

ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF SCHRÖDINGER OPERATORS WITH NONCLASSICAL POTENTIALS

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1. Introduction. Let Δ be the Laplacian in the Euclidean space \mathbf{R}^n , that is, $\Delta = \sum_{i=1}^n \partial^2 / \partial z_i^2$. Let $V(z)$ be a nonnegative function defined on \mathbf{R}^n . Suppose that the set $\{z \in \mathbf{R}^n; V(z) = 0\}$ is an unbounded subset of \mathbf{R}^n . Our aim is to give an estimate for the asymptotic distribution of eigenvalues of the Schrödinger operator $-\Delta + V(z)$. Several results on this problem are known (cf. for example, Robert [8], Simon [10] and Solomyak [12]).

In this paper we restrict our attention to the potential of the form

$$(1.1) \quad V(x, y) = C \prod_{i=1}^p f_i(|x|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y|)^{\beta_j} \cdot |x|^\gamma |y|^\delta,$$

where $x = (x_1, \dots, x_{m_1}) \in \mathbf{R}^{m_1}$, $y = (y_1, \dots, y_{m_2}) \in \mathbf{R}^{m_2}$, $|x| = (\sum_{i=1}^{m_1} x_i^2)^{1/2}$, $|y| = (\sum_{j=1}^{m_2} y_j^2)^{1/2}$ and $m_1 + m_2 = n$ with some conditions on f_i , g_j , α_i , β_j , γ and δ .

Our main result is given in Section 3. Special cases of our estimates are closely related to some results studied by Robert, Simon and others.

The case $V(x, y) = C \prod_{i=1}^p f_i(|x|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y|)^{\beta_j}$ is a classical one and the asymptotic distribution of eigenvalues is given by the well-known formula (cf. Rozenbljum [9]).

The case $V(x, y) = (1 + |x|^2)^\alpha |y|^{2\beta}$ is studied by Robert [8] by means of pseudo-differential operator calculus with operator symbols. Our method is quite different from his. The results will be given as corollaries when $\alpha m_2 \geq \beta m_1$ in Section 3.

The case $V(x, y) = |x|^\alpha |y|^\beta$ is studied by Simon [10] when $m_1 = m_2 = 1$. The case $m_1, m_2 \geq 2$ is included in the results of Solomyak [12]. Our method gives another proof of their results when $\alpha m_2 = \beta m_1$. The result is given in Corollary 3.1.

In order to prove the main theorem we shall use classical Dirichlet-Neumann bracketing method formulated by Edmunds and Evans [2]. We shall also apply a simple modification of Theorem 2 of Fefferman [3; p. 144], where he gives several estimates for the eigenvalues of Schrödinger operators with polynomial potentials. We shall apply Fefferman's theorem to operators with A_∞ -weight potentials and use it in the proof of Lemmas 3.2 and 3.2'.

In Section 2 we shall show some properties of A_∞ -weights. These properties will be used in Sections 3 and 4. In Section 3 we shall state our main theorem and give the

proof assuming several lemmas. In Sections 4 and 5 we shall prove these lemmas in Section 3.

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2. A_∞ -weight potentials. Let Ω be an open set in \mathbf{R}^n . By $L^2(\Omega)$ we shall denote the Lebesgue space of all square integrable functions in Ω . By $H^1(\Omega)$ we shall denote the Sobolev space

$$H^1(\Omega) = \left\{ u \in L^2(\Omega); \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, \dots, n \right\}$$

where $\partial/\partial x_i$ denote distributional derivatives. We put

$$|\nabla u(z)|^2 = \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i}(z) \right|^2$$

for $u \in H^1(\Omega)$ and $z \in \Omega$. By $C_0^\infty(\Omega)$ we shall denote the space of all infinitely differentiable functions with compact support in Ω . For a set S in \mathbf{R}^n , $|S|$ denotes the Lebesgue measure of S . By cubes in \mathbf{R}^n we shall mean closed cubes whose sides are parallel to the coordinate axes.

Let us recall the definition of A_∞ -weights.

DEFINITION. A nonnegative locally integrable function $w(z)$ on \mathbf{R}^n is called an A_∞ -weight on \mathbf{R}^n if there exist positive constants C and δ such that

$$(2.1) \quad \frac{\int_S w(z) dz}{\int_Q w(z) dz} \leq C \left(\frac{|S|}{|Q|} \right)^\delta$$

for all cubes Q in \mathbf{R}^n and for all measurable subsets S of Q . We call the pair (C, δ) of constants A_∞ -constants of w . We denote the space of all A_∞ -weights on \mathbf{R}^n by $A_\infty(\mathbf{R}^n)$ or A_∞ .

We now mention some properties of A_∞ -weights which are useful in proving that our potential V belongs to A_∞ . For the proof we refer to [4; Chap. IV].

LEMMA 2.1. *Let $w(z) \geq 0$ be locally integrable on \mathbf{R}^n . Then the following conditions are equivalent:*

- (1) $w \in A_\infty$.
- (2) There exist $0 < C_1, C_2 < 1$ such that

$$\left| \left\{ z \in Q; w(z) \leq C_1 \frac{1}{|Q|} \int_Q w(y) dy \right\} \right| \leq C_2 |Q|$$

for every cube Q .

(3) There exists $C > 0$ such that

$$\frac{1}{|Q|} \int_Q w(z) dz \leq C \exp\left(\frac{1}{|Q|} \int_Q \log w(z) dz\right)$$

for every cube Q .

(4) There exist $C > 0$ and $\varepsilon > 0$ such that

$$\left(\frac{1}{|Q|} \int_Q w(z)^{1+\varepsilon} dz\right)^{1/(1+\varepsilon)} \leq \frac{C}{|Q|} \int_Q w(z) dz$$

for every cube Q .

REMARK 2.1. By Hölder's and Jensen's inequalities, $w \in A_\infty$ is equivalent to saying that

$$\frac{1}{|Q|} \int_Q w(z) dz \sim \left(\frac{1}{|Q|} \int_Q w(z)^{1+\varepsilon} dz\right)^{1/(1+\varepsilon)} \sim \exp\left(\frac{1}{|Q|} \int_Q \log w(z) dz\right)$$

for every Q , where the bounds are independent of Q .

LEMMA 2.2. Let u and v be A_∞ -weights. Then we have the following:

- (1) If $\alpha, \beta > 0$, then $\alpha u + \beta v \in A_\infty$.
- (2) If $0 < \alpha < 1$, then $u^\alpha \in A_\infty$.
- (3) If $u^2, v^2 \in A_\infty$, then $uv \in A_\infty$.

Lemma 2.2 is a direct consequence of Lemma 2.1 but we give a proof for convenience.

PROOF. (1) follows from the Hardy-Littlewood maximal theorem with weights, but follows directly from the definition of A_∞ -weights. Let (C', δ') and (C'', δ'') be A_∞ -constants of u and v , respectively. Then $C'|S|^{\delta'} \int_Q u dz \geq |Q|^{\delta'} \int_S u dz$ for every subset S of a cube Q and a similar inequality holds for v with constants (C'', δ'') . Thus, adding both sides, we get (2.1) for $\alpha u + \beta v$ with constants $C = \max(C', C'')$ and $\delta = \min(\delta', \delta'')$.

(2) Assume $0 < \alpha < 1$. Fix a cube Q . By Hölder's inequality

$$\frac{1}{|Q|} \int_Q u(z)^\alpha dz \leq \left(\frac{1}{|Q|} \int_Q u(z) dz\right)^\alpha,$$

which, by Lemma 2.1 (3), does not exceed

$$\left(C \exp\left(\frac{1}{|Q|} \int_Q \log u(z) dz\right) \right)^\alpha = C^\alpha \exp\left(\frac{1}{|Q|} \int_Q \log u(z)^\alpha dz\right).$$

Thus $u^\alpha \in A_\infty$.

(3) By Schwartz's inequality

$$\frac{1}{|Q|} \int_Q u(z)v(z) dz \leq \left(\frac{1}{|Q|} \int_Q u(z)^2 dz\right)^{1/2} \left(\frac{1}{|Q|} \int_Q v(z)^2 dz\right)^{1/2}.$$

Applying Lemma 2.1 (3) to each term on the right hand side, we get

$$\begin{aligned} \frac{1}{|Q|} \int_Q u(z)v(z) dz &\leq C \exp\left(\frac{1}{2|Q|} \int_Q (\log u(z)^2 + \log v(z)^2) dz\right) \\ &= C \exp\left(\frac{1}{|Q|} \int_Q \log u(z)v(z) dz\right), \end{aligned}$$

which proves (3).

q.e.d.

