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STABILITY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS AND SOME VARIATIONAL PROBLEMS

TIBOR KRISZTIN*

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1. Introduction. The direct method of Lyapunov for ordinary differential equations has a natural generalization for functional differential equations (FDE) by introducing Lyapunov functionals [3], [5], [7], [9], [10], [11]. Since functions are much simpler to use, sufficient conditions for stability of solutions of FDE are also given in terms of the rate of change of functions on R^n along solutions. This latter approach is referred to as results of Razumikhin type [1], [7], [9], [10], [14], [17], [18].

Barnea [1] obtained a Razumikhin type stability result for autonomous FDE with finite delay. In order to state his result we need the following noation. Let $r>0$ and define $C=C([-r,0], R^n)$ equipped with the sup norm $\|\cdot\|$. For a continuous function $x: [-r, a)\to R^n$ and $t\in[0, a)$, $x_t\in C$ is defined by $x_t(s)=x(t+s)$, $-r\leq s\leq 0$.

Consider the autonomous FDE

$$\text{(E)} \qquad x'(t)=F(x_t),$$

where $F: C\to R^n$ is continuous and $F(0)=0$. Then function $x=x(\varphi)\in C([-r,\omega), R^n)$, $\omega>0$, is a solution of (E) through $(0,\varphi)$, $\varphi\in C$, on $[0,\omega)$ if $x_0=\varphi$ and (E) holds on $[0,\omega)$. We assume that F has additional properties such that for any $\varphi\in C$ equation (E) has a unique solution through $(0,\varphi)$ on $[0,\infty)$ (see e.g. [5], [7]). We remark that in [1], (E) is considered in the space of measurable bounded functions on $[-r, 0]$ instead of C. In any case, $x_t(\varphi)\in C$ for $t\geq r$. So, for the sake of stability investigations, C can be used as a phase space.

The zero solution of (E) is said to be stable if for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\varphi\in C$, $\|\varphi\|<\delta$, $t\geq 0$ imply $|x(\varphi)(t)|<\varepsilon$.

Let $V: R^n\to R$ be a differentiable function satisfying

$$\omega_1(|x|)\leq V(x)\leq\omega_2(|x|) \qquad (x\in R^n)$$

where ω_1 and ω_2 are increasing functions in $C(R^+, R^+)$ such that $\omega_1(0)=\omega_2(0)=0$ and $\omega_1(u)$, $\omega_2(u)\to\infty$ as $u\to\infty$. For $\varphi\in C$ and an integer $k\geq 0$, define

$$\bar{V}(\varphi)=\max_{-r\leq s\leq kr} V(x(\varphi)(s))$$

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and the set

$$H=\{\psi\in C\colon \bar{V}(\psi)=V(x(\psi)(kr))>0,\ \dot{V}(x(\psi)(kr))>0\}\,.$$

Among others, Barnea [1] proved (see also [7], [9], [10]):

THEOREM A. *If H is empty, then the zero solution of* (E) *is stable.*

In addition, in [1], for some interesting examples, the condition $H=\emptyset$ is reduced to certain optimization problems to get stability regions. It is also clear from the procedure of [1] that the results improve monotonically when the integer k increases. On the other hand, the larger k is chosen, the more complicated the solution of the arising minimization problem becomes. The numerical calculations are much too difficult in the case $k\geq 3$ even for the simple equation

$$x'(t)=-ax(t-r)\,, \tag{1}$$

where $a\geq 0$, $r\geq 0$. The theoretical problem also arises according to Barnea's paper: whether the achieved stability regions for a given k approach the entire ones as $k\to\infty$. We emphasize that the entire stability region $0\leq ar\leq\pi/2$ for (1) can be easily obtained from the characteristic equation of (1). Here we show that a Razumikhin type idea gives the region $0\leq ar<\pi/2$. Moreover, this idea works for certain equations where the characteristic equations become very difficult and also for some nonlinear equations.

While we were looking for sufficient conditions for $H=\emptyset$ in some examples, we found certain variational problems playing crucial role in the proofs. These problems are different from those of [1] obtained for special equations in the case $k=2$.

