} = G'$ on H', i.e.

$$\mathrm{Gr}_k^{W^{(1)}} H'_i = 0 \qquad \text{for} \quad i \neq k\,. \tag{3.2.5}$$

We may assume $N_1 = 0$, replacing $H, \boldsymbol{S}, n$ by $P_{N_1}\mathrm{Gr}_k^{W^{(1)}} H, \boldsymbol{S} \circ (\mathrm{id} \otimes N_1^k), n+k$, where the condition (2.8.2) is satisfied by [CKS, (2.11)], [CK, proof of (3.3)]. Then (3.2.5) becomes $H'_i = 0$ $(i \neq 0)$. Put

$$j = \max\{|i| : H'_i \neq 0\}\,.$$

Assume $j > 0$. We have

$$N_2 H'_i = 0 \quad (i<2)\,, \qquad N_2 H''_i \subset H''_{i-2}(-1) \quad (i \geq 2) \tag{3.2.6}$$

by (3.2.4) and (3.2.1–2), because $\mathrm{Gr}_G^k H''_i = 0$ $(i \neq k)$ by (3.2.3) and $N_2 G^i \subset G^{i-2}$. Therefore $j \geq 2$, and for $u \in H'_j$, there exists $v \in H''_j$ such that

$$N_2(u-v) \in H''_{j-2}(-1)\,, \qquad \mathrm{Gr}_G^{j-2}(N_2(u-v)) \in P_{N_2}\mathrm{Gr}_G^{j-2} H''_{j-2}(-1)\,.$$

Then $w := u - v \in P_{N_2} H_j = \mathrm{Ker}\, N_2^{j+1}$, because $\mathrm{Gr}_G^0 H_{-j-2} = H'_{-j-2} = 0$ and

$$\mathrm{Gr}_G^{-j-2}(N_2^{j+1} w) = \mathrm{Gr}_G N_2^j \mathrm{Gr}_G^{j-2}(N_2 w) = 0\,.$$

This implies $w = u = v = 0$, because

$$0 \leq \boldsymbol{S}(W, N_2^j C\bar{w}) = -\boldsymbol{S}(N_2 w, N_2^{j-2} C\overline{N_2 w}) \leq 0$$

by (2.8.2) for H and (2.8.4) for $\mathrm{Gr}_G H''$ (where $i = \sqrt{-1}$ is chosen so that the Tate twists are trivialized on $\boldsymbol{S}$).

To check the condition (3.2.4) in the proof of (3.1), we use:

(3.3) LEMMA. *Let* $\pi : \tilde{X} \to X$ *be a bimeromorphic proper morphism of complex manifolds with* $d_X = \dim X$, *and* U *a Zariski-open dense subset of* X *on which* π *is biholomorphic. Put* $Y = X \setminus U$ *and* $d = \mathrm{codim}\, Y$ (≥ 2). *Assume the decomposition theorem holds*:

$$\pi_*(\boldsymbol{R}_{\tilde{X}}[d_X]) \simeq \bigoplus_{Z,j}(\mathrm{IC}_Z L_Z^j)[-j]\,. \tag{3.3.1}$$

Then

(3.3.2) $L_Z^j = 0$ *for* $|j| > \mathrm{codim}\, Z - 2 \geq 0$, (3.3.3) $\mathrm{Ext}^i(\boldsymbol{R}_X[d_X], \mathrm{IC}_Z L_Z^j)=\mathrm{Ext}^i(\mathrm{IC}_Z L_Z^j, \boldsymbol{R}_X[d_X])=0$ *for* $i<\mathrm{codim}\, Z$,

(3.3.4) $u\in\mathrm{Ext}^i(\boldsymbol{R}_X, \boldsymbol{R}_X)$ *is zero, if its restriction to* U *is zero and* $i<2d$.