LEMMA 2.3. Let $P_{ij}(z)$ be polynomials on \mathbf{R}^n of degrees d_{ij} , where $i=1, \dots, q$ and $j=1, \dots, r$. Let α_{ij}, β_{ij} and γ_{ij} be positive numbers. Let

$$f_i(z) = \sum_{j=1}^r \alpha_{ij} |P_{ij}(z)|^{\beta_{ij}}, \quad i=1, \dots, q,$$

and

$$w(z) = \prod_{i=1}^q f_i(z)^{\gamma_i}.$$

Then $w(z)$ is an A_∞ -weight on \mathbf{R}^n and the A_∞ -constants depend only on $n, d_{ij}, \beta_{ij}, \gamma_{ij}$ and q .

PROOF. First we observe the following: if $P(z)$ is a polynomial on \mathbf{R}^n and $\alpha > 0$, then $|P(z)|^\alpha \in A_\infty$. Indeed, we have

$$(2.2) \quad \frac{1}{|Q|} \int_Q |P(z)| dz \leq \max_{z \in Q} |P(z)| \leq \frac{C}{|Q|} \int_Q |P(z)| dz$$

for every cube Q , where C is a constant depending only on n and the degree of P (cf. [3; p. 146]). Thus Lemma 2.1 (4) holds for every $\varepsilon > 0$. Thus $|P(z)|^\alpha \in A_\infty$ for $\alpha = 1, 2, \dots$. By Lemma 2.2 (2), this holds for every $\alpha > 0$.

Next we observe the following: if $P_j(z), j=1, \dots, h$ are polynomials on \mathbf{R}^n , then $\prod_{j=1}^h |P_j(z)|^{\alpha_j} \in A_\infty$ for every $\alpha_j > 0$. Since $|P_1(z)|^{2\alpha_1} \in A_\infty$ and $|P_2(z)|^{2\alpha_2} \in A_\infty$, we have $|P_1(z)|^{\alpha_1} |P_2(z)|^{\alpha_2} \in A_\infty$ by Lemma 2.2 (3). The case $h > 2$ is shown similarly.

Therefore, by Lemma 2.2 (1), $f_i(z)^\gamma \in A_\infty$ for $\gamma = 1, 2, \dots$. By Lemma 2.2 (2),

$f_i(z)^\gamma \in A_\infty$ for every $\gamma > 0$. Applying the preceding argument, we can show $w(z) \in A_\infty$.
 q.e.d.

COROLLARY TO LEMMA 2.3. *Let $w(z)$ be the function given in Lemma 2.3. Then there exists a positive constant C depending only on $n, d_{ij}, \beta_{ij}, \gamma_i$ and q such that*

$$\frac{1}{|Q|} \int_Q w(z) dz \leq \max_{z \in Q} w(z) \leq C \frac{1}{|Q|} \int_Q w(z) dz$$

for all cubes Q in \mathbb{R}^n .

PROOF. It suffices to show the second inequality. By the definition of w we have

$$\max_{z \in Q} w(z) \leq \prod_{i=1}^q \left(\sum_{j=1}^r \alpha_{ij} \left(\max_{z \in Q} |P_{ij}(z)| \right)^{\beta_{ij}} \right)^{\gamma_i}.$$

Since $|P_{ij}|$ are A_∞ -weights, by (2.2) and Lemma 2.1 (3), the last term does not exceed

$$\begin{aligned} & \prod_{i=1}^q \left(\sum_{j=1}^r \alpha_{ij} \left(C_{1ij} \frac{1}{|Q|} \int_Q |P_{ij}(z)| dz \right)^{\beta_{ij}} \right)^{\gamma_i} \\ & \leq \prod_{i=1}^q \left(\sum_{j=1}^r \alpha_{ij} \left(C_{1ij} C_{2ij} \exp \left(\frac{1}{|Q|} \int_Q \log |P_{ij}(z)| dz \right) \right)^{\beta_{ij}} \right)^{\gamma_i} \\ & \leq C_3 \prod_{i=1}^q \left(\sum_{j=1}^r \alpha_{ij} \left(\exp \left(\frac{1}{|Q|} \int_Q \log |P_{ij}(z)| dz \right) \right)^{\beta_{ij}} \right)^{\gamma_i}, \end{aligned}$$

where C_{1ij} depend only on n and d_{ij} , while $C_3 = \prod_{i=1}^q (\max_j (C_{1ij} C_{2ij})^{\beta_{ij}})^{\gamma_i}$. By Jensen's inequality the last term does not exceed

$$C_3 \prod_{i=1}^q \left(\frac{1}{|Q|} \int_Q \sum_{j=1}^r \alpha_{ij} |P_{ij}(z)|^{\beta_{ij}} dz \right)^{\gamma_i} = C_3 \prod_{i=1}^q \left(\frac{1}{|Q|} \int_Q f_i(z) dz \right)^{\gamma_i}.$$

Note that f_i are A_∞ -weights. Applying Lemma 2.1 (3) again to the last term and arguing similarly as above, we get an estimate

$$\max_{z \in Q} w(z) \leq C_4 \frac{1}{|Q|} \int_Q \prod_{i=1}^q f_i(z)^{\gamma_i} dz \leq C_4 \frac{1}{|Q|} \int_Q w(z) dz,$$

where C_4 depends only on $n, d_{ij}, \beta_{ij}, \gamma_i$ and q .
 q.e.d.

The following Lemmas 2.4 and 2.6 are modifications of Theorems 2 and 3 in Fefferman [3; p. 144], respectively.

LEMMA 2.4. *Let $U(z)$ be an A_∞ -weight on \mathbb{R}^n . Put*

$$\lambda_1 = \inf_{\substack{a > 0 \\ \xi \in \mathbb{R}^n}} \left(a^{-2} + a^{-n} \int_{|z-\xi| < a/2} U(z) dz \right).$$

Suppose that $\lambda_1 > 0$. Then

$$C\lambda_1 \int_Q |v(z)|^2 dz \leq \int_Q (|\nabla v(z)|^2 + U(z)|v(z)|^2) dz$$

for all cubes Q in \mathbf{R}^n with side length $2(\lambda_1)^{-1/2}$ and for all $v \in H^1(\dot{Q})$, where C is a positive constant depending only on n and the A_∞ -constants for $U(z)$, and \dot{Q} denotes the interior of Q .

To prove Lemma 2.4 we use the following lemma.

LEMMA 2.5 (Morimoto [6]). Let Q be a cube in \mathbf{R}^n and let $U(z)$ be a nonnegative measurable function on Q . Suppose that there exist positive constants C_1 and C_2 such that

$$(2.3) \quad C_1|Q| \leq |\{z \in Q; C_2l(Q)^{-2} \leq U(z)\}|,$$

where $l(Q)$ denotes the side length of Q . Then we have

$$Cl(Q)^{-2} \int_Q |v(z)|^2 dz \leq \int_Q (|\nabla v(z)|^2 + U(z)|v(z)|^2) dz$$

for all $v \in H^1(\dot{Q})$, where C is a positive constant depending only on n , C_1 and C_2 .

PROOF OF LEMMA 2.4. Let Q be a cube in \mathbf{R}^n with $l(Q) = 2(\lambda_1)^{-1/2}$ and center z^0 . Put $a = 2\lambda_1^{-1/2}$ and $\xi = z^0$. Then, by the definition of λ_1 , we get

$$\lambda_1 \leq \frac{1}{4} \lambda_1 + \frac{1}{|Q|} \int_Q U dz.$$

Therefore

$$\frac{3}{4} \lambda_1 \leq \frac{1}{|Q|} \int_Q U dz.$$

Thus

$$(2.4) \quad 3l(Q)^{-2} \leq \frac{1}{|Q|} \int_Q U dz.$$

Since U is an A_∞ -weight in \mathbf{R}^n , we have, by Lemma 2.1 (2),

$$C_1|Q| \leq \left| \left\{ z \in Q; C_2 \frac{1}{|Q|} \int_Q U dz \leq U(z) \right\} \right|$$

where C_1 and C_2 are positive constants depending only on n and the A_∞ -constants of U . Combining this with (2.4), we have

$$C_1|Q| \leq |\{z \in Q; 3C_2l(Q)^{-2} \leq U(z)\}|.$$

Therefore U and Q satisfy the inequality (2.3). Thus Lemma 2.4 follows from Lemma 2.5. q.e.d.

Now, in order to consider the distribution of the eigenvalues of Schrödinger operators with A_∞ -weight potentials, we introduce some notation.

