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INVARIANT SUBSETS OF THE LIMIT SET FOR A FUCHSIAN GROUP

Dedicated to Professor Tatsuo Fuji'i'e on his sixtieth birthday

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1. Introduction. Let G be a Fuchsian group acting on the unit disc D. In [2], it was shown that G is finitely generated if and only if the limit set for G consists of only points of approximation and parabolic fixed points. Moreover, it is known that the Lebesgue measure of the set of points of approximation is zero if and only if G is of convergence type. Hence, if G is an infinitely generated Fuchsian group of convergence type and of the first kind, then it necessarily has some other kinds of limit points, the set of which has positive Lebesgue measure. There are some papers where properties of such points are discussed (see, e.g. [5], [6] and [8]). In [4], to study limit points for infinitely generated Fuchsian groups, we have decomposed ∂D into three disjoint sets L_1, L_2 and L_3 (for the definition, see §2). In §2, we study L_1, L_2 and L_3 more closely and consider certain measures on them. In §3, we consider a measure with atoms. In §4, we treat some G-invariant measurable subsets of the limit set for G and estimate the Hausdorff dimensions of those in terms of the exponent of convergence of G. In §5, we consider the Riemann surface D/G, which belongs to a certain class of Riemann surfaces, and give estimates for a lower bound of the exponent of convergence of G.

2. G-invariant subsets of the limit set. Let G be a Fuchsian group acting on the unit disc D. Since the number of elements of G is countable, we consider G as an ordered set $\{g_n\}_{n=1}^{\infty}$ of elements g_n. We assume that each element of G, except the identity, does not fix the origin 0. The isometric circle of a Möbius transformation g which does not fix ∞, is given by $|g'(z)|=1$. By $r(g)$ and $c(g)$ we denote the radius and the center of the isometric circle of g, respectively. We define a subset $L(k)$ of ∂D as follows:

$$L(k)=\{\zeta \mid \limsup_{n\to\infty}\frac{r(g_n)^k}{|\zeta-c(g_n)|}=M,\ 0<M<+\infty\}.$$

Note that $L(k)=\varnothing$ for $k>2$. Hence we only consider the case $0\le k\le 2$. A point in $L(2)$ is called a point of approximation. Ordinary points are contained in $L(0)$. We also note that $L(k)$ is independent of the choice of ordering for the elements of G.

LEMMA 1. *The set $L(k)$ is a G-invariant measurable set.*

PROOF. We set

$$f_k(\zeta)=\limsup_{n\to\infty}\frac{r(g_n)^k}{|\zeta-c(g_n)|}$$

on ∂D. Since the term $r(g_n)^k/|\zeta-c(g_n)|$ is continuous as a function of $\zeta\in\partial D$ for each k, we see that $f_k(\zeta)$ is a measurable function. Hence $L(k)$ is a measurable set.

Elementary calculation shows that

$$r(g_nf^{-1})=|f'(c(g_n))|^{1/2}r(g_n) \quad\text{and}\quad c(g_nf^{-1})=f(c(g_n))$$

for $f\in G$. Hence we have

$$\frac{r(g_nf^{-1})^k}{|\zeta-c(g_nf^{-1})|}=|(f^{-1})'(\zeta)|^{1/2}|f'(c(g_n))|^{(k-1)/2}\frac{r(g_n)^k}{|f^{-1}(\zeta)-c(g_n)|}.$$

Since $|f'(c(g_n))|^{(k-1)/2}$ is positive and bounded both above and below except for finitely many g_n, we conclude that $f^{-1}(\zeta)\in L(k)$ if and only if $\zeta\in L(k)$. This completes the proof.

We set

$$L_1:=\bigcup_{k>1}L(k),\quad L_2:=L(1)\quad\text{and}\quad L_3:=\bigcup_{k<1}L(k).$$

By the definition of the isometric circle of g, we see that

$$|g'(\zeta)|=\frac{r(g)^2}{|\zeta-c(g)|^2}.$$

Hence L_1 consists of horocyclic limit points, i.e. the points ζ which satisfy $\limsup|g'(\zeta)|=+\infty$, while L_3 consists of oricyclic limit points, i.e. the points ζ which satisfy $\limsup|g'(\zeta)|=0$. Hyperbolic fixed points are contained in L_1 and parabolic fixed points are contained in L_2.