The purpose of this paper is twofold. First, we formulate the arising nonstandard variational problems which can be interesting in their own right. Unfortunately, we can solve the proposed problems only in a particular case. Secondly, as a consequence of this particular result, it will be shown that the above mentioned stability region $0\leq ar<\pi/2$ can be obtained for (1) as $k\to\infty$, and more generally, that the condition $\int_{-r}^{0}|s|d\mu(s)<\pi/2$ is sufficient for the stability of the zero solution of the scalar equation

$$x'(t)=-\int_{-r}^{0}x(t+s)d\mu(s)\,,$$

where $\mu\colon[-r,0]\to R$ is nondecreasing on $[-r,0]$ and continuous to the left on $(-r,0)$. In the last section we indicate possible extensions of the idea and the difficulties of the variational problems corresponding to a nonlinar example.

2. Variational problems. First we show how to get a variational problem considering equation (1). Choose $V(x)=x^2/2$ and fix an integer $k\geq 1$. Suppose that H is nonempty, i.e. there is $\varphi\in C$ such that for $x=x(\varphi)$ one has $|x(kr)|=\max_{-r\leq s\leq kr}|x(s)|>0$ and $(d/dt)x^2(kr)/2=-ax(kr)x((k-1)r)>0$. Since $x'(t)$ and $x(t-r)$ have different signs, there must be $t^*>kr$ with

$$|x(t^*)|=\max_{-r\le s\le t^*}|x(s)|>0\,,\qquad \frac{d}{dt}\frac{x^2(t^*)}{2}=-ax(t^*)x(t^*-r)=0\,.$$

Then $x(t^*-r)=0$. From (1) it is also easy to see that $x_{ir}\in C^i([-r,0],R)$, $i=1,2,\cdots$, i.e. x is i-times continuously differentiable on $[(i-1)r, ir]$. Since (1) is linear, without loss of generality we may assume that $x(t^*)=1$. Then, from $|x(t)|\le 1$ for $t\in[-r,t^*]$ and from (1), we clearly have the constraints on x

(2) $$x(t^*-r)=0\,,\quad |x^{(i)}(t)|\le a^i\qquad (i=0,1,\cdots,k-1;\, t\in[t^*-2r,t^*-r])$$

and

(3) $$x(t^*-r)=x'(t^*)=0\,,\quad |x^{(i)}(t)|\le a^i\qquad (i=0,1,\cdots,k;\, t\in[t^*-r,t^*])\,.$$

It is reasonable to expect that (2) and (3) imply the inequalities

$$x(t)\ge y(t)\quad (t\in[t^*-2r,t^*-r])\,,$$
$$x(t)\le z(t)\quad (t\in[t^*-r,t^*])$$

with certain y and z, respectively. Of course, y and z may depend on a, r and k. There are two possibilities to contradict $H\ne\emptyset$. From (1) one obtains

$$1=x(t^*)-x(t^*-r)=\int_{t^*-r}^{t^*}x'(s)ds=-a\int_{t^*-r}^{t^*}x(s-r)ds$$
$$=-a\int_{t^*-2r}^{t^*-r}x(s)ds\le -a\int_{t^*-2r}^{t^*-r}y(s)ds\,.$$

If we can show that either $-a\int_{t^*-2r}^{t^*-r}y(s)ds<1$ or $z(t^*)<1$, then we have a contradiction. Those values of a and r, for which one of the last two inequalities holds, belong to the stability region of (1).

Conditions (2) and (3) yield the motivation to consider the following problems. Let $\{b_i\}_{i=0}^\infty$, $\{c_i\}_{i=0}^\infty$ and T be given sequences of nonnegative numbers and a positive number, respectively. By m we denote either a positive integer or $+\infty$. Define the sets

$$S(T,\{b_i\},\{c_i\},m)=\{f\in C^m([0,T],R)\colon f(0)=0,\ -b_i\le f^{(i)}(t)\le c_i,\ 0\le i\le m,\ t\in[0,T]\}\,,$$
$$S_0(T,\{b_i\},\{c_i\},m)=\{f\in C^m([0,T],R)\colon f(0)=f'(T)=0,$$
$$-b_i\le f^{(i)}(t)\le c_i,\ 0\le i\le m,\ t\in[0,T]\}\,,$$

where $0\le i\le m$ means $i=0,1,\cdots$ when $m=+\infty$.