Let U be an A_∞ -weight. Suppose that an operator $-\Delta + U$ which is defined on $C_0^\infty(\mathbf{R}^n)$ is essentially selfadjoint in $L^2(\mathbf{R}^n)$ and L is a selfadjoint realization of $-\Delta + U$. Assume that L has only discrete spectrum. Let λ be a positive number and let $N(\lambda, U)$ be the number of eigenvalues of L less than λ . Let \mathcal{F}_λ be a tessellation of \mathbf{R}^n by cubes whose side length is $\lambda^{-1/2}$ and whose vertices are points in $\lambda^{-1/2}\mathbf{Z}^n$ where \mathbf{Z} is the set of integers. Let $N_1(\lambda, U)$ be the number of cubes Q in \mathcal{F}_λ such that

$$\frac{1}{|Q|} \int_Q U(z) dz < \lambda.$$

LEMMA 2.6. *Assume that U satisfies the above conditions. Then we have*

$$N_1(C_1\lambda, U) \leq N(\lambda, U) \leq N_1(C_2\lambda, U)$$

for every positive number λ , where C_1 is a constant depending only on n , while C_2 is a constant depending only on n and the A_∞ -constants of U .

We omit the proof of Lemma 2.6. The reader may follow the arguments of the proof of Theorem 3 in [3; p. 148] if he applies Lemma 2.5 in place of Main Lemma in [3; p. 146].

REMARK 2.2. Lemma 2.4 shows that Theorem 2 in [3; p. 144] is also valid for A_∞ -weight potentials. This follows easily from the proof of Theorem 2 in [3].

REMARK 2.3. Let $U(z)$ be an A_∞ -weight on \mathbf{R}^n . Suppose that $-\Delta + U$ defined on $C_0^\infty(\mathbf{R}^n)$ is essentially selfadjoint in $L^2(\mathbf{R}^n)$ and L is a selfadjoint realization of $-\Delta + U$. If $N_1(\lambda, U) < \infty$ for all $\lambda > 0$, then L has only discrete spectrum. This fact is verified in a manner similar to the proof for Remark 4 in Simon [11; p. 215].

REMARK 2.4. Let $w(z)$ be the function given in Lemma 2.3. Let $N_2(\lambda, w)$ be the number of cubes in \mathcal{F}_λ such that

$$\max_{z \in Q} w(z) < \lambda.$$

for $\lambda > 0$. Then

$$N_2(\lambda, w) \leq N_1(\lambda, w) \leq N_2(C\lambda, w)$$

for every positive number λ , where C is a constant independent of λ . The first inequality is obvious and the second inequality follows from Corollary to Lemma 2.3.

3. Main theorem. Let p and q be positive integers. Let

$$f_i(t) = \sum_{k=0}^{d_i} a_{ik} t^k, \quad (i=1, \dots, p),$$

$$g_j(t) = \sum_{s=0}^{h_j} b_{js} t^s, \quad (j=1, \dots, q),$$

where d_i and h_j are nonnegative integers. We assume that $a_{ik}, b_{js} \geq 0, (0 < k < d_i, 0 < s < h_j; 1 \leq i \leq p, 1 \leq j \leq q)$ $a_{i0}, b_{j0} > 0$ and $a_{id_i} = b_{jh_j} = 1$. We put

$$(3.1) \quad V(z) = V(x, y) = C \prod_{i=1}^p f_i(|x|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y|)^{\beta_j} \cdot |x|^\gamma |y|^\delta,$$

where $z = (x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} = \mathbf{R}^n, m_1 > 0, m_2 > 0,$ and $\alpha_i, \beta_j, \gamma$ and δ are nonnegative numbers and $C > 0$ is a constant. To avoid trivial cases, we assume $\sum_{i=1}^p \alpha_i d_i + \gamma \neq 0$ and $\sum_{j=1}^q \beta_j h_j + \delta \neq 0$.

By Lemma 2.3 V is an A_∞ -weight on \mathbf{R}^n . The operator $-\Delta + V$ defined on $C_0^\infty(\mathbf{R}^n)$ is essentially selfadjoint in $L^2(\mathbf{R}^n)$, since $V \geq 0$ and $V \in L^2_{loc}(\mathbf{R}^n)$ (cf. Kato [5]), where $L^2_{loc}(\mathbf{R}^n)$ denotes the set of all functions square integrable on every compact subset of \mathbf{R}^n . Let L be a selfadjoint realization of $-\Delta + V$. Then L has only discrete spectrum. Indeed, we can show easily that $N_2(\lambda, V) < \infty$ for all $\lambda > 0$ where $N_2(\lambda, V)$ is the quantity defined in Remark 2.4. Thus, by Remarks 2.3 and 2.4, the assertion follows.

Now we give an asymptotic formula for $N(\lambda, V)$ which, by definition, is the number of eigenvalues of L less than λ and denoted simply by $N(\lambda)$. Our main result is the following:

THEOREM. *Let V be the potential given by (3.1). Suppose that $\gamma m_2 \leq (\sum_{j=1}^q \beta_j h_j + \delta) m_1$ and $\delta m_1 \leq (\sum_{i=1}^p \alpha_i d_i + \gamma) m_2$. Set $\mu_1 = 2^{-1}(2 + \delta)(\sum_{i=1}^p \alpha_i d_i + \gamma)^{-1}$ and $\mu_2 = 2^{-1}(2 + \gamma)(\sum_{j=1}^q \beta_j h_j + \delta)^{-1}$. Then*

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy \quad \text{as } \lambda \rightarrow \infty,$$

where ω_n is the volume of the unit ball in \mathbf{R}^n and the set A is defined as follows:

(1) *If $\gamma \neq 0$ and $\delta \neq 0$, then*

$$A = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda, |x| \leq C_1 \lambda^{\mu_1}, |y| \leq C_2 \lambda^{\mu_2}\}$$

where C_1 is a positive constant depending only on $m_2, C, d_i, \alpha_i, b_{j0}, \beta_j, \gamma$ and δ , while C_2 is a positive constant depending only on $m_1, C, h_j, \beta_j, a_{i0}, \alpha_i, \gamma$ and δ .

(2) *If $\gamma = 0$ and $\delta \neq 0$, then*

$$A = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda, |x| \leq C_3 \lambda^{\mu_1}\}$$

where C_3 is a positive constant depending only on $m_2, C, d_i, \alpha_i, b_{j0}, \beta_j$ and δ .

(3) *If $\gamma \neq 0$ and $\delta = 0$, then*

$$A = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda, |y| \leq C_4 \lambda^{\mu_2}\}$$

where C_4 is a positive constant depending only on $m_1, C, h_j, \beta_j, a_{i0}, \alpha_i$ and γ .

(4) If $\gamma = 0$ and $\delta = 0$, then

$$A = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda\}.$$

COROLLARY 3.1. Let $\alpha, \beta > 0$ and $\alpha m_2 = \beta m_1$. Let

$$V(x, y) = |x|^\alpha |y|^\beta$$

for $(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$. Then

$$N(\lambda) \sim a \lambda^\theta \log \lambda \quad \text{as } \lambda \rightarrow \infty,$$

where $\theta = n/2 + m_1/\alpha$ and

$$a = \frac{(2 + \alpha + \beta)\Gamma(m_1/\alpha)}{2^{n-1} \alpha \beta \Gamma(m_1/2) \Gamma(m_2/2) \Gamma(n/2 + m_1/\alpha + 1)}.$$

COROLLARY 3.2 ([8; Theorem 3.2 (i)]). Let $\alpha, \beta > 0$ and $\alpha m_2 > \beta m_1$. Let

$$V(x, y) = (1 + |x|^2)^\alpha |y|^{2\beta}$$

for $(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} = \mathbf{R}^n$. Then

$$N(\lambda) \sim a \lambda^\theta \quad \text{as } \lambda \rightarrow \infty,$$

where $\theta = n/2 + m_2/(2\beta)$ and

$$a = \frac{\Gamma(m_2/(2\beta))}{2^n \pi^{m_1/2} \beta \Gamma(m_2/2) \Gamma(\theta + 1)} \int_{\mathbf{R}^{m_1}} (1 + |x|^2)^{-\alpha m_2/(2\beta)} dx.$$

COROLLARY 3.3 ([8; Theorem 3.2 (ii)]). Let $\alpha, \beta > 0$ and $\alpha m_2 = \beta m_1$. Let

$$V(x, y) = (1 + |x|^2)^\alpha |y|^{2\beta}$$

for $(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} = \mathbf{R}^n$. Then

$$N(\lambda) \sim a \lambda^\theta \log \lambda \quad \text{as } \lambda \rightarrow \infty,$$

where $\theta = n/2 + m_2/(2\beta)$ and

$$a = \frac{(1 + \beta)\Gamma(m_2/(2\beta))}{2^n \alpha \beta \Gamma(m_1/2) \Gamma(m_2/2) \Gamma(\theta + 1)}.$$

REMARK. Our constants in the corollaries are different from those in Robert [8]. A careful calculation will lead to our constants.

