Tôhoku Math. J.
42 (1990), 439–455

# RIGIDITY OF SUPERMINIMAL IMMERSIONS OF COMPACT RIEMANN SURFACES INTO $CP^2$

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(Received December 2, 1989, revised March 23, 1990)

**0. Introduction.** The rigidity aspects of minimal hypersurfaces in a Euclidean space or a sphere have constantly drawn authors' attentions, about which we mention the recent conclusive result of Dajczer-Gromoll [12] which states that a complete minimally immersed hypersurface of dimension $\geq 4$ in $S^{n+1}$, or in $R^{n+1}$ if it dose not contain $R^{n-3}$ as a factor, is rigid, even in $R^N \supset R^{n+1}$. On the other hand the failure of this theorem to hold in general for a Riemann surface is well-known, to which we should add the positive result of Barbosa [2] which says that a minimally immersed Riemann sphere in a sphere is rigid, that of Choi-Meeks-White [11] which asserts that a properly embedded minimal surface in $R^3$ with more than one end is rigid, and that of Ramanathan [22] stating that for each compact Riemann surface minimally immersed in $S^3$, there are only finitely many other minimal immersions isometric to it.

Along another line of development, minimal immersions (especially the superminimal ones) of Riemann surfaces into $CP^n$ have recently been extensively studied by several authors [6], [8], [13], [14], [15], [25]. It is the purpose of this paper to look into the rigidity problem for superminimal immersions of compact Riemann surfaces into $CP^2$; to the author's knowledge the only results of this kind are the rigidity theorem of Calabi [7] which says that a holomorphic curve (a special class of superminimal immersions) in $CP^n$ is rigid, the rigidity of totally real superminimal immersions in $CP^n$ in Bolton-Jensen-Rigoli-Woodward [3], and the rigidity of superminimal immersions of constant curvature in [3], Bando-Ohnita [4], and [10]. One different feature of minimal immersions of Riemann surfaces into $CP^2$ from those into $S^3$ is that the immersion is of (real) codimension 2, with respect to which the conclusion of rigidity would be harder to draw in general. However with the given holomorphic data which a superminimal immersion in $CP^2$ enjoys, we are able to assert the rigidity for large classes of superminimal immersions.

After some preliminaries in §1 on the structure of minimal immersions in $CP^n$ through the work in Chern-Wolfson [8], [9], and Eschenberg-Gaudalupe-Tribuzy [15], we establish the result (Lemma 1) in §2 that infers that those points of a given superminimal immersion at which the curvature $K=4$ are exactly those ramified points of index $\geq 2$ of either the holomorphic curve or the dual of the holomorphic curve (but

[1] The author was partially supported by an NSF grant.

not both) which generates the minimal immersion; furthermore the order of a zero of $4-K$ completely determines the index of ramification of the holomorphic curve or its dual at the underlying point. From this follows, by a maximal principle type argument, the rigidity result (Theorem 1) that a superminimal immersion is rigid if it is generated by a holomorphic curve all ramified points of which as well as of its dual are disjoint and of index $\geq 2$. Examples of various genera are constructed, among which we mention those curves projectively equivalent to the Fermat variety of degree 4 (genus $=3$) which generate rigid superminimal immersions among all minimal immersions not even superminimal (Theorem 2).

Although the technique in § 2 fails to give information at ramified points of index $=1$, we shall verify in § 3 that minimal immersions (necessarily superminimal) generated by generic (in the sense made clear in § 3) rational curves of any given degree are rigid, with the aid of the lifting map, of which the Sergre embedding [18] is a special case, constructed in [10] (see also [4]) together with the elimination theory of quasi-projective varieties. In § 4, we prove, incorporating [10] again and some algebraic curve theory, that for each superminimal immersion generated by a nonsingular plane cubic curve, there are only finitely many other superminimal immersions isometric to it. The results in § 3 and § 4 indicate that the lifting map defined in [10] has strong bearings on the rigidity of superminimal immersions, as suggested in that paper.

In contrast to superminimal immersions in $CP^2$, superminimal immersions in $S^4$ studied in Bryant [5] are all rigid in the category of superminimal immersions in $S^4$. This follows from a discussion in the final remark of § 4, in which one transforms these immersions into certain totally real superminimal immersions in $CP^4$.

I would like to thank Peter Hall for sharing insights into part of this paper.

**1. Minimal immersions of compact Riemann surfaces into $CP^2$.** In this section we give a quick review of some facts and formulae that we need in the sequel. The reader is referred to [8], [9], [15] for details. Throughout the paper $M$ is understood to be a compact Riemann surface and $CP^n$ is assumed to be equipped with the Fubini-Study metric $\langle\,,\rangle_{CP^n}$ whose curvature is normalized to be 4. Fix a metric $ds$ in the conformal class of $M$. Let $f_0\colon (M, ds)\to CP^n$ be a weakly conformal and harmonic (or equivalently a branched minimal) immersion, i.e., $f_0^*\langle\,,\rangle_{CP^n}=\lambda\cdot ds$ for some nonnegative function and $\operatorname{tr}(\nabla df_0)=0$. Denote by $\mathscr{L}$ the tautological bundle over $CP^n$. Then $L=f_0^{-1}\mathscr{L}$ inherits a natural holomorphic bundle structure from those of $M$ and $CP^n$ (cf. [9]), and so does $L^\perp$, the hyperplane bundle perpendicular to $L$ in $M\times C^{n+1}$. For a local coordinate system $z$, the Gram-Schmidt process defines a map $G_z\colon X\in L\mapsto \partial X/\partial z-[\langle \partial X/\partial z, X\rangle/\|X\|]X\in L^\perp$, where $\langle\,,\rangle$ denotes the Euclidean inner product on $C^{n+1}$. Then conformality and harmonicity of $f_0$ implies the following:

(1) The well-defined map $f_1$ (denoted $\partial f_0$): $p\mapsto G_z(L_p)$, from $M$ to $CP^n$ is conformal and harmonic.

(2) The map $\partial\colon X\mapsto G_z(X)\otimes dz$, from $L$ to $L^\perp\otimes T^{(1,0)}M$ is a well-defined

holomorphic bundle map.

Clearly the procedures (1) and (2) can be successively carried on so that one obtains

$$f_0 \to f_1 \to f_2 \to \cdots .$$

One sets $L_i = f_i^{-1}\mathscr{L}$. Similarly one can define $f_{\bar{1}}, f_{\bar{2}} \cdots$ and $L_{\bar{i}}$ by replacing $\partial/\partial z$ by $\partial/\partial \bar{z}$ in (1) and (2). Conformality of $f_0$ implies that $L_0$, $L_1$, $L_{\bar{1}}$ are mutually orthogonal. In particular if $n=2$, and if $f_0$ is neither holomorphic nor anti-holomorphic, then we have either of the following:

(a) $0 = f_2 = \partial f_1$ (resp. $f_{\bar{2}} = 0$). It follows that $f_0 = \bar{\partial} f_1$ (resp. $= \partial f_{\bar{1}}$) and $f_1$ (resp. $f_{\bar{1}}$) is anti-holomorphic (resp. holomorphic). $f_0$ is said to be superminimal (we will include holomorphic and anti-holomorphic curves as superminimal immersions as well).

(b) $f_{\bar{1}} = f_2$, so that the $\partial$-process is cyclic. $f_0$ is said to be nonsuperminimal.