PROBLEM 1. *Find or approximate*

$$\sup\{f(t)\colon f\in S(T,\{b_i\},\{c_i\},m)\}\quad (t\in[0,T])\,.$$

PROBLEM 2. *Find or approximate*

$$\sup\{f(t) \colon f \in S_0(T, \{b_i\}, \{c_i\}, m)\} \quad (t \in [0, T]) .$$

We have only the following particular answers for the above problems.

LEMMA 1. *For any $c > 1$ there exists a positive integer $m = m(c)$ such that*

$$\sup\left\{f(t) \colon f \in S\left(\frac{\pi}{2c}, \{1\}, \{1\}, m\right)\right\} \leq \sin ct \quad \left(t \in \left[0, \frac{\pi}{2c}\right]\right) .$$

LEMMA 2. *For $t \in [0, \pi/2]$*

$$\sup\left\{f(t) \colon f \in S\left(\frac{\pi}{2}, \{1\}, \{1\}, \infty\right)\right\} \leq \sin t .$$

LEMMA 3. *For any $c > 1$ there exists a positive integer $m' = m'(c)$ such that if $0 < T \leq \pi/2c$, then*

$$\sup\{f(t) \colon f \in S_0(T, \{1\}, \{1\}, m')\} \leq \sin ct \quad (t \in [0, T]) .$$

LEMMA 4. *If $0 < T \leq \pi/2$, then*

$$\sup\left\{f(t) \colon f \in S_0\left(\frac{\pi}{2}, \{1\}, \{1\}, \infty\right)\right\} \leq \sin t \quad (t \in [0, T]) .$$

Notice that $\sin t \in S_0(\pi/2, \{1\}, \{1\}, \infty) \subset S(\pi/2, \{1\}, \{1\}, \infty)$, and thus Lemmas 2 and 4 give sharp results.

PROOF OF LEMMA 1. Let $c > 1$ be fixed and define the sequence $\{\alpha_i\}_{i=-1}^{\infty}$ by $\alpha_{-1} = 0$ and

$$0 \leq \alpha_i \leq \frac{\pi}{2c}, \quad \left|\frac{d^i}{dt^i} \sin ct \Big|_{t=\alpha_i}\right| = 1 \qquad (i = 0, 1, \cdots) .$$

Then $0 < \alpha_1 < \alpha_3 < \cdots < \pi/2c$, $0 < \cdots < \alpha_2 < \alpha_0 = \pi/2c$ and $\alpha_{2i+1} \to \pi/2c$, $\alpha_{2i} \to 0$ as $i \to \infty$. Define

$$m(c) = 1 + \max\{l \in N \colon \text{if } i, j \text{ are integers with } 0 \leq 2i+1 \leq l \text{ and } 0 \leq 2j \leq l \text{ then } \alpha_{2i+1} < \alpha_{2j}\} .$$

In the following we use the elementary fact that if $u, v \in C^1([t_1, t_2], R)$, $u(t_1) \leq v(t_1)$, $u(t_2) \geq v(t_2)$ and one of the inequalities is strict, then there is $\xi \in (t_1, t_2)$ such that $u'(\xi) > v'(\xi)$.