Let Ω be an open set in \mathbf{R}^n and V be the function given by (3.1). Define

$$t[u, v] = \int_{\Omega} (\nabla u \cdot \nabla v + Vu\bar{v}) dx dy$$

and

$$\|u\|_{t, \Omega}^2 = t[u, u] + \int_{\Omega} |u|^2 dx dy$$

for appropriate functions u and v . By $D_{\mathcal{D}, \Omega}$ and $D_{\mathcal{N}, \Omega}$ we denote the completions with respect to the norm $\| \cdot \|_{t, \Omega}$ of $C_0^\infty(\Omega)$ and the restriction of $C_0^\infty(\mathbf{R}^n)$ to Ω , respectively. Let $t_{\mathcal{D}}$ and $t_{\mathcal{N}}$ be sesquilinear extensions of t to $D_{\mathcal{D}, \Omega}$ and $D_{\mathcal{N}, \Omega}$, respectively. Then $t_{\mathcal{D}}$ and $t_{\mathcal{N}}$ are closed and semibounded forms. Let $T_{\mathcal{D}, \Omega}$ and $T_{\mathcal{N}, \Omega}$ be associated selfadjoint operators with respect to $t_{\mathcal{D}}$ and $t_{\mathcal{N}}$, respectively (cf. [2; p. 139]). Let $\Delta_{\mathcal{D}, \Omega}$ and $\Delta_{\mathcal{N}, \Omega}$ be the Dirichlet and Neumann Laplacian on Ω , respectively. If there is no confusion, we drop the notation Ω , for example, and we denote $T_{\mathcal{D}}$ instead of $T_{\mathcal{D}, \Omega}$.

Let T be a selfadjoint operator in $L^2(\Omega)$. For $\lambda > 0$ let

$$N(\lambda, T, \Omega) = \text{rank} \int_{-\infty}^{\lambda} dE_{\mu}(T),$$

where $E_{\mu}(T)$ is the resolution of the identity corresponding to T .

In this notation we prove the theorem.

PROOF OF THEOREM. First we prove (1). Let λ be a large positive number. Let \mathcal{F}'_{λ} be a tessellation of \mathbf{R}^n by cubes Q whose side length is $\lambda^{-1/2}(\log \lambda)^{1/n}$ and whose vertices are points in $\lambda^{-1/2}(\log \lambda)^{1/n}\mathbf{Z}^n$.

Let $B^i = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; x_i = 0\}$ ($i = 1, \dots, m_1$), $B_j = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; y_j = 0\}$ ($j = 1, \dots, m_2$) and $B = (\bigcup_{i=1}^{m_1} B^i) \cup (\bigcup_{j=1}^{m_2} B_j)$. Let \mathcal{S}_1 be all cubes Q in \mathcal{F}'_{λ} such that $\min_{z \in Q} V(z) \leq \lambda$ and $Q \cap B = \emptyset$. Let \mathcal{S}_2 be all cubes in \mathcal{F}'_{λ} such that $\max_{z \in Q} V(z) \leq \lambda$. Let K_1 and K_2 be positive constants which will be determined later. Let \mathcal{S}_3 be all cubes Q in $\mathcal{F}'_{\lambda} \setminus \mathcal{S}_1$ such that $\min_{z \in Q} V(z) \leq \lambda$ and

$$Q \subset \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; |x| \leq K_1 \lambda^{\mu_1}, |y| \leq K_2 \lambda^{\mu_2}\},$$

where μ_1 and μ_2 are constants defined in the theorem.

Let K_3 and K_4 be positive constants which will be determined later and put

$$F_1 = \left\{ (x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; (x, y) \notin \bigcup_{Q \in \mathcal{S}_1 \cup \mathcal{S}_3} Q, |x_i| < K_3 \lambda^{-1/2}, i = 1, \dots, m_1 \right\},$$

$$F_2 = \left\{ (x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; (x, y) \notin \bigcup_{Q \in \mathcal{S}_1 \cup \mathcal{S}_3} Q, |y_j| < K_4 \lambda^{-1/2}, j = 1, \dots, m_2 \right\},$$

and

$$F_3 = \mathbf{R}^n \setminus \text{the closure of } \left(\left(\bigcup_{Q \in \mathcal{J}_1 \cup \mathcal{J}_3} Q \right) \cup F_1 \cup F_2 \right).$$

Note that if λ is sufficiently large, then $F_1 \cap F_2 = \emptyset$.

Now we estimate $N(\lambda)$. Remark that $N(\lambda) = N(\lambda, L, \mathbf{R}^n)$ by definition. We have $L = T_{\emptyset, \mathbf{R}^n} = T_{\mathcal{N}, \mathbf{R}^n}$ since $-\Delta + V$ defined on $C_0^\infty(\mathbf{R}^n)$ is essentially selfadjoint in $L^2(\mathbf{R}^n)$. Therefore

$$(3.2) \quad N(\lambda) = N(\lambda, T_{\mathcal{N}}, \mathbf{R}^n) = N(\lambda, T_{\emptyset}, \mathbf{R}^n).$$

Let $\Omega_1, \Omega_2, \Omega_3$ and Ω_4 be open sets in \mathbf{R}^n and let Ω be the interior of the closure of $\Omega_1 \cup \Omega_2$. Suppose that $\Omega_1 \cap \Omega_2 = \emptyset, |\Omega \setminus (\Omega_1 \cup \Omega_2)| = 0$ and $\Omega_3 \subset \Omega_4$. By an argument similar to that in Edmunds and Evans [2; p. 143], we get

$$\begin{aligned} N(\lambda, T_{\mathcal{N}}, \Omega) &\leq N(\lambda, T_{\mathcal{N}}, \Omega_1) + N(\lambda, T_{\mathcal{N}}, \Omega_2), \\ N(\lambda, T_{\emptyset}, \Omega) &\geq N(\lambda, T_{\emptyset}, \Omega_1) + N(\lambda, T_{\emptyset}, \Omega_2), \end{aligned}$$

and

$$N(\lambda, T_{\emptyset}, \Omega_3) \leq N(\lambda, T_{\emptyset}, \Omega_4).$$

Therefore, by (3.2),

$$(3.3) \quad \sum_{Q \in \mathcal{J}_2} N(\lambda, T_{\emptyset}, Q) \leq N(\lambda) \leq \sum_{Q \in \mathcal{J}_1} N(\lambda, T_{\mathcal{N}}, Q) + \sum_{Q \in \mathcal{J}_3} N(\lambda, T_{\mathcal{N}}, Q) + \sum_{i=1}^3 N(\lambda, T_{\mathcal{N}}, F_i).$$

We have the following three estimates for $N(\cdot, \cdot, \cdot)$.

LEMMA 3.1. $N(\lambda, T_{\mathcal{N}}, F_3) = 0$.

LEMMA 3.2. $N(\lambda, T_{\mathcal{N}}, F_1) = N(\lambda, T_{\mathcal{N}}, F_2) = 0$.

LEMMA 3.3.

$$\begin{aligned} \sum_{Q \in \mathcal{J}_2} N(\lambda, T_{\emptyset}, Q) &\sim \sum_{Q \in \mathcal{J}_1} N(\lambda, T_{\mathcal{N}}, Q) + \sum_{Q \in \mathcal{J}_3} N(\lambda, T_{\mathcal{N}}, Q) \\ &\sim \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy \quad \text{as } \lambda \rightarrow \infty \end{aligned}$$

where

$$A = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda, |x| \leq K_1 \lambda^{\mu_1}, |y| \leq K_2 \lambda^{\mu_2}\}.$$

We shall postpone the proof of these three lemmas to the following sections. By (3.3), Lemmas 3.1, 3.2 and 3.3

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy \quad \text{as } \lambda \rightarrow \infty.$$

Thus the proof of (1) is complete.

We prove (2). Let \mathcal{F}_1 and \mathcal{F}_2 be the subsets of \mathcal{F}'_λ defined in the proof of (1). Let K'_1 be a positive constant which will be determined later. Let \mathcal{F}'_3 be the set of all cubes Q in $\mathcal{F}'_\lambda \setminus \mathcal{F}_1$ such that $\min_{z \in Q} V(z) \leq \lambda$ and

$$Q \subset \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; |x| \leq K'_1 \lambda^{\mu_1}\},$$

where μ_1 is a constant defined in the theorem. Let K'_4 be a positive constant which will be determined later and put

$$F'_2 = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; (x, y) \notin \bigcup_{Q \in \mathcal{F}'_1 \cup \mathcal{F}'_3} Q, |y_j| < K'_4 \lambda^{-1/2}, j=1, \dots, m_2\},$$

and

$$F'_3 = \mathbf{R}^n \setminus \text{the closure of } \left(\left(\bigcup_{Q \in \mathcal{F}'_1 \cup \mathcal{F}'_3} Q \right) \cup F'_2 \right).$$

An argument similar to that in the proof of (1) shows that

$$(3.4) \quad \sum_{Q \in \mathcal{F}_2} N(\lambda, T_\varnothing, \dot{Q}) \leq N(\lambda) \\ \leq \sum_{Q \in \mathcal{F}'_1} N(\lambda, T_{\mathcal{N}}, \dot{Q}) + \sum_{Q \in \mathcal{F}'_3} N(\lambda, T_{\mathcal{N}}, \dot{Q}) + \sum_{i=2}^3 N(\lambda, T_{\mathcal{N}}, F'_i).$$