Let μ be a finite Borel measure on ∂D with the property

$$(*)\qquad d\mu(\zeta)=\frac{1}{|g'(\zeta)|}d\mu(g(\zeta))\qquad\text{for all}\quad g\in G.$$

Through the Poisson integral we see that there exists a one-to-one correspondence between the set of such measures and the set of positive harmonic functions on D/G.

By m we denote the Lebesgue measure on ∂D. Let A be a G-invariant measurable set with $m(A)>0$. Then $\chi_A dm$ satisfies $(*)$, where χ_A is the characteristic function of A.

THEOREM 1. *Let G be a Fuchsian group acting on D. Then the following four statements hold*:

(1) *Each of L_1, L_2 and L_3 is a G-invariant measurable set.*

(2) *Let μ be a finite Borel measure satisfying $(*)$ and with support on L_1. Then L_1 is a conservative piece with respect to μ.*

(3) $m(L_2)=0$.

(4) *Let μ be a finite Borel measure satisfying* $(*)$ *and with support on L_3. Then L_3 is a dissipative piece with respect to μ.*

REMARK. For the Lebesgue measure, the statements (2) and (4) in the above theorem are already in [9].

PROOF. (1) By the definition of L_1, L_2 and L_3, each of them is G-invariant. Since $L_1=\{\zeta\in\partial D\,|\,f_1(\zeta)=+\infty\}$, $L_2=\{\zeta\in\partial D\,|\,0<f_1(\zeta)<+\infty\}$ and $L_3=\{\zeta\in\partial D\,|\,f_1(\zeta)=0\}$, we see that L_1, L_2 and L_3 are measurable.

(2) Recall that L_1 is said to be conservative with respect to μ if, for an arbitrary measurable set $A\subset L_1$ with $\mu(A)>0$, there exist infinitely many $g\in G$ such that $\mu(A\cap g(A))>0$.

First we show that $\sum_{g\in G}\mu(g(A))=+\infty$ implies $\mu(A\cap g(A))>0$ for infinitely many $g\in G$. We assume that there exists $\{g_j\}_{j=0}^{n}$ with $g_0=\mathrm{id}$ such that $\mu(A\cap g(A))=0$ for all $g\in G\setminus\{g_j\}_{j=0}^{n}$. Let ζ be a hyperbolic fixed point of a hyperbolic element h. Since $|h'(\zeta)|\neq 1$, there exists no atomic measure on ζ with the property $(*)$. We set $A_1=A\setminus\{\text{fixed points of } g_j\}_{j=1}^{n}$. Then we have $\mu(A_1)=\mu(A)$. For each $\zeta\in A_1$, there exists an open interval I_ζ such that $I_\zeta\cap g_j(I_\zeta)=\emptyset$ for all g_j, $1\leq j\leq n$. Suppose that $\mu(A_1\cap I_\zeta)=0$, for all $\zeta\in A_1$. The set $\{I_\zeta\}$ is a covering of A_1. Since A_1 is the Lindelöf space, there exists $\{I_{\zeta(n)}\}_{n=1}^{\infty}$ such that $A_1\subset\bigcup_{n=1}^{\infty}I_{\zeta(n)}$. We have

$$0<\mu(A_1)=\mu\left(A_1\cap\left(\bigcup_{n=1}^{\infty}I_{\zeta(n)}\right)\right)\leq\sum_{n=1}^{\infty}\mu(A_1\cap I_{\zeta(n)})=0.$$

This is a contradiction. Hence there exists $\zeta\in A_1$ with $\mu(A_1\cap I_\zeta)>0$. We set $A_2=A_1\cap I_\zeta$. Then we have $\mu(A_2\cap g(A_2))=0$ for all $g\in G\setminus\{\mathrm{id}\}$. Hence we get $0<\sum_{g\in G}\mu(g(A_2))\leq\mu(L_1)<+\infty$.