We also need the quantitative description of these. Pick orthogonal unit vectors $Z_0$, $Z_1$, $Z_2$ spanning $L_0$, $L_1$, $L_2$. Let $\varphi = \theta_1 + \sqrt{-1}\theta_2$ be the complexified dual form of an orthonormal frame $(e_1, e_2)$ with respect to $ds$ on $M$. Then $d\varphi = \sqrt{-1}\omega \wedge \varphi$ with $\omega$ the connection form and $K$ the curvature of $ds$. Then (b) says (cf. [8])

$$\begin{aligned} dZ_0 &= \psi_0 Z_0 + s\varphi Z_1 + \overline{t\varphi} Z_2 \\ dZ_1 &= -\overline{s\varphi} Z_0 + \psi_1 Z_1 + c\varphi Z_2 \\ dZ_2 &= -t\varphi Z_0 - \overline{c\varphi} Z_1 + \psi_2 Z_2 , \end{aligned} \tag{1.1}$$

where $\psi_0$, $\psi_1$, $\psi_2$ are the connection forms of the bundles $L_0$, $L_1$, $L_2 = L_{\bar{1}}$. Note that $f_0$ is superminimal precisely when $c \equiv 0$. Furthermore, the holomorphy of the map $\partial: L_0 \to L_1 \otimes T^{(1,0)}M$, i.e., $Z_0 \mapsto sZ_1 \otimes \varphi$, infers that the difference between the first Chern classes of $L_1 \otimes T^{(1,0)}M$ and $L_0$ is the ramification index of $\partial$, or equivalently,

$$\Delta(\log|s|)\varphi \wedge \bar{\varphi} = K\varphi \wedge \bar{\varphi} + 2d(\Psi_0 - \Psi_1) ,$$

or,

$$\Delta(\log|s|) = K + 2(|t|^2 - 2|s|^2 + |c|^2) . \tag{1.2}$$

Similarly by considering $\partial: L_1 \to L_2 \otimes T^{(1,0)}M$, where $Z_1 \mapsto cZ_2 \otimes \varphi$, and $\partial: L_2 \to L_0 \otimes T^{(1,0)}M$, where $Z_2 \mapsto -tZ_0 \otimes \varphi$, one deduces

$$\Delta(\log|c|) = K + 2(|s|^2 + |t|^2 - 2|c|^2) , \tag{1.3}$$

$$\Delta(\log|t|) = K + 2(|s|^2 - 2|t|^2 + |c|^2) . \tag{1.4}$$

One introduces the Kaehler angle $\alpha$, $0 \le \alpha \le \pi$ (cf. [8]) such that

$$\begin{aligned} \cos(\alpha/2) &= |s|/(|s|^2 + |t|^2)^{1/2} , \\ \sin(\alpha/2) &= |t|/(|s|^2 + |t|^2)^{1/2} . \end{aligned} \tag{1.5}$$

Notice that in case $f_0$ is an isometric immersion, the first equation of (1.1) implies

$e_1 = sZ_1 + \bar{t}Z_2$, and $e_2 = \sqrt{-1}(sZ_1 - \bar{t}Z_2)$. Let $e_3 = -\bar{t}Z_1 + sZ_2$, and $e_4 = \sqrt{-1}(\bar{t}Z_1 + sZ_2)$ be in the normal bundle of $M$ in $CP^2$. With the aid of the connection of $CP^2$ given by $\nabla Z_i = \sum_{j=1}^{2} \omega_i^j Z_j$, $1 \le i \le 2$, where $\omega_i^i = \Psi_i - \Psi_0$, and $\omega_1^2 = c\varphi$, one verifies (cf. [8], [15])

$$2\omega_1^2 = \langle \nabla(e_1 - \sqrt{-1}e_2), e_3 \rangle_{CP^2} - \sqrt{-1}\langle \nabla(e_1 + \sqrt{-1}e_2), e_4 \rangle_{CP^2},$$

so that

$$4|c|^2 = (h_{11}^3 + h_{12}^4)^2 + (h_{11}^4 + h_{12}^3)^2,$$

where $h_{ij}^r$ denote the components of the second fundamental form. Hence

$$\|B\|^2 = 4|c|^2 + (h_{11}^3 - h_{12}^4)^2 + (h_{11}^4 - h_{12}^3)^2. \tag{1.6}$$

One also concludes from the Gauss equation that

$$\|B\|^2 = 2[4 - 3 \cdot \sin^2(\alpha) - K]. \tag{1.7}$$

**2. Rigidity of superminimal immersions of higher order singularities in $CP^2$.** The following theorem reduces the study of the rigidity of superminimal immersions in $CP^2$ to those immersions of the form $\partial g$ for some holomorphic curve $g$.

THEOREM 0. *Let $f: M \to CP^2$ be a nondegenerate holomorphic curve. If $f$ is isometric to a superminimal immersion $g: M \to CP^2$, then, up to complex conjugation, $g$ is holomorphic, and hence is unitarily equivalent to $f$.*

PROOF. Recall (cf. [10]) that if $F$ is a holomorphic curve from $M$ into $CP^2$ which generates a superminimal immersion $\partial F$, let $\hat{F}$ be its dual curve and let $L$ and $\hat{L}$ be the bundles pulled back via $F$ and $\hat{F}$ from the universal bundle of $CP^2$. Then $L \otimes \hat{L}$ gives rise to a holomorphic curve, denoted $F \otimes \hat{F}$, from $M$ into $CP^8$ whose pull-back metric is identical with that of the superminimal immersion $\partial F$. Explicitly, if $\psi$ and $\psi \wedge \psi'$ are two local lifts of $F$ and $\hat{F}$, respectively, then $\psi \otimes (\psi \wedge \psi')$ is a local lift of the curve induced by $L \otimes \hat{L}$.

Suppose now $g = \partial G$ for some nondegenerate holomorphic cruve $G$ from $M$ into $CP^2$. Then by the above remark $G \otimes \hat{G}: M \to CP^8$ is isometric to $f: M \to CP^2 \subset CP^8$. Since $G \otimes \hat{G}$ and $f$ are both holomorphic, the theorem of Calabi infers that $G \otimes \hat{G}$ and $f$ are unitarily equivalent in $CP^8$; in particular, $G \otimes \hat{G}$ is contained in certain 2-dimensional linear subspace of $CP^8$. Recall that locally $G$ can be put in the canonical form $[1 : z^{1+l}(a_0 + a_1 z + \cdots) : z^{2+l+m}(b_0 + b_1 z + \cdots)]$, where $l$ and $m$ are called the *ramification indexes* at $z=0$ of $G$ and its dual, respectively. A straightforward computation gives

$$G \otimes \hat{G} = [1 : z^{1+l} : z^{1+m} : z^{2+l+m} : z^{2+l+m} : z^{2+l+m} : z^{3+l+2m} : z^{3+2l+m} : z^{4+2l+2m}],$$

where in each slot of the homogeneous coordinates, we only display the zero part of the Taylor expansion, e.g., $z^{1+m}$ should really be $z^{1+m}(a_0+a_1z+\cdots)$ with $a_0\neq 0$, etc. A look at the above expression for $G\otimes\hat{G}$ confirms that $G\otimes\hat{G}$ can be put in the canonical form

$$[1:z^{1+\alpha}:z^{2+\alpha+\beta}:z^{3+\alpha+\beta+\gamma}:\cdots],$$

where again we only display the zero part in each slot. However this implies that $G\otimes\hat{G}$ can not possibly lie in a 2-dimensional linear subspace of $CP^8$. Such a contradiction forces the map $g$ to be either holomorphic or anti-holomorphic, which may be assumed to be the former by applying complex conjugation. The theorem of Calabi then asserts that $f$ is unitarily equivalent to $g$. Q.E.D.