As before we have:

LEMMA 3.1'.

$$N(\lambda, T_{\mathcal{N}}, F'_3) = 0.$$

LEMMA 3.2'.

$$N(\lambda, T_{\mathcal{N}}, F'_2) = 0.$$

LEMMA 3.3'.

$$\sum_{Q \in \mathcal{F}_2} N(\lambda, T_\varnothing, \dot{Q}) \sim \sum_{Q \in \mathcal{F}'_1} N(\lambda, T_{\mathcal{N}}, \dot{Q}) + \sum_{Q \in \mathcal{F}'_3} N(\lambda, T_{\mathcal{N}}, \dot{Q}) \\ \sim \frac{\omega_n}{(2\pi)^n} \int_{A'} (\lambda - V)^{n/2} dx dy \quad \text{as } \lambda \rightarrow \infty$$

where

$$A' = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda, |x| \leq K'_1 \lambda^{\mu_1}\}.$$

We shall postpone the proof of these three lemmas again to the following sections. By (3.4), Lemmas 3.1', 3.2' and 3.3'

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int_{A'} (\lambda - V)^{n/2} dx dy \quad \text{as } \lambda \rightarrow \infty .$$

Thus the proof of (2) is complete.

We get the proof for (3) if we interchange x and y in the definition of $V(x, y)$.

We now prove (4). Let \mathcal{F}_1 and \mathcal{F}_2 be the subsets of \mathcal{F}'_λ defined in the proof of (1). Let \mathcal{F}''_3 be the set of all cubes Q in $\mathcal{F}'_\lambda \setminus \mathcal{F}_1$ such that $\min_{z \in Q} V(z) \leq \lambda$. Let

$$F''_3 = \mathbf{R}^n \setminus \text{the closure of } \bigcup_{Q \in \mathcal{F}_1 \cup \mathcal{F}''_3} Q .$$

An argument similar to that in the proof of (1) shows that

$$(3.5) \quad \sum_{Q \in \mathcal{F}_2} N(\lambda, T_Q, \mathcal{Q}) \leq N(\lambda) \leq \sum_{Q \in \mathcal{F}_1} N(\lambda, T_Q, \mathcal{Q}) + \sum_{Q \in \mathcal{F}''_3} N(\lambda, T_Q, \mathcal{Q}) .$$

LEMMA 3.3''.

$$\begin{aligned} \sum_{Q \in \mathcal{F}_2} N(\lambda, T_Q, \mathcal{Q}) &\sim \sum_{Q \in \mathcal{F}_1} N(\lambda, T_Q, \mathcal{Q}) + \sum_{Q \in \mathcal{F}''_3} N(\lambda, T_Q, \mathcal{Q}) \\ &\sim \frac{\omega_n}{(2\pi)^n} \int_{A''} (\lambda - V)^{n/2} dx dy \quad \text{as } \lambda \rightarrow \infty \end{aligned}$$

where

$$A'' = \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda\} .$$

This lemma is proved in Section 5. By (3.5) and Lemma 3.3''

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int_{A''} (\lambda - V)^{n/2} dx dy \quad \text{as } \lambda \rightarrow \infty .$$

Thus the proof of (4) is complete.

q.e.d.

4. Proof of Lemmas 3.1, 3.2, 3.1' and 3.2'.

PROOF OF LEMMA 3.1. First we assume that

$$(4.1) \quad V(x, y) > \lambda \quad \text{for all } (x, y) \in F_3 .$$

Then we obviously have

$$\int_{F_3} (|\nabla u|^2 + V|u|^2) dx dy > \lambda \int_{F_3} |u|^2 dx dy ,$$

for all $u \in H^1(F_3)$, $u \neq 0$. This proves Lemma 3.1.

Now we prove (4.1). Suppose contrarily that there exists a point (x_0, y_0) in F_3 such that

$$(4.2) \quad V(x_0, y_0) \leq \lambda.$$

Then there exists a cube Q in \mathcal{F}'_λ such that $(x_0, y_0) \in Q$ and $\min_{z \in Q} V(z) \leq \lambda$. By the definition of F_3 this cube Q does not belong to $\mathcal{F}_1 \cup \mathcal{F}_3$. Therefore, there exists a point $(x_1, y_1) \in Q$ such that $|x_1| > K_1 \lambda^{\mu_1}$ or $|y_1| > K_2 \lambda^{\mu_2}$, where K_1, K_2, μ_1 and μ_2 are constants given in the definition of \mathcal{F}_3 .

Suppose $|x_1| > K_1 \lambda^{\mu_1}$. Since the side length of Q is $\lambda^{-1/2}(\log \lambda)^{1/n}$,

$$\inf\{|x|; (x, y) \in Q\} \geq |x_1| - m_1^{1/2} \lambda^{-1/2}(\log \lambda)^{1/n}.$$

Observe that the right hand side is not less than

$$K_1 \lambda^{\mu_1} - m_1^{1/2} \lambda^{-1/2}(\log \lambda)^{1/n} > (K_1/2) \lambda^{\mu_1}$$

if λ is sufficiently large. Therefore

$$(4.3) \quad \inf\{|x|; (x, y) \in Q\} \geq (K_1/2) \lambda^{\mu_1}.$$

Thus, by (4.2) and the assumptions on f_i and g_j ,

$$\lambda \geq V(x_0, y_0) = C \prod_{i=1}^p f_i(|x_0|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y_0|)^{\beta_j} \cdot |x_0|^\gamma |y_0|^\delta \geq C \prod_{j=1}^q b_{j0}^{\beta_j} \cdot |x_0|^{\sum \alpha_i d_i + \gamma} |y_0|^\delta.$$

By (4.3) the last term is not less than

$$C \prod_{j=1}^q b_{j0}^{\beta_j} \cdot ((K_1/2) \lambda^{\mu_1})^{\sum \alpha_i d_i + \gamma} |y_0|^\delta = C_1 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1 + \delta/2} |y_0|^\delta,$$

where $C_1 = C \prod_{j=1}^q b_{j0}^{\beta_j} \cdot 2^{-(\sum \alpha_i d_i + \gamma)}$. Therefore

$$|y_0| \leq C_2 K_1^{-(\sum \alpha_i d_i + \gamma)/\delta} \lambda^{-1/2}$$

where $C_2 = C_1^{-1/\delta}$.

If we choose K_1 and K_4 so that

$$(4.4) \quad C_2 K_1^{-(\sum \alpha_i d_i + \gamma)/\delta} < K_4,$$

then

$$|y_0| < K_4 \lambda^{-1/2}.$$

Thus, for all components y_{0j} ($j=1, \dots, m_2$) of y_0 we have $|y_{0j}| < K_4 \lambda^{-1/2}$. Hence $(x_0, y_0) \in F_2$. This contradicts $(x_0, y_0) \in F_3$.

If $|y_1| > K_2 \lambda^{\mu_2}$, then a similar argument shows that

$$|x_{0i}| < K_3 \lambda^{-1/2}, \quad x_0 = (x_{01}, \dots, x_{0m_1}),$$

under the condition

$$(4.5) \quad C_3 K_2^{-(\sum \beta_j h_j + \delta)/\gamma} < K_3,$$

where $C_3 = (C \prod_{i=1}^p a_{i0}^{\alpha_i} \cdot 2^{-(\sum \beta_j \mu_j + \delta)})^{-1/\gamma}$. Therefore $(x_0, y_0) \in F_1$ and this contradicts $(x_0, y_0) \in F_3$. Thus (4.1) holds under the conditions (4.4) and (4.5). We shall give exact values of K_1, K_2, K_3 and K_4 satisfying (4.4) and (4.5) later. q.e.d.

PROOF OF LEMMA 3.2. We prove $N(\lambda, T_{\mathcal{N}}, F_2) = 0$. First we show

$$(4.6) \quad \inf\{|x|; (x, y) \in F_2\} > (K_1/2)\lambda^{\mu_1}.$$

Let $(x, y) \in F_2$. Choose Q in \mathcal{F}'_λ so that $(x, y) \in Q$. Since

$$|y_j| < K_4 \lambda^{-1/2}, \quad y = (y_1, \dots, y_{m_2})$$

by the definition of F_2 and since the side length of Q is $\lambda^{-1/2}(\log \lambda)^{1/n}$, we have $(x, 0) \in Q$ if λ is sufficiently large. Therefore $0 = \min_{z \in Q} V(z) \leq \lambda$. Since $Q \notin \mathcal{F}_1 \cup \mathcal{F}_3$, there exists a point $(x_0, y_0) \in Q$ such that

$$(4.7) \quad |x_0| > K_1 \lambda^{\mu_1}$$

or

$$(4.8) \quad |y_0| > K_2 \lambda^{\mu_2}.$$

(4.8) is impossible if λ is sufficiently large. Therefore (4.7) holds and

$$|x| \geq |x_0| - m_1^{1/2} \lambda^{-1/2} (\log \lambda)^{1/n} > (K_1/2)\lambda^{\mu_1}$$

if λ is sufficiently large. Thus we have (4.6).