Next we show that $\sum_{g\in G}\mu(g(A))=+\infty$ for an arbitrary measurable set $A\subset L_1$ with $\mu(A)>0$. For an arbitrary $M>0$, set

$$C(g,M):=\{\zeta\in\partial D\,|\,|g'(\zeta)|>M\}.$$

Since $A\subset L_1$ we have $A\subset\bigcup_{g\in G}C(g,M)$. Hence we get

$$\sum_{g\in G}\mu(g(A))=\sum_{g\in G}\int_A|g'(\eta)|\,d\mu(\eta)\geq\sum_{g\in G}M\mu(C(g,M)\cap A)\geq M\mu(A)>0.$$

Since M is arbitrary, we get $\sum_{g\in G}\mu(g(A))=+\infty$.

(3) We set $D':=D\setminus\{\text{elliptic fixed points of } G\}$. For $z\in D'$ we denote by F_z the Dirichlet fundamental polygon with center at z. Let

$$O:=\{\zeta\in\partial D\,|\,\zeta\in\overline{g(F_z)}\quad\text{for all}\quad z\in D'\quad\text{and for some}\quad g\in G\}.$$

In [4], we showed that $L_3\subset O\subset L_2\cup L_3$ and that if $\zeta\in L_2$, then ζ is a Garnett point (for the definition, see, e.g. [6]) or the point which is in O. In [6], it was shown that the Lebesgue measure of the set of Garnett points is zero. Hence, if $m(L_2)>0$, then $m(L_2\cap O)>0$. On the other hand, [8] showed that $m(O\setminus L_3)=0$. It is a contradiction. Hence we get $m(L_2)=0$.

(4) Recall that L_3 is said to be dissipative with respect to μ, if there exists a measurable set B such that $B\cap g(B)=\varnothing$ for all $g\in G\setminus\{\mathrm{id}\}$ and such that $\mu(L_3)=\mu(\bigcup_{g\in G} g(B))$. Let

$$B_1:=\{\zeta\in\partial D\,|\,|g'(\zeta)|\le 1 \quad \text{for all} \quad g\in G\}\cap L_3=\partial D\cap\bar{F}_0\cap L_3\,.$$

The set B_1 is measurable. Since $L_3\subset O\subset\bigcup_{g\in G} g(\partial D\cap\bar{F}_0)$, we have $L_3=\bigcup_{g\in G} g(B_1)$. Let

$$K:=\{\zeta\in L_3\,|\,\text{there exist at least two elements } g_1 \text{ and } g_2 \text{ with } \zeta\in\overline{g_1(F_0)}\cap\overline{g_2(F_0)}\}\,.$$

In [6], it was shown that $\zeta\in\overline{g_1(F_0)}\cap\overline{g_2(F_0)}$ for $g_1\neq g_2$ if and only if $g_1(0)$ and $g_2(0)$ are on the same horocycle whose point at infinity is ζ. Hence the cardinality of K is at most countable. If μ has atoms, then let $\tilde{P}$ be a set of points on which μ has atoms. Since the cardinality of $\tilde{P}$ is countable, we choose a subset P of $\tilde{P}$ such that points of P are not mutually G-equivalent and that $\tilde{P}=\bigcup_{g\in G} g(P)$. We set $B=(B_1\setminus(K\cup\tilde{P}))\cup P$. Then B is measurable. It is clear that $B\cap g(B)=\varnothing$ for all $g\in G\setminus\{\mathrm{id}\}$ and $\mu(L_3)=\mu(\bigcup_{g\in G} g(B))$. Thus we complete the proof.

3. An atomic part of a measure with the property (*).

First we consider a measure with support on L_3.

LEMMA 2. *Let μ be a finite Borel measure satisfying $(*)$. If μ has an atom at $\eta\in L_3$, then $\sum_{g\in G}|g'(\eta)|<+\infty$.*

Conversely, if $\sum_{g\in G}|g'(\eta)|<+\infty$, then there exists a finite Borel measure satisfying $(*)$ and having an atom at η.

PROOF. We set $E:=\{g(\eta)\,|\,g\in G\}$. Since L_3 contains no fixed point, the points of E are mutually distinct. By the property $(*)$, we have

$$+\infty>\int_E d\mu(\zeta)=\sum_{g\in G}|g'(\eta)|\int_\eta d\mu(\zeta)>0\,.$$

Hence $\sum_{g\in G}|g'(\eta)|<\infty$.