In light of Theorem 0, we will assume from now on, unless otherwise stated, that the superminimal immersions we deal with are of the form $\partial g$ for some holomorphic curve $g: M\to CP^2$. We will also assume without loss of generality that $g$ is nondegenerate, since otherwise the pull-back metric for the map $\partial g$ from $M$ to $CP^1$ is of constant curvature, and the rigidity follows from [10]. Choose a metric $ds$ in the conformal class of $M$. Set $p=|s|$, $q=|t|$ in (1.2) and (1.4) ((1.3) is vacuous since $c\equiv 0$ now). Then

$$g^*\langle\,,\rangle_{CP^2}=q^2\cdot ds^2 \tag{2.1}$$

by the third equation of (1.1). Let $\hat{g}$ be the dual curve of $g$, i.e., $\hat{g}$ be the tangent lines of $g$ in $CP^2$. Then

$$\hat{g}^*\langle\,,\rangle_{CP^2}=p^2\cdot ds^2 \tag{2.2}$$

by the second equation of (1.1), and

$$f^*\langle\,,\rangle=(p^2+q^2)\cdot ds^2 \tag{2.3}$$

by the first equation of (1.1). Given locally the canonical form $[1:z^{1+l}(a_0+a_1z+\cdots):z^{2+l+m}(b_0+b_1z+\cdots)]$ of a holomorphic curve in $CP^2$, it is not hard to see that $l$ (resp. $m$) is the order of zero of $q$ (resp. $p$) at the underlying point (cf. [17]). Also (1.2) and (1.4) now read

$$\Delta(\log p)=K+2(q^2-2p^2)\,, \tag{2.4}$$

$$\Delta(\log q)=K+2(p^2-2q^2)\,. \tag{2.5}$$

Note that $p^2+q^2\neq 0$ everywhere, since $f$ is an isometric immersion.

LEMMA 1. *Let $f: M\to CP^2$ be an isometric superminimal immersion so that $f=\partial g$ for some nondegenerate holomorphic curve with $\hat{g}$ the dual curve, and let $K_f$ be the curvature of $M$ with respect to the pull-back metric of $f$. Then $K_f(x_0)=4$ if and only if $x_0$ is a ramified point of index $\geq 2$ of either $g$ or $\hat{g}$ (but not both since $f$ is an isometric immersion). Furthermore at such points the ramification index is determined by the order of zero of $4-K_f$.*

PROOF. Suppose $K_f=4$. By (1.7) we have $K_f\leq 4-3\cdot\sin^2(\alpha)$. Hence $K_f=4$ implies $\sin(\alpha)=0$, i.e., either $p=0$ or $q=0$. Assume $p\neq 0$, $q=0$ at the point without loss of generality so that the Kaehler angle $\alpha=0$. By choosing the metric $ds=\hat{g}^*\langle\,,\rangle_{CP^2}$ in the conformal class of $M$ whose curvature we denote by $\hat{K}$, one gets $p=1$ in (2.2) with respect to the chosen $ds$. Therefore (2.4) reads

$$0=\Delta(\log p)=\hat{K}+2q^2-4\,,$$

i.e.,

$$\hat{K}=4-2q^2=4-2\cdot\tan^2(\alpha/2)\,, \tag{2.6}$$

so that $\hat{K}=4$ at $x_0$. Now $\tan^2(\alpha/2)=q^2/p^2=|z|^2\cdot r(z)$ in terms of any local coordinate system $z=x+\sqrt{-1}y$ with $z=0$ representing the point $x_0$, where $r(0)=a\neq 0$ if and only if the ramification index of $g$ is 1. Since $\tan^2(\alpha/2)$ is real analytic, one can write

$$\tan^2(\alpha/2)=(x^2+y^2)(a+bx+cy+\cdots)\,. \tag{2.7}$$

Now with respect to the metric $ds^2=f^*\langle\,,\rangle_{CP^2}$ in the conformal class of $M$, (2.4) can be rewritten as

$$\begin{aligned} K_f=4p^2-2q^2+\Delta(\log p)&=\hat{K}\cdot\cos^2(\alpha/2)-\frac{1}{2}\cdot\Delta(\log p^{-2}) \\ &=\hat{K}\cdot\cos^2(\alpha/2)-\frac{1}{2}\cdot\Delta[\log(1+\tan^2(\alpha/2))]\,. \end{aligned} \tag{2.8}$$

A simple calculation gives that at $z=0$, $\Delta\log(1+\tan^2(\alpha/2))=a/s$ for some positive number $s$ upon incorporating (2.7). Hence one infers that $K_f=4-a/s$ at $z=0$, or interchangeably at $x_0$. This forces $a=0$ since $K_f=4$ at $x_0$; thus the ramification index $\geq 2$. Conversely if the ramification index of $g$ at $x_0$ is $\geq 2$ then $a=0$ and so $K_f=4$.

Now around $z=0$, $\tan^2(\alpha/2)=|z|^{2m}\cdot r(z)$, $m\geq 2$, $r(0)\neq 0$, and in view of (2.6) $\hat{K}=4+|z|^{2m}+\cdots$ (ignoring coefficients in the Taylor expansion), while $\cos^2(\alpha/2)=1+\cdots$. Consequently, one sees from (2.8) that $K_f=4+(x^2+y^2)^{(m-1)}+$higher order terms, which says that the ramification index $(=m)$ can be determined by the order of the zero of $4-K_f$ $(=2(m-1))$. Q.E.D.

THEOREM 1. *Let $g: M\to CP^2$ be a nondegenerate holomorphic curve with the dual curve $\hat{g}$. If the ramified points of $g$ and $\hat{g}$ are disjoint and are all of index $\geq 2$, then $\partial g$ is a superminimal immersion which is rigid among all superminimal immersions, i.e., any other superminimal immersion with the same pull-back metric as that of $\partial g$ must be unitarily equivalent to $\partial g$ up to complex conjugation.*