PROOF. For (3.3.2) it is enough to show the vanishing for $j>\mathrm{codim}\, Z-2\geq 0$ by duality. By (3.3.1) and proper base change theorem, we have

$$\mathscr{H}^{-d_Z}(\mathrm{IC}_Z L_Z^j)_y\subset H^{j+\mathrm{codim}\, Z}(X_y, \boldsymbol{R})$$

and $\dim X_y\leq\mathrm{codim}\, Z-1$, if y is a generic point of Z. Therefore $L_Z^j=0$ for $j+\mathrm{codim}\, Z>2\,\mathrm{codim}\, Z-2$.

The assertions (3.3.3–4) follow from the adjoint relation

$$\mathrm{Hom}(i^*K, K')=\mathrm{Hom}(K, i_*K')$$

and duality. In fact, (3.3.4) is reduced to

$$\mathrm{Ext}^j(\boldsymbol{R}_Y, i_Y^!\boldsymbol{R}_X)=0 \qquad \text{for} \quad j<2\,\mathrm{codim}\, Y,$$

where $i_Y: Y\to X$. It is clear in the case Y smooth, because

$$i_Y^!\boldsymbol{R}_X=\boldsymbol{R}_Y(-d)[-2d].$$

In general we can proceed by induction on $d=\mathrm{codim}\, Y$, using

$$\to i_Z^!\boldsymbol{R}_X\to i_Y^!\boldsymbol{R}_X\to j_*j^{-1}i_Y^!\boldsymbol{R}_X\xrightarrow{+1}$$

where $Z=\mathrm{Sing}\, Y, j\colon Y\setminus Z\to Y$.

(3.4) REMARK. The above argument cannot be generalized to the X singular case, replacing $\boldsymbol{R}_X[d_X]$ by $\mathrm{IC}_X\boldsymbol{R}$. In fact let V be a smooth projective variety in $\boldsymbol{P}^n$, X the affine cone in $\boldsymbol{C}^{n+1}$, $\pi\colon \tilde{X}\to X$ the blow-up of the origine, and $D=\pi^{-1}(0)$ $(\simeq V)$ with $i\colon D\to X$. Then $-D$ is a π-ample divisor, and $-l\in\mathrm{Ext}^2(\boldsymbol{R}_{\tilde{X}}, \boldsymbol{R}_{\tilde{X}}(1))$ is the composition:

$$\boldsymbol{R}_{\tilde{X}}\xrightarrow{i^*}\boldsymbol{R}_D\simeq\mathscr{H}_D^2\boldsymbol{R}_{\tilde{X}}(1)\simeq i_*i^!\boldsymbol{R}_{\tilde{X}}(1)[2]\to\boldsymbol{R}_{\tilde{X}}(1)[2]. \tag{3.4.1}$$

On the other hand we have

$$\pi_*(\boldsymbol{R}_{\tilde{X}}[d_X])=\mathrm{IC}_X\boldsymbol{R}\oplus\left(\bigoplus_{0<k\leq j} P_lH^{d_X-1-j}(D, \boldsymbol{R})(-k)[1+j-2k]\right)$$

$$i_0^*(\mathrm{IC}_X\boldsymbol{R})=\bigoplus_j P_lH^{d_X-1-j}(D, \boldsymbol{R})[1+j]$$

$$i_0^!(\mathrm{IC}_X\boldsymbol{R})=\bigoplus_j P_lH^{d_X-1-j}(D, \boldsymbol{R})(-j-1)[-j-1]$$

and π_* of the middle isomorphism of (3.4.1):

$$i_0^*\pi_*(R_{\tilde{X}}[d_X]) \xrightarrow{\sim} i_0^!\pi_*(R_{\tilde{X}}[d_X])(1)[2]$$

is given by the identity on

$$\bigoplus_{0 \le k \le j} P_l H^{d_X-1-j}(D, \boldsymbol{R})(-k)[1+j-2k].$$

Therefore its restriction to $i_0^*(\mathrm{IC}_X\boldsymbol{R}) \to i_0^!(\mathrm{IC}_X\boldsymbol{R})(1)[2]$ is not zero, if $P_l H^{d_X-1}(D, \boldsymbol{R}) \neq 0$.