Assume that Lemma 1 is false for c with the above defined $m = m(c)$, i.e. there exist $f \in S(\pi/2c, \{1\}, \{1\}, m)$ and $t_0 \in (0, \pi/2c)$ such that $f(t_0) > \sin ct_0$. Let $t_{0,1} = 0$, $t_{0,2} = t_0$ and $t_{0,3} = \alpha_0 = \pi/2c$. Then

$$0 = t_{0,1} < t_{0,2} < t_{0,3} = \alpha_0, \; f(t_{0,1}) = \sin ct_{0,1}, \; f(t_{0,2}) > \sin ct_{0,2}, \; f(t_{0,3}) \leq \sin ct_{0,3}$$

(see Figure 1). Thus, there are $t_{1,2} \in (t_{0,1}, t_{0,2})$, $t_{1,3} \in (t_{0,2}, t_{0,3})$ and $t_{1,1}$ such that

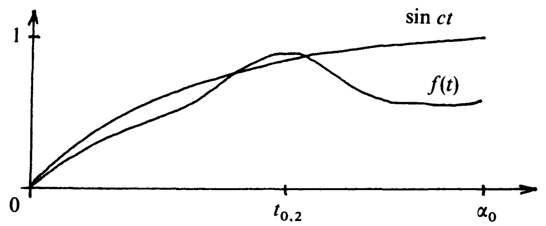

FIGURE 1.

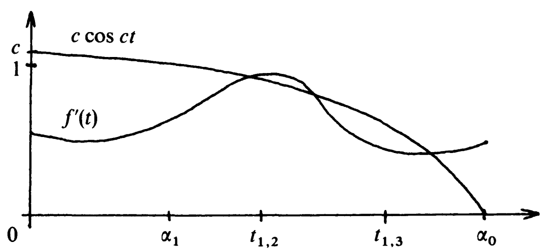

FIGURE 2.

$$\alpha_1 = t_{1,1} < t_{1,2} < t_{1,3} < \alpha_0,\ f'(t_{1,1}) \le c \cos ct_{1,1}\ ,$$
$$f'(t_{1,2}) > c \cos ct_{1,2},\ f'(t_{1,3}) < c \cos ct_{1,3}$$

because of the definition of α_1 (see Figure 2). The procedure can be continued up to $m-1$ to obtain $t_{i,1}$, $t_{i,2}$ and $t_{i,3}$, $i=0, 1, \cdots, m-1$, such that one of the conditions

$$(4)\qquad \begin{aligned}
&\alpha_{i-1} \le t_{i,1} < t_{i,2} < t_{i,3} = \alpha_i,\ f^{(i)}(t_{i,1}) \le c^i \sin ct_{i,1},\\
&\quad f^{(i)}(t_{i,2}) > c^i \sin ct_{i,2},\ f^{(i)}(t_{i,3}) \le c^i \sin ct_{i,3}\ ;\\
&\alpha_i = t_{i,1} < t_{i,2} < t_{i,3} < \alpha_{i-1},\ f^{(i)}(t_{i,1}) \le c^i \cos ct_{i,1},\\
&\quad f^{(i)}(t_{i,2}) > c^i \cos ct_{i,2},\ f^{(i)}(t_{i,3}) < c^i \cos ct_{i,3}\ ;\\
&\alpha_{i-1} < t_{i,1} < t_{i,2} < t_{i,3} = \alpha_i,\ f^{(i)}(t_{i,1}) > -c^i \sin ct_{i,1},\\
&\quad f^{(i)}(t_{i,2}) < -c^i \sin ct_{i,2},\ f^{(i)}(t_{i,3}) \ge -c^i \sin ct_{i,3}\ ;\\
&\alpha_i = t_{i,1} < t_{i,2} < t_{i,3} < \alpha_{i-1},\ f^{(i)}(t_{i,1}) \ge -c^i \cos ct_{i,1},\\
&\quad f^{(i)}(t_{i,2}) < -c^i \cos ct_{i,2},\ f^{(i)}(t_{i,3}) > -c^i \cos ct_{i,3}
\end{aligned}$$

holds according as $i=4l$, $i=4l+1$, $i=4l+2$ or $i=4l+3$ (see Figures 1–4 for $i=0, 1$, 2, 3).