PROOF. Set $f=\partial g$. Choose $ds=f^*\langle\,,\rangle_{CP^2}$ in the conformal class of $M$. Then $p^2+q^2=1$ in (2.3), which, when integrated over $M$, yields (cf. [17])

$$\frac{1}{\pi}\cdot \text{Area}(M)=\deg(g)+\deg(\hat{g})=r(g)+r(\hat{g})-4(g(M)-1)\,, \tag{2.9}$$

where $r(g)$ (resp. $r(\hat{g})$) denotes the total ramification index of $g$ (resp. $\hat{g}$) and $g(M)$ is the genus of $M$. Now suppose we are given a holomorphic curve $G: M\to CP^2$ which generates a superminimal immersion $F=\partial G$ isometric to $f$. On the one hand, since $F$ is isometric to $f$, Lemma 1 asserts that the set of ramified points of $G$ and $\hat{G}$ of index $\geq 2$ coincides with the set of ramified points of $g$ and $\hat{g}$, counting multiplicities. On the other hand, if there is a ramified point of index 1 for $G$ or $\hat{G}$, then $r(G)+r(\hat{G})>r(g)+r(\hat{g})$, which contradicts (2.9). As a result, the set of the ramified points of $G$ and $\hat{G}$ is identical with that of $g$ and $\hat{g}$, counting multiplicities; if one denotes the associated quantities of $G$ parallel to those defined for $g$ by the same letters with a upperscript "*", one concludes that $pq/p^*q^*=\sin(\alpha)/\sin(\alpha^*)$ is a nowhere vanishing function on $M$. However, one sees, summing up (2.4) and (2.5), that

$$\Delta(\log \sin(\alpha))=2K-2\,;$$

therefore

$$\Delta(\log (pq/p^*q^*))=0$$

since $f$ and $F$ are isometric. Hence $pq/p^*q^*=\mu$, a constant by the maximal principle. We claim that $\mu=1$ in fact. We may assume $\mu\leq 1$ without loss of generality. If there exists a point $y$ at which $\sin(\alpha(y))=1$, we are done; for then $1/\sin(\alpha^*(y))=\sin(\alpha(y))/\sin(\alpha^*(y))=\mu\leq 1$ would imply $\sin(\alpha^*(y))=1$. Otherwise there is no point at which $\sin(\alpha)=1$. Then either $\sup_{y\in M}\alpha(y)=\alpha_0<\pi/2$, or $\inf_{y\in M}\alpha(y)=\alpha_0>\pi/2$. Now subtracting (2.4) from (2.5) one obtains

$$\Delta(\log \tan(\alpha/2))=6\cdot\cos(\alpha)>0$$

(resp. $<0$) if $\alpha_0<\pi/2$ (resp. $>\pi/2$), while $\tan(\alpha/2)\leq\tan(\alpha_0/2)$ (resp. $\geq\tan(\alpha_0/2)$). Hence $\tan(\alpha/2)$ is a constant by the maximal principle, which forces $\cos(\alpha)=0$, a contradiction. Hence $\mu=1$, i.e., $pq=p^*q^*$. This together with $p^2+q^2=p^{*2}+q^{*2}=1$ establishes that either $(p,q)=(p^*,q^*)$, or $(p,q)=(q^*,p^*)$, which may be assumed to be the former by complex conjugation so that $q=q^*$. Consequently, $g^*\langle\,,\rangle_{CP^2}=G^*\langle\,,\rangle_{CP^2}$ by (2.1). The theorem of Calabi [7] on the rigidity of isometric holomorphic curves in $CP^n$ then shows that $g$ is unitarily equivalent to $G$, and so is $\partial g$ to $\partial G$. Q.E.D.

Example 1. Let $g$ be an algebraic curve in $CP^2$ projectively equivalent to $h=[1:x:y]$, $y^m=x^{m+1}$, or parametrically $h=[1:z^m:z^{m+1}]$: $CP^1\to CP^2$, $m\geq 3$. $h$ has a cusp of order $m-1$ at $z=0$ and $\hat{h}=[1:z:z^{m+1}]$ has a cusp of order $m-1$ at $\infty$. The rigidity of $\partial g$ among superminimal immersions then follows from Theorem 1.

Corollary 1. *$\partial g$ is rigid among all superminimal immersions if $g$ is a nonsingular algebraic curve in $CP^2$ whose flexes are all of the form $[1:z:z^m]$ with $m\geq 4$.*

PROOF. Since $g$ has no ramified points, the ramified points of $g$ and $\partial g$ are trivially disjoint. Furthermore the ramified points of $\hat{g}$ are exactly the flexes of $g$, whose ramification index is therefore $m-2\geq 2$. Q.E.D.

EXAMPLE 2. Let $g$ be a plane algebraic curve projectively equivalent to the Fermat variety $h=[x:y:z]$, $x^m+y^m+z^m=0$, $m\geq 4$. $h$ is a nonsingular algebraic curve of genus $(m-1)(m-2)/2$ (cf. [19]), which has $3m$ flexes at $[1:\varepsilon:0]$, $[0:1:\varepsilon]$, $[\varepsilon:0:1]$, $\varepsilon^m=-1$, each of index $m-2$. The rigidity of $\partial g$ among superminimal immersions then follows from Corollary 1.

EXAMPLE 3. Let $g$ be a plane algebraic curve projectively equivalent to $h=[x:y:z]$, $x^my^m+y^mz^m+z^mx^m=0$, $m\geq 3$. $h$ is the quadratic transform of the Fermat variety of degree $m$, and has three ordinary $m$-ple points at $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ each of which is of the form $[1:z:z^{m+1}]$. $h$ has no cusps and a computation on the Hessian of $h$ shows that $h$ has no flexes. $\partial g$ is rigid among superminimal immersions by Theorem 1. Notice that the surface is a torus when $m=3$.

EAMPLE 4. Let $g$ be an irreducible plane conic (i.e., a rational normal curve). $g$ is known to be totally unramified and projectively equivalent to $[1:x:y]$, $y=x^2$, (cf. [17]); therefore $pq=\sin(\alpha)$ is nowhere vanishing. If $\partial g$ is isometric to $\partial G$, then $\deg(G)=2$ by (2.9); thus $G$ is a conic as well so that $p^*q^*$ is nowhere vanishing. One concludes as in Theorem 1 that $\partial g$ is unitarily equivalent to $\partial G$.

We conclude this section with a class of Riemann surfaces of genus 3 which are rigid among all minimal immersions not neccessarily superminimal.

THEOREM 2. *Superminimal immersions (including the holomorphic and anti-holomorphic ones) generated by the plane algebraic curves projectively equivalent to the Fermat variety of degree 4 (genus = 3) are rigid among all minimal immersions not necessarily superminimal in* $CP^2$.

PROOF. Let $F$ be any nonsuperminimal immersion in $CP^2$ isometric to $\partial g$ generated by a curve $g$ as given in the statement of the theorem. Denote $p=|s|$, $q=|t|$, $r=|c|$ for the minimal immersion $F$. Summing (1.2) through (1.4) yields

$$\Delta(\log pqr)=3K. \tag{2.10}$$

Integrating (2.10) one gets

$$\#(pqr)=-3\chi(M)=12,$$

where $\#(pqr)$denotes the number of zeros of $pqr$, counting multiplicities, and $\chi(M)$ is the Euler characteristic of $M$. In particular, $\#(pq)=\#(\sin(\alpha))\leq 12$; here $\alpha$ is the Kaehler angle of $F$. However, (1.7) says $\sin(\alpha)=0$ at $K=4$, which therefore asserts that $\sin(\alpha)=0$ at the 12 flexes (of index 2) of the quartic curve $g$ in view of Theorem 1. Consequently, $\#(\sin(\alpha))=12$, and so $\#(r)=0$ because $\#(pqr)=12$. On the other hand at those 12 flexes

$\|B\|^2=0$ by (1.7), and so $r=0$ by (1.6). We see then $\#(r)\neq 0$. This contradiction establishes that $r\equiv 0$, i.e., $F$ is superminimal. Therefore the theorem follows from Theorems 0 and 1. Q.E.D.

**3. Generic rational curves generate rigid minimal immersions.** Lemma 1 fails to hold true if the holomorphic curve (or its dual) which generates the superminimal immersion has a ramified point of index 1. However, by incorporating what is developed in [10], we are able to prove the rigidity of a minimal immersion (which must be superminimal) generated by a "generic" plane rational curve of any given degree.