Let us denote by ν_η the unit Dirac measure at η, and let

$$d\mu_{g(\eta)}(\zeta)=\frac{d\nu_{g(\eta)}(\zeta)}{|(g^{-1})'(\zeta)|}\,.$$

We easily see that

$$d\mu_{g(\eta)}(h(\zeta))=|h'(\zeta)|d\mu_{h^{-1}g(\eta)}(\zeta)$$

for $h \in G$. Let

$$d\mu(\zeta) := \sum_{g \in G} d\mu_{g(\eta)}(\zeta).$$

Then $\mu(\{\eta\}) > 0$ and μ is a finite Borel measure satisfying $(*)$. This completes the proof.

Next we consider parabolic fixed points. These points are in L_2.

LEMMA 3. *Let G be a Fuchsian group of convergence type and let h be a parabolic element of G with $h(\eta) = \eta$. Then there exists a finite atomic measure satisfying $(*)$ and with support $\{g(\eta) \mid g \in G\}$.*

PROOF. Let $G' = \{g_n\}$ be a set of representatives of the left cosets with respect to the cyclic subgroup $\langle h \rangle$ generated by h. First we show that $\sum_{g_n \in G'} |(g_n^{-1})'(\eta)| < +\infty$ if and only if G is of convergence type. Note that the value $|(g_n^{-1})'(\eta)|$ is independent of the choice of a representative. We assume $\eta = 1$ and set $\varphi(z) = i(z+1)/(-z+1)$. Then $\tilde{G} = \varphi G \varphi^{-1}$ acts on the upper half plane and $\tilde{h} = \varphi h \varphi^{-1}$ is a parabolic element with $\tilde{h}(\infty) = \infty$. We can represent g and $\tilde{g} = \varphi g \varphi^{-1}$ in terms of matrices in $SL(2, \boldsymbol{C})$ and $SL(2, \boldsymbol{R})$ as

$$\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

respectively. Then we get

$$\alpha = \pm\left(\frac{a+d}{2} + i\frac{b-c}{2}\right) \quad \text{and} \quad \beta = \pm\left(\frac{a-d}{2} + i\frac{b+c}{2}\right).$$

Using these relations, we have

$$|(g^{-1})'(1)| = \frac{1}{|-\beta+\alpha|^2} = \frac{1}{c^2+d^2} = |\tilde{g}'(i)|.$$

It is clear that $\varphi G' \varphi^{-1} = \tilde{G}' = \{\tilde{g}_n\}$ is a set of representatives of the left cosets with respect to the subgroup $\langle \tilde{h} \rangle$ of $\tilde{G}$. In [1], it was proved that $\tilde{G}$ is of convergence type if and only if $\sum_{\tilde{g}_n \in \tilde{G}'} |\tilde{g}_n'(i)| < +\infty$. Hence we are done.

Next we construct an atomic measure on η. As in the proof of Lemma 2, let

$$d\mu(\zeta) := \sum_{g_n \in G'} d\mu_{g_n^{-1}(\eta)}(\zeta).$$

For all $f \in G$, $\{g_n f\}$ is again a set of representatives of the left cosets with respect to $\langle h \rangle$. Thus μ satisfies $(*)$. This completes the proof.

As was stated in the proof of Theorem 1, (2), on a hyperbolic fixed point there exists no atomic part of a measure satisfying $(*)$. Moreover, since $\limsup |g'(\zeta)| = M > 0$, for $\zeta \in L_1 \cup L_2$, the infinite sum $\sum_{g \in G} |g'(\zeta)|$ diverges. Thus, by Lemmas 2 and 3, we have the following:

PROPOSITION 1. *Let G be a Fuchsian group of convergence type. A finite Borel measure with the property $(*)$ has an atom at ζ if and only if ζ is a parabolic fixed point or the point which satisfies $\sum_{g\in G}|g'(\zeta)|<+\infty$.*

The following theorem is a generalization of [8, Theorem 1].

THEOREM 2. *Let μ be a finite Borel measure satisfying $(*)$ and with support on L_3. Then almost all points ζ of L_3 with respect to μ satisfy $\sum_{g\in G}|g'(\zeta)|<+\infty$.*

PROOF. By Theorem 1, (4), there exists a measurable subset B which satisfies $B\cap g(B)=\varnothing$ for all $g\in G\setminus\{\mathrm{id}\}$ and $\mu(L_3)=\mu(\bigcup_{g\in G}g(B))$. Hence we have

$$+\infty>\mu(L_3)=\sum_{g\in G}\int_{g(B)}d\mu(\zeta)=\int_B\sum_{g\in G}|g'(\zeta)|\,d\mu(\zeta)\,.$$

Thus we get $\sum_{g\in G}|g'(\zeta)|<+\infty$ for almost all $\zeta\in B$ with respect to μ. Thus our theorem is proved.