Assume that $m-1=4l$. Then (4) with $i=m-1$, implies the existence of $\tau \in (t_{m-1,1}, t_{m-1,2})$ such that

$$(5)\qquad f^{(m)}(\tau) < c^m \cos c\tau\ .$$

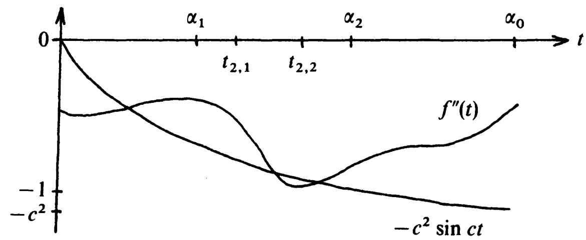

FIGURE 3.

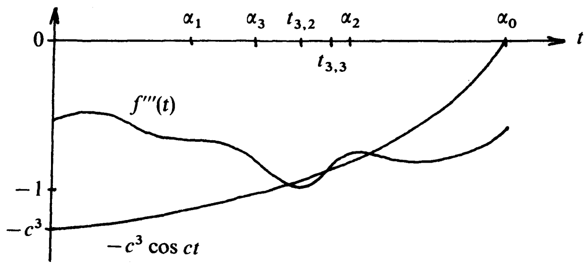

FIGURE 4.

From the definition of m and (4) it follows that $\tau < \alpha_{m-1} \leq \alpha_m$. On the other hand, $c^m \cos c\alpha_m = (d^m/dt^m) \sin ct|_{t=\alpha_m} = 1$. Therefore, by (5) and $\tau < \alpha_{m-1}$, one obtains $f^{(m)}(\tau) > 1$, which contradicts $f \in S(\pi/2c, \{1\}, \{1\}, m)$. The other cases, $m-1 = 4l+1$, $4l+2$ and $4l+3$ also lead to a contradiction. The proof is complete.

PROOF OF LEMMA 3. Let $c > 1$ and $T \in (0, \pi/2c]$ be fixed and define the sequence $\{\beta_i\}_{i=-1}^{\infty}$ and $m' = m'(c)$ by

$$\beta_{2i+1} = \alpha_{2i+1}, \quad \beta_{2i} = \min\{T, \alpha_{2i}\} \qquad (i = -1, 0, 1, \cdots),$$

$$m' = 1 + \max\{l \in N\colon \text{if } i, j \text{ are integers with } 0 \leq 2i+1 \leq l,\ 0 \leq 2j \leq l \text{ then } \beta_{2i+1} < \beta_{2j}\},$$

where $\{\alpha_i\}$ is the same as in the proof of Lemma 1.

If the statement is not true for c and T with m', then there are $f \in S_0(T, \{1\}, \{1\}, m')$ and $t_0 \in (0, T)$ such that $f(t_0) > \sin ct_0$. From $f(0) = 0$, $f(t_0) > \sin ct_0$ it follows that there exists $t_{1,2} \in (0, t_0)$ such that $f'(t_{1,2}) > c \cos ct_{1,2}$. Choosing $t_{1,1} = \beta_1$ and $t_{1,3} = T = \beta_0$, we obtain

$$\beta_1 = t_{1,1} < t_{1,2} < t_{1,3} = \beta_0,\ f'(t_{1,1}) \leq c \cos ct_{1,1},\ f'(t_{1,2}) > c \cos ct_{1,2},\ f'(t_{1,3}) \leq c \cos ct_{1,3},$$

since $\beta_1 = \alpha_1$, $f'(t_{1,3}) = 0$ and $c \cos ct_{1,3} \geq 0$. From this point, in order to get a contradiction, one can follow the same procedure as in the proof of Lemma 1. The proof is complete.

Lemmas 2 and 4 can be obtained by letting $c \to 1$ in Lemmas 1 and 3, respectively.