Regard $CP^1$ as $C\cup\{\infty\}$. Then a plane rational curve of degree $n$, $[p_1:p_2:p_3]$, where $p_1$, $p_2$, $p_3$ are relatively prime polynomials, can be lifted over $C$ into $C^3$ as

$$\psi=A_0+A_1z+A_2z^2+\cdots+A_nz^n, \tag{3.1}$$

where $A_i\in C^3$ up to a constant factor. Since two polynomials in $z$ have a common factor if and only if their resultant is 0, it is clear that $p_1$, $p_2$, $p_3$ not being relatively prime gives algebraic relations among the coordinates of $A_i$. Hence the space of plane rational curves of degree $n$ is a Zariski open set in $CP^{3(n+1)-1}$. The dual curve of a plane rational curve has a lift over $C$ into $C^3$ as

$$\psi\wedge\psi'=\sum_\sigma\left(\sum_{j+k=\sigma+1}k\cdot A_j\wedge A_k\right)z^\sigma. \tag{3.2}$$

We identify $A_j\wedge A_k$ with a vector in $C^3$ in the usual way. In the following we refer to an open and dense set as being "generic", so that when we say a generic plane rational curve of degree $n$, we mean a generic point in $CP^{3(n+1)-1}$ which corresponds to the curve as described above.

LEMMA 2. *For a generic plane rational curve of degree $n$, $\|\psi\|^2$ and $\|\psi\wedge\psi'\|^2$ are irreducible polynomials in $z$ and $\bar z$.*

PROOF. We have

$$\|\psi\|^2=\langle\psi,\psi\rangle=\sum_{\rho,\sigma}\langle A_\rho, A_\sigma\rangle z^\rho\bar z^\sigma, \tag{3.3}$$

where $\langle\,,\rangle$ denotes the inner product of $C^3$. If $\|\psi\|^2=p_1p_2\cdots p_m$ is the irreducible decomposition, then so is $\|\psi\|^2=\bar p_1\bar p_2\cdots\bar p_m$ the same decomposition. It follows that either $p_i=\bar p_i$ or both $p_i$ and $\bar p_i$ appear in $\|\psi\|^2$. We may therefore assume $\|\psi\|^2=GH$ ($H=1$ if $\|\psi\|^2=p_1\bar p_1$, $p_1\neq\bar p_1$), where

$$G=\sum_{i,j\leq k}A_{ij}z^i\bar z^j \tag{3.4}$$

and

$$H = \sum_{u,v \leq l} B_{uv} z^u \bar{z}^v \tag{3.5}$$

with $A_{ij} = \bar{A}_{ji}$, $B_{ij} = \bar{B}_{ji}$, $A_{kk} \neq 0$, $B_{ll} \neq 0$, and $k + l = n$.

Consider the map $f: CP^{3(n+1)-1} \times CP^{3(n+1)-1} \to CP^{(n+1)^2-1}$ defined by

$$f: [A_0 : \cdots : A_n] \times [B_0 : \cdots : B_n] \mapsto [\cdots : (A_i, B_j) : \cdots], \tag{3.6}$$

where $A_s$ and $B_s$ are vectors in $C^3$, and $(A_i, B_j)$ denotes the symmetric product $\langle A_i, \bar{B}_j \rangle$. One also defines the map $g_k$: $CP^{(k+1)^2-1} \times CP^{(l+1)^2-1} \to CP^{(n+1)^2-1}$ by

$$g_k: [\cdots : A_{ij} : \cdots] \times [\cdots : B_{st} : \cdots] \mapsto \left[\cdots : \sum_{i+s=\rho, j+t=\sigma} A_{ij} B_{st} : \cdots\right]. \tag{3.7}$$

Then $f$ and $g_k$ are regular maps between projective varieties (cf. [16]) so that the set $\mathscr{S}_k = \{(x, y): f(x) = g_k(y)\}$ is Zariski closed in $CP^{3(n+1)-1} \times CP^{3(n+1)-1} \times CP^{(k+1)^2-1} \times CP^{(l+1)^2-1}$; let $\pi$ be the projection from this product space to the product of the first two summands. Then $\pi(\mathscr{S}_k)$ is also Zariski closed defined by algebraic functions $F_1^{(k)}, F_2^{(k)}, \cdots$.

Now from (3.3) through (3.7), one sees that the set of plane rational curves of degree $n$ for which $\|\psi\|^2$ is reducible is a subset of $\mathscr{T} = \bigcup_k \mathscr{T}_k = \bigcup_k \{(x, \bar{x}) \in \pi(\mathscr{S}_k)\}$, where $\bar{x}$ is the complex conjugate of $x$. It is thus clear that each $\mathscr{T}_k$ is a (real) Zariski closed set defined by the same functions $F_1^{(k)}, F_2^{(k)}, \cdots$ above with the (real) algebraic substitutions $B_i = \bar{A}_i$ if $[A_0 : A_1 : \cdots : A_n] \times [B_0 : B_1 : \cdots : B_n]$ parametrizes $CP^{3(n+1)-1} \times CP^{3(n+1)-1}$. If after the substitution $B_i = \bar{A}_i$ the relations $F_1^{(k)}, F_2^{(k)}, \cdots$ become trivially true, then it implies that for every $\psi \neq 0$ in (3.1) $\|\psi\|^2$ is reducible of the form in (3.3) and (3.4). This is impossible by Lemma 3 below. Therefore there must be at least one of $F_1^{(k)}, F_2^{(k)}, \cdots$ not trivial after the substitution $B_i = \bar{A}_i$, which implies that the complement of $\mathscr{T}$ is a nontrivial (real) Zariski open set, on which $\|\psi\|^2$ is irreducible, proving the lemma for $\|\psi\|^2$. For the irreducibility of $\|\psi \wedge \psi'\|^2$, one considers the quasi-projective variety

$$\mathscr{V} = \left\{[A_0 : \cdots : A_n] \in CP^{3(n+1)-1} : \sum_{j+k=\sigma+1} k \cdot A_j \wedge A_k \neq 0 \text{ for some } \sigma.\right\},$$

and the map $f_1 : \mathscr{V} \times \mathscr{V} \to CP^{(2n-1)^2-1}$ defined by

$$f_1: [A_0 : A_1 : \cdots : A_n] \times [B_0 : B_1 : \cdots : B_n]$$
$$\mapsto \left[\cdots : \left(\sum_{j+k=\rho+1} k \cdot A_j \wedge A_k, \sum_{j+k=\sigma+1} k \cdot A_j \wedge A_k\right) : \cdots\right],$$

where again ( , ) is the symmetric product between two vectors in $C^3$. For $k + l = 2n - 2$, one defines the map $g_1: CP^{(k+1)^2-1} \times CP^{(l+1)^2-1} \to CP^{(2n-1)^2-1}$ similar to (3.6) above. Then once more $\mathscr{S}_k = \{(x, y): f_1(x) = g_1(y)\}$ is Zariski closed, and so is its image under the projection map taking $CP^{(k+1)^2-1} \times CP^{(l+1)^2-1} \times \mathscr{V} \times \mathscr{V}$ to $\mathscr{V} \times \mathscr{V}$, because the projection map $X \times Y \to Y$ takes a Zariski closed set to a Zariski closed set, if $X$ is a projective variety and $Y$ is a quasi-projective variety (cf. [18], [23]). The rest of the argument then proceeds identically as before. Q.E.D.