Applying arguments similar to those in the proof of Lemma 3.1, we have, by (4.6),

$$\begin{aligned} V(x, y) &= C \prod_{i=1}^p f_i(|x|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y|)^{\beta_j} \cdot |x|^\gamma |y|^\delta \geq C \prod_{j=1}^q b_{j0}^{\beta_j} \cdot |x|^{\sum \alpha_i d_i + \gamma} \cdot |y|^\delta \\ &\geq C \prod_{j=1}^q b_{j0}^{\beta_j} \cdot ((K_1/2)\lambda^{\mu_1})^{\sum \alpha_i d_i + \gamma} \cdot |y|^\delta = C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1 + \delta/2} |y|^\delta, \end{aligned}$$

for all $(x, y) \in F_2$, where $C_4 = C \prod_{j=1}^q b_{j0}^{\beta_j} \cdot 2^{-(\sum \alpha_i d_i + \gamma)}$. Therefore

$$(4.9) \quad \begin{aligned} \int_{F_2} (|\nabla u|^2 + V|u|^2) dx dy &\geq \int_{F_2} (|\nabla u|^2 + C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1 + \delta/2} |y|^\delta |u|^2) dx dy \\ &\geq \int_{F_{2x}} \left(\int_G (|\nabla_y u|^2 + C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1 + \delta/2} |y|^\delta |u|^2) dy \right) dx \end{aligned}$$

for all $u \in H^1(F_2)$ where $|\nabla_y u|^2 = \sum_{j=1}^{m_2} |\partial u / \partial y_j|^2$, $F_{2x} = \{x \in \mathbf{R}^{m_1}; (x, y) \in F_2\}$ and $G = \{y \in \mathbf{R}^{m_2}; |y_j| < K_4 \lambda^{-1/2}, j = 1, \dots, m_2\}$.

Remark that the function $C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1 + \delta/2} |y|^\delta$ is an A_∞ -weight on \mathbf{R}^{m_2} by Lemma 2.3. Set

$$\lambda_1 = \inf_{\substack{a > 0 \\ \xi \in \mathbb{R}^{m_2}}} \left(a^{-2} + a^{-m_2} \int_{|x-\xi| < a/2} C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1+\delta/2} |y|^\delta dy \right).$$

Then, by elementary calculus,

$$(4.10) \quad \lambda_1 = C_5 K_1^{1/\mu_1} \lambda,$$

where

$$C_5 = 2^{-2/(2+\delta)} (2+\delta) \delta^{-\delta/(2+\delta)} \{2^{-(\delta+m_2)} C_4 m_2 (\delta+m_2)^{-1} \omega_{m_2}\}^{2/(2+\delta)}$$

and ω_{m_2} is the volume of the unit ball in \mathbb{R}^{m_2} .

By Lemma 2.4

$$(4.11) \quad \int_{G'} (|\nabla_y v|^2 + C_4 K_1^{\sum \alpha_i d_i + \gamma} \lambda^{1+\delta/2} |y|^\delta |v|^2) dy \geq C_6 \lambda_1 \int_{G'} |v|^2 dy$$

for all $v \in H^1(G')$, where C_6 is a constant depending only on m_2 and δ , while $G' = \{y \in \mathbb{R}^{m_2}; |y_j| < \lambda_1^{-1/2}, j=1, \dots, m_2\}$.

Choosing K_1 and K_4 so that

$$(4.12) \quad C_5 K_1^{1/\mu_1} = K_4^{-2},$$

we get $G = G'$ by (4.10). Therefore we have

$$(4.13) \quad \int_{F_2} (|\nabla u|^2 + V|u|^2) dx dy \geq C_5 C_6 K_1^{1/\mu_1} \lambda \int_{F_2} |u|^2 dx dy$$

for all $u \in H^1(F_2)$. Choose K_1 so that

$$(4.14) \quad C_5 C_6 K_1^{1/\mu_1} > 1.$$

Then we have

$$\int_{F_2} (|\nabla u|^2 + V|u|^2) dx dy > \lambda \int_{F_2} |u|^2 dx dy$$

for all $u \in H^1(F_2)$, $u \neq 0$. Hence $N(\lambda, T_{\mathcal{N}}, F_2) = 0$.

Similar arguments show that $N(\lambda, T_{\mathcal{N}}, F_1) = 0$ if we choose K_2 and K_3 so that

$$(4.15) \quad C_7 K_2^{1/\mu_2} = K_3^{-2}$$

and

$$(4.16) \quad C_7 C_8 K_2^{1/\mu_2} > 1,$$

where C_7 is a positive constant depending only on $m_1, \gamma, C, \beta_j, h_j, \alpha_i$ and a_{i0} , while C_8 is the constant given in Lemma 2.4 for the function $|x|^\gamma$.

Now we choose K_1, K_2, K_3 and K_4 so that they satisfy (4.4), (4.5), (4.12), (4.14),

(4.15) and (4.16). We may put

$$(4.17) \quad K_1 = \max\{(C_2 C_5^{1/2})^{\delta\mu_1}, (C_5 C_6)^{-\mu_1}\} + 1,$$

$$(4.18) \quad K_2 = \max\{(C_3 C_7^{1/2})^{\gamma\mu_2}, (C_7 C_8)^{-\mu_2}\} + 1,$$

and define K_3 and K_4 so that they satisfy (4.12) and (4.15), respectively. Then all conditions in the proofs of Lemmas 3.1 and 3.2 are satisfied. q.e.d.

PROOF OF LEMMA 3.1'. If we set $\gamma=0$ and replace $F_2, F_3, \mathcal{J}_3, K_1, K_4$ in the proof of Lemma 3.1 by $F'_2, F'_3, \mathcal{J}'_3, K'_1, K'_4$, respectively, then we get the proof of Lemma 3.1'. The different point is that the argument on the inequality $|y_1| > K_2 \lambda^{\mu_2}$ does not occur. The condition on K'_1 and K'_4 is

$$(4.4)' \quad C_9 K_1'^{-\sum \alpha_i d_i / \delta} < K'_4,$$

where C_9 is a positive constant corresponding to C_2 . We shall give exact values of K'_1 and K'_4 later. q.e.d.

PROOF OF LEMMA 3.2'. If we set $\gamma=0$ and replace $F_2, \mathcal{J}_3, K_1, K_4$ in the proof of $N(\lambda, T_{\mathcal{N}}, F_2)=0$ in Lemma 3.2 by $F'_2, \mathcal{J}'_3, K'_1, K'_4$, respectively, then we get the proof of Lemma 3.2'. The different point is that the inequality (4.8) does not occur. The conditions on K'_1 and K'_4 are

$$(4.12)' \quad C_{10} K_1'^{1/\mu_1} = K_4'^{-2}$$

and

$$(4.14)' \quad C_{10} C_{11} K_1'^{1/\mu_1} > 1,$$

where C_{10} and C_{11} are positive constants corresponding to C_5 and C_6 . If we put

$$(4.17)' \quad K_1 = \max\{(C_9 C_{10}^{1/2})^{\delta\mu_1}, (C_{10} C_{11})^{-\mu_1}\} + 1,$$

then all conditions (4.4)', (4.12)' and (4.14)' are satisfied. q.e.d.

5. Proof of Lemmas 3.3, 3.3' and 3.3''. First we prove Lemma 3.3. Let l be the side length of cubes in \mathcal{F}'_λ , that is, $l = \lambda^{-1/2}(\log \lambda)^{1/n}$. In order to prove Lemma 3.3, we show the following three inequalities:

$$(1) \quad \sum_{Q \in \mathcal{J}_1} N(\lambda, T_{\mathcal{N}}, \hat{Q}) \leq \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy + O(M_1 (\log \lambda)^{1-1/n})$$

as $\lambda \rightarrow \infty$, where $M_1 = \# \mathcal{J}_1$.

$$(2) \quad \sum_{Q \in \mathcal{J}_3} N(\lambda, T_{\mathcal{N}}, \hat{Q}) \leq O(M_3 \log \lambda)$$

as $\lambda \rightarrow \infty$, where $M_3 = \# \mathcal{J}_3$.

$$(3) \quad \sum_{Q \in \mathcal{F}_2} N(\lambda, T_Q, \dot{Q}) \geq \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy - m_1 \lambda^{n/2} |S^1| - m_2 \lambda^{n/2} |S_1| - O(M_2 (\log \lambda)^{1-1/n})$$

as $\lambda \rightarrow \infty$, where $M_2 = \# \mathcal{F}_2$, $S^1 = \{(x, y) \in A; |x_1| < l\}$ and $S_1 = \{(x, y) \in A; |y_1| < l\}$.