Now, returning to equation (1), assume $0 \leq ar < \pi/2$ and choose $c > 1$ and $T > 0$ such that $ar \leq T < \pi/2c$. Introducing the functions

$$f(t) = -x\left(-\frac{2cr}{\pi}t + t^* - r\right) \quad \left(t \in \left[0, \frac{\pi}{2c}\right]\right),$$

$$g(t) = x\left(\frac{r}{T}t + t^* - r\right) \qquad (t \in [0, T]),$$

from (2) and (3) it follows that $f \in S(\pi/2c, \{1\}, \{1\}, k-1)$ and $g \in S_0(T, \{1\}, \{1\}, k)$. If $k \geq \max\{m(c)+1, m'(c)\}$, then Lemmas 1 and 3 imply $f(t) \leq \sin ct$, $0 \leq t \leq \pi/2c$, and $g(t) \leq \sin ct$, $0 \leq t \leq T$, that is,

$$x(t) \geq y(t) := \sin\frac{\pi}{2r}(t + r - t^*) \quad (t \in [t^* - 2r, t^* - r]),$$

$$x(t) \leq z(t) := \sin\frac{cT}{r}(t + r - t^*) \quad (t \in [t^* - r, t^*]).$$

It can be easily seen that $-a\int_{t^*-2r}^{t^*-r} y(s)ds = 2ar/\pi < 1$ and $z(t^*) = \sin cT < 1$. Therefore, $0 \leq ar < \pi/2$ implies the stability of the zero solution of (1).

Remark that $\{a \geq 0, r \geq 0: ar \leq \pi/2\}$ is the entire region of stability for (1). Barnea [1] obtained the region $\{a \geq 0, r \geq 0: ar < 3/2\}$ by choosing $k = 2$ (see also [7]).

3. An example with distributed delay. The stability region $\{a \geq 0, r \geq 0: ar \leq \pi/2\}$ for (1) can be obtained easily by using pole locations. Now we consider the equation

$$x'(t) = -\int_{-r}^{0} x(t+s)d\mu(s), \tag{6}$$

which is still linear and autonomous. Nevertheless, the pole location becomes extremely difficult, and neither the entire region of stability nor a good approximation of it is given explicitly (as far as we know). In (6) we assume that $r > 0$, $\mu: [-r, 0] \to R$ is nondecreasing on $[-r, 0]$ and continuous to the left on $(-r, 0)$. By introducing a linear transformation in t, if necessary, it can be assumed that $\int_{-r}^{0} d\mu = 1$. For the stability region of the zero solution of (6) we have:

THEOREM 1. *If* $\int_{-r}^{0} |s| d\mu(s) < \pi/2$, *then the zero solution of* (6) *is stable.*

PROOF. Theorem A will be applied. Choose $c > 1$ so that

$$\int_{-r}^{0} |s| d\mu(s) < \frac{\pi}{2c} + 1 - \frac{1}{c}. \tag{7}$$

Let $V(x) = x^2/2$ and $k = 2 + \max\{m, m'\}$, where $m = m(c)$ and $m' = m'(c)$ are given by Lemmas 1 and 3, respectively. It suffices to show that $H = \varnothing$ by Theorem A. Suppose that H with the above prescribed V and k is nonempty, i.e. there is $\varphi\in C$ such that for $x=x(\varphi)$

$$|x(kr)|=\max_{-r\le s\le kr}|x(s)|>0\,,\qquad \frac{d}{dt}\frac{x^2(kr)}{2}=-x(kr)\int_{-r}^{0}x(kr+s)d\mu(s)>0\,.$$

By (6), one cannot have $x'(t)x(t+s)>0$ for all $s\in[-r,0]$. Hence, a $t^*>kr$ can be obtained such that

$$|x(t^*)|=\max_{-r\le s\le t^*}|x(s)|>0\,,\qquad \frac{d}{dt}\frac{x^2(t^*)}{2}=-x(t^*)\int_{-r}^{0}x(t^*+s)d\mu(s)=0\,.$$