The following lemma is alluded to in Lemma 2 and is interesting in its own right.

LEMMA 3. *One can choose a generic nonsingular* $3 \times 3$ *lower triangular matrix* $A$ *over* $C^3$ *such that for the projective cubic curve* $A \cdot [1 : z : z^3]$ *and its dual,* $\|\psi\|^2$ *and* $\|\psi \wedge \psi'\|^2$ *are both irreducible in* $z$ *and* $\bar{z}$, *where* $\psi$ *is the lift* $\psi = A \cdot (1, z, z^3)$ *of the curve.*

PROOF. Let

$$A = \begin{pmatrix} \gamma, & 0, & 0 \\ \beta, & \alpha, & 0 \\ c, & b, & a \end{pmatrix}$$

be a nonsigular lower trianglar matrix. Let $\psi = A \cdot (1, z, z^3)$. By a direct computation, $\|\psi\|^2 = p(z)\bar{z}^3 + q(z)\bar{z} + r(z)$, where

$$p(z) = |a|^2 z^3 + \bar{a}bz + \bar{a}c\,,$$

$$q(z) = a\bar{b}z^3 + (|\alpha|^2 + |b|^2)z + (\bar{\alpha}\beta + \bar{b}c)\,,$$

$$r(z) = a\bar{c}z^3 + (\alpha\bar{\beta} + b\bar{c})z + (|c|^2 + |\beta|^2 + |\gamma|^2)\,.$$

Supposing $\|\psi\|^2$ is reducible, let

$$\|\psi\|^2 = [s(z)\cdot\bar{z} + t(z)]\cdot[u(z)\cdot\bar{z}^2 + v(z)\cdot\bar{z} + w(z)]\,,$$

where $\deg(s(z)) = 1$, $\deg(t(z)) \leq 1$, $\deg(u(z)) = 2$, and $\deg(v(z))$, $\deg(w(z)) \leq 2$, in view of (3.3) through (3.5). Equating, one sees that

$$s(z)\cdot u(z) = p(z)\,, \tag{3.8}$$

$$s(z)\cdot v(z) + t(z)\cdot u(z) = 0\,, \tag{3.9}$$

$$s(z)\cdot w(z) + t(z)\cdot v(z) = q(z)\,, \tag{3.10}$$

$$t(z)\cdot w(z) = r(z)\,. \tag{3.11}$$

We now impose the condition that $4b^3 + 27ac^2 \neq 0$, i.e., that $p(z)$ has no repeated roots so that $s(z)$ and $u(z)$ are relatively prime in view of (3.8). This implies that $s(z)$ divides $t(z)$ by (3.9); hence

$$t(z) = \lambda\cdot s(z) \tag{3.12}$$

for some constant $\lambda \neq 0$ by degree count. Solving $v(z)$ and $w(z)$ in terms of $s(z)$ and $t(z)$

by (3.9) and (3.11), substituting them into (3.10), and equating the coefficients with the aid of (3.12), one ends up with

$$a\bar{c}-\lambda^3|a|^2=\lambda a\bar{b}\,, \tag{3.13}$$

$$\alpha\bar{\beta}+b\bar{c}-\lambda^3\bar{a}b=\lambda(|\alpha|^2+|b|^2) \tag{3.14}$$

$$(|c|^2+|\beta|^2+|\gamma|^2)-\lambda^3\bar{a}c=\lambda(\bar{\alpha}\beta+\bar{b}c)\,. \tag{3.15}$$

From (3.13), one derives $\bar{c}=\lambda^3\bar{a}+\lambda\bar{b}$. Substitution of this into (3.14) results in $\bar{\beta}=\lambda\bar{\alpha}$, which, when inserted into (3.15), gives $|\gamma|=0$. This contradicts the fact that the matrix $A$ is nonsingular. Hence we conclude that $\|\psi\|^2$ is irreducible if $4b^3+27ac^2\neq 0$, which is a generic condition.

Before proceeding further, we remark that by the same reasoning a cubic curve of the form

$$\varphi=\begin{pmatrix} a, & b, & c \\ 0, & \alpha, & \beta \\ 0, & 0, & \gamma \end{pmatrix}\cdot(1, z, z^3) \tag{3.16}$$

has the property that $\|\varphi\|^2$ is irreducible if

$$4(\alpha\bar{\beta}+b\bar{c})^3+27(|c|^2+|\beta|^2+|\gamma|^2)(a\bar{c})^2\neq 0\,. \tag{3.17}$$

Now a computation gives

$$\psi\wedge\psi'=\begin{pmatrix} \alpha\gamma, & 0, & 0 \\ -b\gamma, & -3a\gamma, & 0 \\ b\beta-\alpha c, & 3a\beta, & 2a\alpha \end{pmatrix}\cdot(1, z^2, z^3)\,,$$

which can be transformed into the same form (in the new variable $t=z^{-1}$) as $\varphi$ in (3.16). Hence by (3.17), $\|\psi\wedge\psi'\|^2$ is irreducible if

$$0\neq 4[3a\bar{b}|\gamma|^2+3a\beta(\overline{b\beta}-\overline{\alpha c})]^3+27[|b\beta-\alpha c|^2+|b\gamma|^2+|\alpha\gamma|^2][2a\alpha(\overline{b\beta}-\overline{\alpha c})]^2\,,$$

which is a generic condition as well. Hence the lemma is proved. Q.E.D.

THEOREM 3. *A minimal immersion (necessarily superminimal) generated by a generic plane rational curve of any given degree is rigid among all minimal immersions.*

PROOF. We remark that any minimal immersion from a Riemann sphere into $CP^n$ is superminimal (cf. [6], [13], [14], [25]). Also in view of (3.2) generic plane rational curves of any degree carry nonsingular pull-back metric, so that the generated superminimal immersion is nowhere branched by (2.1) through (2.3). Let $F$ be a generic plane rational curve of degree $n$ which generates the superminimal immersion $\partial F$. Let $G$ be another rational curve for which the generated superminimal immersion $\partial G$ is isometric to $\partial F$. Let $\psi_F$ and $\psi_G$ be lifts of $F$ and $G$ over $C$ into $C^3$ as given in (3.1).

Let $\hat{F}$ and $\hat{G}$ be the dual curves of $F$ and $G$, respectively. Then by an earlier remark $F\otimes\hat{F}$ is isometric to $G\otimes\hat{G}$ in $\boldsymbol{CP}^8$. Hence the rigidity theorem of Calabi implies that $F\otimes\hat{F}$ is unitarily equivalent to $G\otimes\hat{G}$ in $\boldsymbol{CP}^8$. Over $\boldsymbol{C}$ this means that there is one constant unitary matrix $U$ over $\boldsymbol{C}^9$ such that

$$\psi_G\otimes(\psi_G\wedge\psi'_G)^* = e^r\cdot U\cdot\psi_F\otimes\psi_F\wedge\psi'_F\,,$$

where $(\psi_G\wedge\psi'_G)^*$ denotes $\psi_G\wedge\psi'_G$ divided by the polynomial which is the greatest common factor of all the components of the vector $\psi_G\wedge\psi'_G$, and $r$ is an entire holomorphic function. However, since the components of $\psi_G\otimes(\psi_G\wedge\psi'_G)^*$ and $\psi_F\otimes\psi_F\wedge\psi'_F$ are polynomials, one infers that $e^r$ is a constant. Thus up to a constant one may assume

$$(3.18)\qquad \|\psi_G\|^2\|(\psi_G\wedge\psi'_G)^*\|^2 = \|\psi_G\otimes(\psi_G\wedge\psi'_G)^*\|^2 = \|\psi_F\|^2\|\psi_F\wedge\psi'_F\|^2\,.$$