(3.5) REMARK. For the proof of (3.1) we need [KK1], [KK2] only in the semi-simple monodromy (of finite order) case, where the proof is rather trivial. We need also the elementary properties in [S2] (e.g. 2.5.6, 3.3.17, 3.4.12, 5.2.14, etc.) and in § 1 of this paper (except for (1.20–21)) as well as the calculation of vanishing cycle functors in the normal crossing case in [S5, (3.a)], but not the deep results like (0.2), (0.3). In the constant sheaf case, (2.5) is also replaced by the natural morphism

$$(\mathcal{O}_X, F) \to \pi_*(\mathcal{O}_{\tilde{X}}, F) \qquad (\text{i.e. } (\Omega_X, F) \to \pi_*(\Omega_{\tilde{X}}, F), \text{ cf. [S2, § 2]})$$

compatible with $\boldsymbol{R}_X \to \pi_* \boldsymbol{R}_{\tilde{X}}$ in $D_c^b(\boldsymbol{C}_X)$, because they induce the splitting, combined with the duality in [S2, 2.5] and the octahedral axiom of derived category.

As a corollary of (0.6), we get the following (cf. [BBD] in the algebraic case):

(3.6) COROLLARY (local invariant cycle theorem). *Let $f: X \to Y$ be a proper surjective morphism of complex analytic spaces. Assume X smooth Kähler. Let $U \subset Y$ be the Zariski-open dense smooth subset of Y, on which f is smooth. Then for $y \in Y$ there exists a sufficiently small neighborhood Y_y of y such that for $y' \in U_y := U \cap Y_y$ the natural morphism*

$$H^j(X_y, \boldsymbol{R}) \to H^j(X_{y'}, \boldsymbol{R})^{\pi_1(U_y, y')} \tag{3.6.1}$$

is surjective, where $X_y := f^{-1}(y)$.

PROOF. The natural morphism (3.6.1) exists once the neighborhood Y_y is sufficiently small, by proper base change and constructibility of $R^j f_* \boldsymbol{R}$. Then by (0.7–8) it is enough to show the surjectivity of

$$\mathcal{H}^{-d}(\mathrm{IC}_Z L)_y \to \mathcal{H}^{-d}(\mathrm{IC}_Z L)_{y'}^{\pi_1(U_y, y')} \tag{3.6.2}$$

where Z is the local irreducible component of Y at y containing y' and $d = \dim Z$. Replacing L by its maximal constant subsheaf, we may assume that L is constant, and then $L = \boldsymbol{R}$, because $\mathrm{IC}_Z L$ is functorial for L. We have the natural morphism $\boldsymbol{R}_Z[d] \to \mathrm{IC}_Z \boldsymbol{R}$ and the assertion follows from the commutative diagram:

$$\begin{array}{ccc}\mathscr{H}^{-d}(\boldsymbol{R}_Z[d])_y & \xrightarrow{\sim} & \mathscr{H}^{-d}(\boldsymbol{R}_Z[d])_{y'} \\ \downarrow & & \downarrow \wr \\ \mathscr{H}^{-d}(\mathrm{IC}_Z\boldsymbol{R})_y & \longrightarrow & \mathscr{H}^{-d}(\mathrm{IC}_Z\boldsymbol{R})_{y'}\,.\end{array}$$

By a similar argument (replacing $\mathscr{H}^{-d}(\mathrm{IC}_Z L)_y$ by $H^{-d}(Z, \mathrm{IC}_Z L)$ with Z the globally irreducible component of Y containing y), we get:

(3.7) COROLLARY (global invariant cycle theorem). *Let $f: X \to Y$ and U be as above (e.g. X is smooth Kähler). Then for $y \in U$ the natural morphism*

$$H^j(X, \boldsymbol{R}) \to H^j(X_y, \boldsymbol{R})^{\pi_1(U,y)} \tag{3.7.1}$$

is surjective.