Without loss of generality we may assume $x(t^*)=1$, since (6) is linear. Then, from (6) it follows that there is $T\in(0,r]$ satisfying $x(t^*-T)=0$ and $x(t)>0$ for $t\in(t^*-T,t^*]$. We have $t^*-T-r\ge(k-2)r$, and thus $x\in C^{k-2}([t^*-T-r,t^*],R)$. Equation (6), $x(t^*)=1$, $\int_{-r}^{0}d\mu=1$ and $\max_{-r\le s\le t^*}|x(s)|\le 1$ imply the inequalities

$$|x^{(i)}(t)|\le 1\qquad (i=0,1,\cdots,k-2;\ t\in[t^*-T-r,t^*])\,.$$

Defining

$$f(t)=x(t^*-T+t)\qquad (0\le t\le T)\,,$$
$$g(t)=1-x(t^*-t)\qquad \left(0\le t\le\frac{\pi}{2c}\right),$$
$$h(t)=-x(t^*-T-t)\qquad \left(0\le t\le\frac{\pi}{2c}\right),$$

it is clear that $f\in S_0(T,\{1\},\{1\},m')$. If $T<\pi/2c$, then by Lemma 3, $1=x(t^*)=f(T)\le \sin cT<\sin(\pi/2)=1$, a contradiction. Therefore, $T\ge\pi/2c$, and consequently, $g,h\in S(\pi/2c,\{1\},\{1\},m)$.

From Lemma 1 we obtain $g(t),h(t)\le\sin ct$, $0\le t\le\pi/2c$. Then it follows that x can be estimated from below:

$$x(t)\ge y(t)\quad(-r\le t\le t^*)\,,$$

where

$$y(t)=\begin{cases}-1 & \text{if}\quad -r\le t\le t^*-T-\pi/2c\\ \sin c(t-(t^*-T)) & \text{if}\quad t^*-T-\pi/2c\le t\le t^*-T\\ 0 & \text{if}\quad t^*-T\le t\le t^*-\pi/2c\\ 1+\sin c(t-t^*) & \text{if}\quad t^*-\pi/2c\le t\le t^*\,.\end{cases}$$

Integrating (6) over $[t^*-T,t^*]$ and taking into account $x(t^*-T)=0$, $x(t^*)=1$, we conclude

$$1=\int_{t^*-T}^{t^*} x'(t)dt=-\int_{t^*-T}^{t^*}\int_{-r}^{0} x(t+s)d\mu(s)dt \tag{8}$$
$$=-\int_{-r}^{0}\int_{t^*-T+s}^{t^*+s} x(t)dtd\mu(s)\leq-\int_{-r}^{0}\int_{t^*-T+s}^{t^*+s} y(t)dtd\mu(s)\,.$$

Now we estimate the integral $I(s)=\int_{t^*-T+s}^{t^*+s} y(t)dt$. For any fixed $s\in[-r,0]$, the interval $[t^*-T+s, t^*+s]$ can be decomposed into subintervals on which y has simple primitive functions and the integrals are easily evaluated. Omitting certain tedious (but elementary) calculations, we get

$$I(s)=s+\frac{\pi}{2c}-\frac{1}{c}\quad(s\in[-T,0])\,.$$

If $r\leq T+\pi/2c$, then

$$I(s)=s+\frac{\pi}{2c}-\frac{1}{c}\cos(s+T)\geq s+\frac{\pi}{2c}-\frac{1}{c}\quad(s\in[-r,T])\,.$$

If $r>T+\pi/2c$, then

$$I(s)=s+\frac{\pi}{2c}-\frac{1}{c}\cos(s+T)\geq s+\frac{\pi}{2c}-\frac{1}{c}\quad\left(s\in\left[-T-\frac{\pi}{2c},-T\right]\right)$$

and

$$I(s)=-T>-T-\frac{\pi}{2c}+\frac{\pi}{2c}-\frac{1}{c}\geq s+\frac{\pi}{2c}-\frac{1}{c}\quad\left(s\in\left[-r,-T-\frac{\pi}{2c}\right]\right).$$

Therefore, by using inequality (8) and $\int_{-r}^{0}d\mu=1$,

$$1\leq-\int_{-r}^{0}\left(s+\frac{\pi}{2c}-\frac{1}{c}\right)d\mu(s)=\frac{1}{c}-\frac{\pi}{2c}+\int_{-r}^{0}|s|d\mu(s)$$

follows, contradicting (7). Thus, $\int_{-r}^{0}|s|d\mu(s)<\pi/2$ implies stability.