Lemma 3 then asserts either $\|\psi_F\|^2=\|\psi_G\|^2$, or $\|\psi_F\|^2=\|(\psi_G\wedge\psi_G)^*\|^2$. This implies that either the two holomorphic curves $F$ and $G$ have the same induced metric, or $F$ and $\hat{G}$ have the same induced metric, by the fact that the pull-back metric of $F$ is $4\cdot\partial^2/\partial z\partial\bar{z}\,(\log\|\psi_F\|^2)|\,dz\,|^2$. Hence up to complex conjugation $F$ is unitarily equivalent to $G$ in $\boldsymbol{CP}^2$ by Calabi's result, and so is $\partial F$ unitarily equivalent to $\partial G$. Q.E.D.

As a special case, when $n=3$, it is known (cf. [20]) that any singular cubic curve in $\boldsymbol{CP}^2$ is projectively equivalent to either $y^2=x^3$ in affine coordinates, i.e., parametrically $[1:z^2:z^3]$, which is a curve with a cusp, or to $y^2=x^2(x+1)$, i.e., parametrically $[1:z^2-1:z(z^2-1)]$, which is a curve with a double point. It is easy to see via the parametric representation that those cubic curves with a double point constitute a generic set in the space of plane rational curves of degree 3; therefore generic such curves generate rigid superminimal immersions by Theorem 3.

On the other hand although the cubic curves with a cusp are "nongeneric" among all the singular cubic curves, Lemma 3 says that among themselves generic such curves generate rigid superminimal immersions.

It is interesting to note that for the cubic curves $[1:\mu z:\nu z^3]$, where $\mu$, $\nu$ are constants, $\|\psi_F\|^2$ of their lifts $\psi_F=(1,\,\mu z,\,\nu z^3)$ are reducible, and these curves fall precisely in the "nongeneric" category in the sense of Lemma 3 among cubic curves with a cusp, because $b^3+27ac^2=0$ for the matrix $A$ in that lemma. However these curves still generate rigid superminimal immersions as can be seen by the following argument: Set $t=|\,z\,|^2$. Then $\|\psi_F\|^2$ and $\|\psi_F\wedge\psi'_F\|^2$ are cubic polynomials in $t$ with real coefficients. (3.18) now forces $\|\psi_G\|^2$ to be a cubic polynomial in $t$ as well, which implies that $\psi_G$ must be of the form $A_0+A_1z+A_2z^2+A_3z^3$, where $A_0$ up to $A_4$ are mutually orthogonal, and so either $A_1$ or $A_2$ is zero; we may assume $A_2=0$ by applying the complex conjugation. Then by applying the unitary transformation sending $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ to $A_0/\|A_0\|$, $A_1/\|A_1\|$, $A_3/\|A_3\|$ one may assume that $\psi_G=(1,\,\mu_1 z,\,\nu_1 z^3)$. Now (3.18) gives $|\,\mu\,|=|\,\mu_1\,|$, and $|\,\nu\,|=|\,\nu_1\,|$, which establishes the rigidity.

**4. Rigidity of superminimal immersions generated by nonsingular plane cubic curves.** In this final section we look into the rigidity of those superminimal immersions generated by nonsingular plane cubic curves of genus 1, and will prove that for each of such superminimal immersions there are at most finite other isometric superminimal immersions from the same torus into $CP^2$, up to unitary equivalence and complex conjugation.

LEMMA 4. *Let $g: M \to CP^2$ be biholomorphic to a nonsingular plane cubic which generates the superminimal immersion $\partial g$. If $G: M \to CP^2$ is another holomorphic curve which generates the superminimal immersion $\partial G$ isometric to $\partial g$, then $G$ itself is biholomorphic to a nonsingular plane cubic such that there is an automorphism $\Psi$ of $M$ and an automorphism $U$ of $CP^2$ such that $G = U \cdot g \circ \Psi$.*

PROOF. Let $\hat{g}$ and $\hat{G}$ be the dual of $g$ and $G$, respectively. Then $\deg(\hat{g}) = 6$ (cf. [17]). Since $\deg(G) + \deg(\hat{G}) = \deg(g) + \deg(\hat{g}) = 9$, we may assume, up to complex conjugation, that $\deg(G) = 3$ or 4. Supposing $\deg(G) = 4$, let $C$ be the image of $M$ via $G$. Then $C$ is an irreducible algebraic curve of degree 2 or 4. If $\deg(C) = 2$, then $C$ is a rational normal curve and so is its dual, which means that $\deg(\hat{G})$ must be a multiple of $2 = \deg(\hat{C})$. This is impossible since $\deg(\hat{G}) = 5$. Hence $\deg(C) = 4$. Two cases occur: (a) $C$ has only traditional singularties, or (b) $C$ has higher order singularities. In Case (a), Plücker's formula (cf. [17]) says $2\delta + 3\kappa = 7$, where $\delta$ denotes the number of double points and $\kappa$ the number of cusps, since $\deg(C) = 4$ and $\deg(\hat{C}) = 5$. So $\delta = 2$ and $\kappa = 1$. But then the genus formula (cf. [17]) implies $g(M) = (4-1)(4-2)/2 - \delta - \kappa = 0$, contradicting $g(M) = 1$, where $g(M)$ stands for the genus of $M$. In Case (b), $C$ has only one triple point and no other singularities. Three cases then occur: (i) the singular point is an ordinary triple point, (ii) it is an ordinary cusp meeting a regular point, or (iii) it is a higher order cusp. However by the Riemann-Hurwitz formula (cf. [17])

$$2 - 2g(M) = 2\deg(C) - \deg(\hat{C}) - i = 3 - i,$$

where $i$ is the branching index over the unique singular point with respect to a projection from a generic point onto a generic projective line. Now it is directly checked that $i = 0, 1, 2$ for Case (i), (ii), (iii), respectively, and thus $g(M) = 0$ for Case (ii) and is vacuous otherwise. In any event this is contradictory. What we have concluded is then $\deg(C) \neq 4$. Thus $\deg(G) = \deg(C) = 3$, so that $C$ is a nonsingular plane cubic and so $G: M \to C$ is bi-holomorphic. Since $g(M)$ is also a nonsingular plane cubic, one concludes that $C = G(M)$ and $g(M)$ are projectively equivalent (cf. [21]). The rest of the lemma then follows as a consequence. Q.E.D.