(3.8) REMARK. The theorem (3.1) and the corollaries (3.6–7) hold under the assumption that X is smooth and f is cohomologically Kähler, cf. (2.1). If X is singular and irreducible, and satisfies the assumption of (0.6), the assertions of (3.6–7) hold with $H^j(X_y, \boldsymbol{R})$, $H^j(X, \boldsymbol{R})$ replaced by $H^j(X_y, \mathrm{IC}_X\boldsymbol{R}|_{X_y})$, $IH^{j+\dim X}(X, \boldsymbol{R}) = H^j(X, \mathrm{IC}_X\boldsymbol{R})$, where U is a Zariski-open dense smooth subset of Y on which $R^jf_*\mathrm{IC}_X\boldsymbol{R}$ are local systems in this case. We have also

$$\mathrm{IC}_X\boldsymbol{R}|_{X_y} = \mathrm{IC}_{X_y}\boldsymbol{R}[d_Y] \qquad \text{for} \quad y \in U\,,$$

if we further restrict U so that $f^{-1}(U)$ has a stratification whose strata are smooth over U. In this case (3.6.1), (3.7.1) become the surjectivity of the natural morphisms

$$H^j(X_y, \mathrm{IC}_X\boldsymbol{R}|_{X_y}) \to IH^{j+\dim X}(X_{y'}, \boldsymbol{R})^{\pi_1(U_y,y')} \tag{3.8.1}$$

$$IH^j(X, \boldsymbol{R}) \to IH^j(X_y, \boldsymbol{R})^{\pi_1(U,y)} \tag{3.8.2}$$

respectively.

(3.9) REMARK. In the assumption of (0.6) the condition $\tilde{X}$ Kähler may be replaced by: the restrictions of π and $f\pi$ to any relatively compact open subsets of X and Y are cohomologically Kähler with Kähler classes extendable to $H^2(\tilde{X}, \boldsymbol{R}(1))$ (in particular X (singular) Kähler is enough, cf. (2.1)), because the decomposition theorem holds for $f_*\boldsymbol{R}_X[d_X]$, if $f: X \to Y$ is proper, X is smooth and the restriction of f to any relatively compact open subset U of Y is cohomologically Kähler with Kähler class in the image of $H^2(X, \boldsymbol{R}(1)) \to H^2(f^{-1}(U), \boldsymbol{R}(1))$. In fact, by [D3] it is enough to show the E_2-degeneration of the spectral sequence

$$E_2^{ij} = H^i({}^p\mathscr{H}^j f_*(\boldsymbol{R}_X[d_X])) \Rightarrow H^{i+j}(f_*\boldsymbol{R}_X[d_X])$$

for any cohomological functor $H^i: D_c^b(\boldsymbol{R}_Y) \to \mathrm{Mod}(\boldsymbol{Z})$. But we have the strict support decomposition ${}^p\mathscr{H}^j f_*(\boldsymbol{R}_X[d_X]) = \bigoplus_Z \mathrm{IC}_Z L_Z^j$ which induces the direct product decomposition $E_2^{ij} = \prod_Z H^i(\mathrm{IC}_Z L_Z^j)$, and for any Z_1, Z_2 there exists l such that $l^j: E_2^{i,-j} \to E_2^{ij}(j)$

induces isomorphisms

$$l^j\colon H^i(\mathrm{IC}_{Z_a}L_{Z_a}^{-j}) \xrightarrow{\sim} H^i(\mathrm{IC}_{Z_a}L_{Z_a}^{j})(j) \qquad \text{for} \quad a=1,2 \quad \text{and} \quad j>0\,.$$

Here l induces a morphism of spectral sequences (with shift of index j by 2), and preserves the strict support decomposition. Then we get the vanishing of the restrictions of d_r to $H^i(\mathrm{IC}_{Z_1}L_{Z_1}^{j}) \to H^{i+r}(\mathrm{IC}_{Z_2}L_{Z_2}^{j-r+1})$ for any Z_1, Z_2 by induction on r using the primitive decomposition as in [D3].

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