REMARK. If $m(L_3)=0$ and if there exists a measure on L_3 satisfying $(*)$, then the Riemann surface D/G has an unbounded positive harmonic function on it. Hence the Fuchsian group G is of convergence type and of the first kind. But the author does not know whether such a Fuchsian group G exists or not.

4. The Hausdorff dimension and the exponent of convergence. We denote by H-dim(A) the Hausdorff dimension of a set A, while by $\delta(G)$ we denote the exponent of convergence of G, i.e., for an arbitrary $\varepsilon>0$, $\sum_{g\in G}r(g)^{2(\delta(G)+\varepsilon)}$ converges and $\sum_{g\in G}r(g)^{2(\delta(G)-\varepsilon)}$ diverges. We set

$$E(k):=\bigcup_{a\geq k}L(a)\,.$$

In the same way as in the proof of Theorem 1, (1), we see that $E(k)$ is a G-invariant measurable set.

THEOREM 3. *Let G be a Fuchsian group. Then, for $k>0$,*

$$\text{H-dim}\,(E(k))\leq\frac{2\delta(G)}{k}\,.$$

PROOF. We may assume that the origin is not an elliptic fixed point of G. We choose and fix an order $\{g_n\}_{n=0}^{\infty}$ for the elements of G as follows: If $|g_n(0)|<|g_m(0)|$, we let $n<m$. If $|g_n(0)|=|g_m(0)|$, but $\operatorname{Arg}g_n(0)<\operatorname{Arg}g_m(0)$, we again let $n<m$, where $\operatorname{Arg}z=\theta$ for $z=r\exp(i\theta)$, $r\in\boldsymbol{R}$ and $0\leq\theta<2\pi$. We choose and fix an arbitrary positive number λ. For an arbitrary positive number ε, there exists N such that

$$\sum_{n\geq N} r(g)^{2\delta(G)(1+\lambda)} < \varepsilon .$$

We set $t=2\delta(G)(1+\lambda)^2/k$ and $k'=k/(1+\lambda)$. Let ρ be a positive number. Since $k'<k$ and ζ is a limit point, there exists $n(\zeta,\rho)>N$ for each $\zeta\in E(k)$ such that

$$\frac{r(g_{n(\zeta,\rho)})^{k'}}{|\zeta-c(g_{n(\zeta,\rho)})|}>1$$

and $r(g_{n(\zeta,\rho)})^{k'}<\rho$. We set $A_{n(\zeta,\rho)}=\{\eta\,|\,|\eta-c(g_{n(\zeta,\rho)})|<r(g_{n(\zeta,\rho)})^{k'}\}$. Then $\{A_{n(\zeta,\rho)}\}_{\zeta\in E(k)}$ is an open covering of $E(k)$. In the following sum, we consider only distinct $n(\zeta,\rho)$.

$$\sum r(g_{n(\zeta,\rho)})^{k't}=\sum r(g_{n(\zeta,\rho)})^{2\delta(G)(1+\lambda)}<\varepsilon ,$$

For all $\rho>0$, the above inequality holds. Hence we have

$$H_t(E(k))<\varepsilon .$$

where $H_t(\cdot)$ denote the t-dimensional Hausdorff measure. Thus we get H-dim$(E(k))\leq 2\delta(G)/k$. The proof is complete.

Since $L(k)\subset E(k)$, we have:

COLLORARY 1. *Let G be a Fuchsian group. Then, for $k>0$,*

$$\text{H-dim}\,(L(k))\leq\frac{2\delta(G)}{k}.$$

Furthermore, we have:

COLLORARY 2. *Let G be a Fuchsian group with $\delta(G)\leq 1/2$. Then G is of fully accessible type.*

REMARK. In [7, Theorem 3], Patterson proved the above in the case where $\delta(G)<1/2$.