THEOREM 4. *Given a holomorphic curve $F: M \to CP^2$ which is biholomorphic to a nonsingular cubic, let $\partial F$ be the superminimal immersion generated by $F$. Then there are at most finite superminimal immersions from $M$ into $CP^2$ which are isometric to $\partial F$, up to unitary congruence and complex conjugation.*

PROOF. Without loss of generality we may assume that $F$ is projectively equivalent to $[1:\wp(z):\wp'(z)]$, where $\wp(z)$ is the Weierstrass $\wp$-function; a natural lift is $\psi_F=(1, \wp(z), \wp'(z))$. Supposing $G: M\to CP^2$ is another holomorphic curve which generates the superminimal immersion $\partial G$ isometric to $\partial F$, then by Lemma 4 there is a constant nonsingular $3\times 3$ matrix and constants $\lambda$ and $\mu$ (to be specified later) such that

$$\psi_G=U\cdot\psi_F(\lambda z+\mu) \tag{4.1}$$

over the complex plane $C$ is a lift of $G$. In fact, it will follow from Remark 1 below that one can assume $\lambda=1$ without loss of generality, which is to be adapted from now on. Since $\wp(z)=1/z^2+O(z^2)$ (cf. [1]), a computation shows that $\psi_F\otimes\psi_F\wedge\psi'_F=1/z^9+$ higher order terms. Similarly $\psi_G\otimes\psi_G\wedge\psi'_G$ is doubly periodic with a pole of order 9 at $-\mu$. Let $h_F$ and $h_G$ be two entire functions whose zeros are the poles of $\psi_F\otimes\psi_F\wedge\psi'_F$ and $\psi_G\otimes\psi_G\wedge\psi'_G$, respectively (cf. [1]). Then $h_F\cdot\psi_F\otimes\psi_F\wedge\psi'_F$ and $h_G\cdot\psi_G\otimes\psi_G\wedge\psi'_G$ are two holomorphic lifts of the isometric holomorphic curves $F\otimes\hat{F}$ and $G\otimes\hat{G}$ in $CP^8$. Then the rigidity theorem of Calabi asserts that there is a unitary matrix $U$ in $C^9$ and an entire holomorphic function $r$ such that

$$h_G\cdot\psi_G\otimes\psi_G\wedge\psi'_G=e^r\cdot U\cdot h_F\cdot\psi_F\otimes\psi_F\wedge\psi'_F\,. \tag{4.2}$$

Let $U\cdot\psi_F\otimes\psi_F\wedge\psi'_F=(f_0, f_1, \cdots)$, and $\psi_G\otimes\psi_G\wedge\psi'_G=(g_0, g_1, \cdots)$, where we assume that $f_0$ is doubly periodic with pole $z=0$ of order 9. Then (4.2) implies that the zeros of $f_0$ are identical with those of $g_0$. On the other hand, it is well-known (cf. [1]) that the sum of poles is equal to the sum of zeros modulo lattice; thus one derives that the sum of poles of $f_0$ is equal to the sum of poles of $g_0$, i.e., modulo lattice,

$$0=-9\mu\,. \tag{4.3}$$

Let 1 and $\omega$ be the generators of the lattice without loss of generality, where $\omega=x+\sqrt{-1}y$ with $-1/2\leq x\leq 1/2$, $x^2+y^2\geq 1$, and $y>0$. Then $\mu=a+b\omega$, $0\leq a, b\leq 1$ (cf. [20], [24]). Hence one concludes, in view of (4.3), that $a=m/9$ and $b=n/9$, where $m$ and $n$ are suitable integers, and so there are at most 81 such automorphisms $z+\mu$.

To finish the proof, one notices that if there are given 82 mutually noncongruent superminimal immersions $\partial G$ isometric to $\partial F$, then there are two of such immersions, say $\partial G_1$ and $\partial G_2$, which share the same automorphism in view of the above and Lemma 4, i.e., $G_1=U_1\cdot F\circ\Psi$, and $G_2=U_2\cdot F\circ\Psi$ for two automorphisms $U_1$ and $U_2$ of $CP^2$. Then $G_2=U_2\cdot U_1^{-1}\cdot G_1$. However this implies that $G_1$ and its dual have the same ramified points, counting multiplicities, as $G_2$ and its dual, because projective automorphisms preserve ramification indexes. The arguments in Theorem 1 then show that $G_1$ is unitarily equivalent to $G_2$, which contradicts the mutual noncongruence of $\partial G_1$ and $\partial G_2$.

Q.E.D.

REMARK 1. We observe that if $\mu$ in (4.3) satisfies $3\mu=0$ modulo lattice, then $\partial G$

is in fact unitarily equivalent to $\partial F$. This follows from the general fact that if $3(\mu/\lambda)=0$, then the automorphism $\lambda z+\mu$ will leave the linear system $|3(0)|$ invariant, where $3(0)$ denotes the divisor $0+0+0$ on $M$ (cf. [20]), and so induces an automorphism $A$ of $CP^2$ such that $[1:\wp(\lambda z+\mu):\wp'(\lambda z+\mu)]=A\cdot[1:\wp(z):\wp'(z)]$. It follows from this and (4.1) that $G$ is in fact projectively equivalent to $F$, and hence the rigidity holds again by the same argument as in the last paragraph of Theorem 4. This also implies that the 80 possible automorphisms which came up in Theorem 4 can be reduced to $81-9=72$.

In particular, setting $\mu=0$, one sees that for an automorphism of the form $\lambda z$ on $M$, $[1:\wp(\lambda z):\wp'(\lambda z)]=A\cdot[1:\wp(z):\wp'(z)]$ for some automorphism $A$ of $CP^2$. Incorporating this with (4.1), one may therefore assume that $\lambda=1$, as was done there.

REMARK 2. In contrast to $CP^2$, superminimal immersions in $S^4$ as defined and studied in [5] are all rigid among superminimal immersions in $S^4$. This can be seen as follows. Let $\{e_0, e_1, \cdots, e_4\}$ denote an orthonormal frame of $S^4$ with $e_0$ the position vector. Imbed $S^4$ in $CP^4$ in the standard way. Let $Z_0=e_0$, $Z_1=(e_1+\sqrt{-1}e_2)/\sqrt{2}$, $Z_2=\bar{Z}_1$, $Z_3=(e_3+\sqrt{-1}e_4)/\sqrt{2}$, and $Z_4=\bar{Z}_3$. As derived in [8],

$$dZ_0=\frac{1}{\sqrt{2}}(Z_1\bar{\varphi}+Z_2\varphi)\,,$$

$$dZ_1=\frac{1}{\sqrt{2}}Z_0\varphi-\sqrt{-1}Z_1\theta_{12}-\frac{1}{2}[(H_3-\sqrt{-1}H_4)Z_3+(H_3+\sqrt{-1}H_4)Z_4]\bar{\varphi}\,, \tag{4.4}$$

$$dZ_3=\frac{1}{2}(\bar{H}_3+\sqrt{-1}\bar{H}_4)Z_1\varphi+\frac{1}{2}(H_3+\sqrt{-1}H_4)Z_2\bar{\varphi}-\sqrt{-1}Z_3\theta_{34}\,,$$

where $H_r=h^r_{11}+\sqrt{-1}h^r_{12}$. The immersion is said to be superminimal in $S^4$ if and only if $(H_3)^2+(H_4)^2=0$; one may assume $H_3+\sqrt{-1}H_4=0$. One then reads from (4.4) that $Z_0\xrightarrow{\partial}Z_2\xrightarrow{\partial}Z_4\xrightarrow{\partial}0$. Therefore the immersion is superminimal when it is viewed as an immersion into $CP^4$. Since the immersion is totally real in $CP^4$ (Kaehler angle$=\pi/2$), one concludes that the immersion is rigid among superminimal immersions in $S^4$ (cf. [3]).

QUESTION. Are all superminimal immersions rigid in the category of superminimal immersions in $CP^2$?

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