PROOF OF (1). Let Q be a cube in \mathcal{F}_1 . Since

$$\int_Q (|\nabla u|^2 + V|u|^2) dx dy \geq \int_Q (|\nabla u|^2 + \min_Q V \cdot |u|^2) dx dy$$

for all $u \in H^1(\dot{Q})$,

$$N(\lambda, T_Q, \dot{Q}) \leq N(\lambda - \min_Q V, -\Delta_Q, \dot{Q})$$

by the min-max principle in Reed-Simon [7; p. 78]. Following Edmunds and Evans [2; p. 143], we get

$$N(\lambda - \min_Q V, -\Delta_Q, \dot{Q}) \leq \frac{\omega_n}{(2\pi)^n} |Q| \left(\lambda - \min_Q V \right)^{n/2} + C_1 \{1 + (|Q| \lambda^{n/2})^{1-1/n}\},$$

where C_1 is a positive constant depending only on m_1 and m_2 . Therefore

$$(5.1) \quad \sum_{Q \in \mathcal{F}_1} N(\lambda, T_Q, \dot{Q}) \leq \frac{\omega_n}{(2\pi)^n} \sum_{Q \in \mathcal{F}_1} |Q| \left(\lambda - \min_Q V \right)^{n/2} + C_1 \{M_1 + M_1 (\log \lambda)^{1-1/n}\},$$

since the side length of Q is $l = \lambda^{-1/2} (\log \lambda)^{1/n}$.

Let ξ_1, \dots, ξ_n be positive integers. Let Q be a cube in \mathcal{F}_1 with center $(l(\xi_1 + 1/2), \dots, l(\xi_n + 1/2))$ and let Q' be a cube in \mathcal{F}'_λ with center $(l(\xi_1 - 1/2), \dots, l(\xi_n - 1/2))$. Then

$$V(x, y) = C \prod_{i=1}^p f_i(|x|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y|)^{\beta_j} \cdot |x|^\gamma |y|^\delta \leq \min_Q V \leq \lambda$$

for all $(x, y) \in Q'$. Therefore $Q' \in \mathcal{F}_2$ and

$$|Q| \left(\lambda - \min_Q V \right)^{n/2} \leq \int_{Q'} (\lambda - V)^{n/2} dx dy.$$

Note that $Q \rightarrow Q'$ is a one-to-one correspondence from cubes in \mathcal{F}_1 with centers in the first orthant to cubes in \mathcal{F}_2 with centers in the first orthant. Then we get, by the symmetry property of V ,

$$(5.2) \quad \sum_{Q \in \mathcal{F}_1} |Q| \left(\lambda - \min_Q V \right)^{n/2} \leq \int_I (\lambda - V)^{n/2} dx dy,$$

where $I = \bigcup_{Q \in \mathcal{J}_2} Q$. Note that

$$(5.3) \quad I \subset A.$$

Indeed, by the definition of \mathcal{J}_2 ,

$$I \subset \{(x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}; V(x, y) \leq \lambda\}.$$

Furthermore, following the argument in the proof of Lemma 3.1, we get $\mathcal{J}_2 \subset \mathcal{J}_1 \cup \mathcal{J}_3$. Thus we get (5.3). Hence

$$\sum_{Q \in \mathcal{J}_1} |Q| \left(\lambda - \min_Q V \right)^{n/2} \leq \int_A (\lambda - V)^{n/2} dx dy.$$

Applying this to (5.1), we get

$$\sum_{Q \in \mathcal{J}_1} N(\lambda, T_{\mathcal{N}}, \dot{Q}) \leq \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy + O(M_1 (\log \lambda)^{1-1/n}),$$

where the bound of the error term is independent of λ . q.e.d.

PROOF OF (2). Applying the argument in the proof of (1), we get

$$\begin{aligned} \sum_{Q \in \mathcal{J}_3} N(\lambda, T_{\mathcal{N}}, \dot{Q}) &\leq \sum_{Q \in \mathcal{J}_3} N(\lambda, -\Delta_{\mathcal{N}}, \dot{Q}) \leq \frac{\omega_n}{(2\pi)^n} \sum_{Q \in \mathcal{J}_3} |Q| \lambda^{n/2} + C_1 \{M_3 + M_3 (\log \lambda)^{1-1/n}\} \\ &= O(M_3 \log \lambda). \end{aligned}$$

q.e.d.

PROOF OF (3). Let Q be a cube in \mathcal{J}_2 . Since

$$\int_Q (|\nabla u|^2 + V|u|^2) dx dy \leq \int_Q (|\nabla u|^2 + \max_Q V \cdot |u|^2) dx dy$$

for all $u \in H^1(\dot{Q})$,

$$N(\lambda, T_{\mathcal{Q}}, \dot{Q}) \geq N\left(\lambda - \max_Q V, -\Delta_{\mathcal{Q}}, \dot{Q}\right)$$

by the min-max principle. Following Edmunds and Evans [2; p. 143] as before, we get

$$N\left(\lambda - \max_Q V, -\Delta_{\mathcal{Q}}, \dot{Q}\right) \geq \frac{\omega_n}{(2\pi)^n} |Q| \left(\lambda - \max_Q V \right)^{n/2} - C_2 \{1 + (|Q| \lambda^{n/2})^{1-1/n}\},$$

where C_2 is a positive constant depending only on m_1 and m_2 . Therefore

$$(5.4) \quad \sum_{Q \in \mathcal{J}_2} N(\lambda, T_{\mathcal{Q}}, \dot{Q}) \geq \frac{\omega_n}{(2\pi)^n} \sum_{Q \in \mathcal{J}_2} |Q| \left(\lambda - \max_Q V \right)^{n/2} - C_2 \{M_2 + M_2 (\log \lambda)^{1-1/n}\}.$$

Applying an argument similar to that in the proof of (1), we get

$$(5.5) \quad \sum_{Q \in \mathcal{F}_2} |Q| \left(\lambda - \max_Q V \right)^{n/2} \geq \int_J (\lambda - V)^{n/2} dx dy,$$

where $J = \{(x, y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}; (x, y) \in \bigcup_{Q \in \mathcal{F}_1} Q, V(x, y) \leq \lambda\}$. Recall the definition of \mathcal{F}_1 and apply the argument in the proof of Lemma 3.1. Then we get

$$\left(\bigcup_{Q \in \mathcal{F}_1} Q \right) \cap \{(x, y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}; |x| > K_1 \lambda^{\mu_1} \text{ or } |y| > K_2 \lambda^{\mu_2}\} = \emptyset.$$

Therefore, by the definition of A ,

$$\begin{aligned} J &= \{(x, y) \in A; |x_i| \geq l, i = 1, \dots, m_1, |y_j| \geq l, j = 1, \dots, m_2\} \\ &= A \setminus \left(\bigcup_{i=1}^{m_1} \{(x, y) \in A; |x_i| < l\} \cup \bigcup_{j=1}^{m_2} \{(x, y) \in A; |y_j| < l\} \right) \\ &= A \setminus \left(\bigcup_{i=1}^{m_1} S^i \cup \bigcup_{j=1}^{m_2} S_j \right), \text{ say.} \end{aligned}$$

Thus by (5.5)

$$\begin{aligned} \sum_{Q \in \mathcal{F}_2} |Q| \left(\lambda - \max_Q V \right)^{n/2} &\geq \int_A (\lambda - V)^{n/2} dx dy - \sum_{i=1}^{m_1} \int_{S^i} (\lambda - V)^{n/2} dx dy - \\ &\sum_{j=1}^{m_2} \int_{S_j} (\lambda - V)^{n/2} dx dy \geq \int_A (\lambda - V)^{n/2} dx dy - \lambda^{n/2} \sum_{i=1}^{m_1} |S^i| - \lambda^{n/2} \sum_{j=1}^{m_2} |S_j| \\ &\geq \int_A (\lambda - V)^{n/2} dx dy - \lambda^{n/2} m_1 |S^1| - \lambda^{n/2} m_2 |S_1|, \end{aligned}$$

where we used the symmetry property of V . Therefore, by (5.4),

$$\begin{aligned} \sum_{Q \in \mathcal{F}_2} N(\lambda, T_\vartheta, Q) &\geq \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V)^{n/2} dx dy \\ &\quad - m_1 \lambda^{n/2} |S^1| - m_2 \lambda^{n/2} |S_1| - O(M_2 (\log \lambda)^{1-1/n}). \end{aligned} \quad \text{q.e.d.}$$

Therefore, by (1), (2) and (3), Lemma 3.3 follows from the following three lemmas.