PROOF. Recall that a Fuchsian group G is said to be of fully accessible type, if $m(L_3)=2\pi$. If $\delta(G)\leq 1/2$, then by Theorem 3 we have H-dim$(E(k))\leq 1/k<1$ for $k>1$. This implies $m(E(k))=0$ for $k>1$. Since $E(k)$ increases as k decreasing to 1 and $L_1=\bigcup_{k>1}E(k)$, by the continuity of the Lebesgue measure we get $m(L_1)=0$. Hence we have $m(L_2\cup L_3)=2\pi$. By Theorem 1, (3), we see $m(L_2)=0$. Therefore, we get $m(L_3)=2\pi$. The proof is complete.

5. Harmonic functions on D/G. We say that a G-invariant measurable set $A\subset\partial D$ with $m(A)>0$ is an indivisible set under G, if there exists no G-invariant measurable subset B of A with $m(A)>m(B)>0$. If $m(L_3)>0$, then, by Theorem 1, (4), there exists a measurable set B such that $B\cap g(B)=\varnothing$ for all $g\in G\setminus\{\text{id}\}$ and that $m(L_3)=m(\bigcup_{g\in G}g(B))$.

Let A be a G-invariant measurable set in L_3 with $m(A)>0$. Then $A'=A\cap B$ satisfies $A'\cap g(A')=\varnothing$ for all $g\in G\setminus\{\mathrm{id}\}$ and $m(A)=m(\bigcup_{g\in G}g(A'))$. Since m has no atom, there exists a measurable subset C' of A' with $m(A')>m(C')>0$. We set $C=\bigcup_{g\in G}g(C')$. Then C is a G-invariant measurable set with $m(A)>m(C)>0$. Hence A is not an indivisible set under G. We saw $m(L_2)=0$ already. Thus, if an indivisible set exists, then it must be in L_1. There exists a one-to-one correspondence between indivisible sets under G and minimal bounded harmonic functions on D/G (see [3]). By $O_{\mathrm{HB}N}$ ($N\in\boldsymbol{N}$) we denote the class of Riemann surfaces on which there exist at most N linearly independent bounded harmonic functions. We observe that $O_{\mathrm{HB}1}=O_{\mathrm{HB}}$ is the class of Riemann surfaces on which there exists no non-constant bounded harmonic function. If a Riemann surface belongs to $O_{\mathrm{HB}N}\setminus O_{\mathrm{HB}(N-1)}$ ($N>2$), then there exist N linearly independent minimal bounded harmonic functions on it. From the above argument, we have:

PROPOSITION 2. *Let G be a Fuchsian group. If $D/G\in O_{\mathrm{HBN}}$, then $m(L_1)=2\pi$.*

The converse is not true. M. Taniguchi pointed out that [8, Example 1] is the group for which the converse of Proposition 2 does not hold.

THEOREM 4. *Let G be a Fuchsian group. If $D/G\in O_{\mathrm{HB}}$, then there exists k, $2\delta(G)\geq k>1$, such that $m(L(k))=2\pi$.*

PROOF. By Proposition 2, $m(L_1)=2\pi$. If there is a G-invariant measurable set A, then, by the assumption $D/G\in O_{\mathrm{HB}}$, we have $m(A)=2\pi$ or $m(A)=0$. Since $E(k)$ is a G-invariant measurable set and $E(k+\varepsilon)\subset E(k)$ for all $\varepsilon>0$, there exists $k>1$ such that $m(E(k))=2\pi$ and $m(E(k+\varepsilon))=0$ for all $\varepsilon>0$. Hence $m(L(k))=2\pi$. This shows that H-dim$(L(k))=1$. Thus, by Corollary 1 to Theorem 3, we get $2\delta(G)\geq k>1$. This completes the proof.

THEOREM 5. *Let G be a Fuchsian group. Suppose that a Riemann surface D/G has a minimal bounded harmonic function on it. Then $\delta(G)>1/2$.*

PROOF. Since D/G has a minimal bounded harmonic function on it, there exists an indivisible set A under G on ∂D. Such a set A is contained in L_1. Each $L(k)$ is a G-invariant measurable set. Hence there exists $k>1$ such that $m(A\cap L(k))=m(A)>0$. Thus we have H-dim$(L(k))=1$. By Corollary 1 to Theorem 3, we get $\delta(G)\geq k/2>1/2$. The proof is complete.

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