4. Remarks. 1. Gopalsamy [6] used pole location to get the stability region $r\int_{-r}^{0}d\mu<\pi/2$ for a special case of (6). In [12] we applied a Razumikhin type technique (which corresponds to the case $k=1$ of this paper) to obtain stability regions for nonautonomous and nonlinear equations. Especially [12] gives the stability region $\int_{-r}^{0}|s|d\mu(s)\leq 3/2$ for (6). For related results see [7], [15], [16], [20], [21], [22], [23].

2. Asymptotic stability results can also be obtained by introducing standard modifications in the above technique. For example, $0<\int_{-r}^{0}|s|d\mu(s)<\pi/2$ implies

asymptotic stability for (6).

3. If (E) is a linear equation and $\det D(z)=0$ is its characteristic equation, then $\det D(z)\neq 0$ for $\operatorname{Re} z\geq 0$ implies asymptotic stability. The technique of this paper can be applied for pole location, i.e. to prove that $\det D(z)\neq 0$ for $\operatorname{Re} z\geq 0$. If there is z_0 with $\operatorname{Re} z_0\geq 0$ and $\det D(z_0)=0$, then $e^{z_0 t}v$, $v\in C^n$, $v\neq 0$, is a C^∞ solution of (E) on $(-\infty, \infty)$. The existence of the solution $e^{z_0 t}v$ leads to Problems 1 and 2 with $m=\infty$. The solution of these problems, especially Lemmas 2 and 4, can be used to get a contradiction similarly to that of Section 3.

This remark allows us to show that $\int_{-r}^{0}|s|d\mu(s)=\pi/2$ also belongs to the stability region of (6). So, $\int_{-r}^{0}|s|d\mu(s)\leq\pi/2$ implies the stability of the zero solution of (6) by Theorem 1. The constant $\pi/2$ is the best possible one in this result. However, $\int_{-r}^{0}|s|d\mu(s)\leq\pi/2$ is not the entire region of stability for (6) whenever (6) is different from $x'(t)=-ax(t-r)$.

It is also possible to obtain stability results for linear autonomous infinite delay equations and Volterra equations of convolution type. As an example, for the scalar equation

$$x'(t)=-ax(t)-\int_0^t b(t-s)x(s)ds\,,$$

where $a\geq 0$, $b\in C(R_+, R_+)$, one can show that $a+\int_0^\infty b(s)ds>0$ and $\int_0^\infty sb(s)<\pi/2$ imply uniform asymptotic stability of the zero solution. This improves certain results of [2], [4], [8], [13].

4. The motivation of this paper was to consider not only linear but also nonlinear equations. We illustrate the difficulties on the nonlinear scalar equation

$$x'(t)=-\alpha(e^{x(t-1)}-1)\,, \tag{9}$$

where $\alpha\geq 0$. Assume that x is a solution of (9) on $[0, 2]$ and $-m\leq x(t)\leq M$ for $t\in[-1, 1]$, $m>0$, $M>0$. Then

$$\begin{aligned} &-\alpha(e^M-1)\leq x'(t)\leq\alpha(1-e^{-m}) \quad (t\in[1,2])\,,\\ &-\alpha^2e^M(1-e^{-m})\leq x''(t)\leq\alpha^2e^M(e^M-1) \quad (t\in[1,2])\,. \end{aligned} \tag{10}$$

It is possible to continue the procedure to get inequalities also for higher derivatives of $x(t)$. But (10) already shows that the main difficulty here is that the sequences $\{b_i\}$, $\{c_i\}$ in Problems 1 and 2 become much more complicated than those for (1) and (6). We can solve Problem 1 corresponding to equation (9) only for $m=2$. This gives asymptotic stability of the zero solution of (9) for $0<\alpha<1+1/e$, which is far from $0<\alpha\leq 3/2$ proved by Wright [19] and from $0<\alpha<\pi/2$ conjectured by many authors.

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Bolyai Institute
Aradi vértanúk tere 1.
6720 Szeged
Hungary