LEMMA 5.1.

$$M_1 (\log \lambda)^{1-1/n} = M_2 (\log \lambda)^{1-1/n} = o\left(\int_A (\lambda - V)^{n/2} dx dy \right) \quad \text{as } \lambda \rightarrow \infty.$$

LEMMA 5.2.

$$\lambda^{n/2} |S^1| = o\left(\int_A (\lambda - V)^{n/2} dx dy \right) \quad \text{as } \lambda \rightarrow \infty,$$

and

$$\lambda^{n/2} |S_1| = o\left(\int_A (\lambda - V)^{n/2} dx dy\right) \quad \text{as } \lambda \rightarrow \infty.$$

LEMMA 5.3.

$$M_3(\log \lambda) = o\left(\int_A (\lambda - V)^{n/2} dx dy\right) \quad \text{as } \lambda \rightarrow \infty.$$

To prove Lemmas 5.1, 5.2 and 5.3, we use the following lemma, where $f(\lambda) \approx g(\lambda)$ means that $f(\lambda) = O(g(\lambda))$ and $g(\lambda) = O(f(\lambda))$ as $\lambda \rightarrow \infty$.

LEMMA 5.4. *Let V be the function defined by (3.1). Set $v_1 = n/2 + m_1(\sum_{i=1}^p \alpha_i d_i + \gamma)^{-1}$, $v_2 = n/2 + m_2(\sum_{j=1}^q \beta_j h_j + \delta)^{-1}$, $v_3 = m_1/2 + 2^{-1}(2 + \delta)m_1(\sum_{i=1}^p \alpha_i d_i + \gamma)^{-1}$, and $v_4 = m_2/2 + 2^{-1}(2 + \gamma)m_2(\sum_{j=1}^q \beta_j h_j + \delta)^{-1}$.*

(1) *If $\gamma m_2 < (\sum_{i=1}^p \beta_j h_j + \delta)m_1$, $\delta m_1 < (\sum_{i=1}^p \alpha_i d_i + \gamma)m_2$ and $v_1 \neq v_2$, then*

$$\int_A (\lambda - V)^{n/2} dx dy \approx \lambda^{v_1} + \lambda^{v_2}.$$

(2) *If $\delta m_1 > (\sum_{i=1}^p \alpha_i d_i + \gamma)m_2$, then*

$$\int_A (\lambda - V)^{n/2} dx dy \approx \lambda^{v_3}.$$

(3) *If $\gamma m_2 > (\sum_{j=1}^q \beta_j h_j + \delta)m_1$, then*

$$\int_A (\lambda - V)^{n/2} dx dy \approx \lambda^{v_4}.$$

(4) *In the other cases,*

$$\int_A (\lambda - V)^{n/2} dx dy \approx (\lambda^{v_1} + \lambda^{v_2}) \log \lambda.$$

These estimates are given by elementary calculus, so we omit the proof of Lemma 5.4.

As a consequence of Lemma 5.4, we get

$$(5.6) \quad \int_A (\lambda - V)^{n/2} dx dy = O((\lambda^{v_1} + \lambda^{v_2}) \log \lambda + \lambda^{v_3} + \lambda^{v_4}).$$

Remark that an easy calculation shows that the order of $\int_A (\lambda - V)^{n/2} dx dy$ is the same as that of $\lambda^{n/2} |A|$.

PROOF OF LEMMA 5.1. Since the argument before (5.2) shows that $M_1 = M_2$, it suffices to estimate $M_2(\log \lambda)^{1-1/n}$.

Since the side length of $Q \in \mathcal{F}_2$ is $l = \lambda^{-1/2}(\log \lambda)^{1/n}$,

$$M_2(\log \lambda)^{1-1/n} = l^{-n}(\log \lambda)^{1-1/n} \left| \bigcup_{Q \in \mathcal{F}_2} Q \right| = (\log \lambda)^{-1/n} \lambda^{n/2} |I|.$$

By (5.3) the term on the right hand side does not exceed $(\log \lambda)^{-1/n} \lambda^{n/2} |A|$. Since $\lambda^{n/2} |A| = O(\int_A (\lambda - V)^{n/2} dx dy)$, the assertion of Lemma 5.1 is valid. q.e.d.

PROOF OF LEMMA 5.2. First we prove

$$(5.7) \quad \lambda^{n/2} |S^1| = o\left(\int_A (\lambda - V)^{n/2} dx dy\right).$$

If $m_1 > 1$, then

$$\lambda^{n/2} |S^1| \leq 2\lambda^{n/2} l |S'| = 2(\log \lambda)^{1/n} \lambda^{(n-1)/2} |S'|,$$

where S' is the set of all points $(x', y) \in \mathbf{R}^{m_1-1} \times \mathbf{R}^{m_2}$ such that

$$C \prod_{i=1}^p f_i(|x'|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y|)^{\beta_j} \cdot |x'|^{\nu_1} |y|^{\nu_2} \leq \lambda,$$

$$|x'| \leq K_1 \lambda^{\mu_1} \quad \text{and} \quad |y| \leq K_2 \lambda^{\mu_2},$$

where K_1, K_2, μ_1 and μ_2 are constants given in the definition of \mathcal{F}_3 . By an argument similar to that in the note after Lemma 5.4, we can show that the order of $\lambda^{(n-1)/2} |S'|$ is the same as that of $\int_{S'} (\lambda - V'(x', y))^{(n-1)/2} dx' dy$, where $V'(x', y) = V(0, x', y)$. If we replace m_1 by $m_1 - 1$ in Lemma 5.4, we get the order of $\int_{S'} (\lambda - V')^{(n-1)/2} dx' dy$. Thus, replacing m_1 by $m_1 - 1$ in (5.6), we get

$$(5.8) \quad (\log \lambda)^{1/n} \int_{S'} (\lambda - V)^{(n-1)/2} dx' dy = O((\lambda^{\nu_1} + \lambda^{\nu_2})(\log \lambda)^{1+1/n} + (\lambda^{\nu_3} + \lambda^{\nu_4})(\log \lambda)^{1/n}),$$

where $\nu_1 = (n-1)/2 + (m_1-1)(\sum_{i=1}^p \alpha_i d_i + \gamma)^{-1}$, $\nu_2 = (n-1)/2 + m_2(\sum_{j=1}^q \beta_j h_j + \delta)^{-1}$, $\nu_3 = (m_1-1)/2 + 2^{-1}(2+\delta)(m_1-1)(\sum_{i=1}^p \alpha_i d_i + \gamma)^{-1}$, and $\nu_4 = m_2/2 + 2^{-1}(2+\gamma)m_2(\sum_{j=1}^q \beta_j h_j + \delta)^{-1}$. If we compare the order of $\int_A (\lambda - V)^{n/2} dx dy$ in Lemma 5.4 with the one on the right hand side of (5.8), then we get

$$\lambda^{n/2} |S^1| = o\left(\int_A (\lambda - V)^{n/2} dx dy\right) \quad \text{as } \lambda \rightarrow \infty.$$

If $m_1 = 1$, then, by the definition of S^1 ,

$$\lambda^{(1+m_2)/2} |S^1| \leq C \lambda^{(1+m_2)/2} l^{\mu_2 \mu_2} = C \lambda^{m_2/2 + m_2 \mu_2} (\log \lambda)^{1/n},$$

where C is a constant independent of λ . Therefore, by Lemma 5.4, we can show

$$\lambda^{n/2} |S^1| = o\left(\int_A (\lambda - V)^{n/2} dx dy\right).$$

Thus we get (5.7).

Similarly, we can prove

$$\lambda^{n/2} |S_1| = o\left(\int_A (\lambda - V)^{n/2} dx dy\right).$$

q.e.d.

PROOF OF LEMMA 5.3. Let B^i and B_j be the subsets of \mathbb{R}^n and \mathcal{S}_3 be the set of cubes as defined in the proof of the Theorem. Let $\{i_1, \dots, i_s\}$ and $\{j_1, \dots, j_t\}$ be subsets of $\{1, \dots, m_1\}$ and $\{1, \dots, m_2\}$, respectively. For $\{i_1, \dots, i_s\}$ and $\{j_1, \dots, j_t\}$, denote

$$\mathcal{Q}^{i_1, \dots, i_s} = \{Q \in \mathcal{S}_3; Q \cap B^i \neq \emptyset, i = i_1, \dots, i_s, Q \cap B^i = \emptyset, i \neq i_1, \dots, i_s, Q \cap B_j = \emptyset, j = 1, \dots, m_2\},$$

$$\mathcal{Q}_{j_1, \dots, j_t} = \{Q \in \mathcal{S}_3; Q \cap B_j \neq \emptyset, j = j_1, \dots, j_t, Q \cap B_j = \emptyset, j \neq j_1, \dots, j_t, Q \cap B^i = \emptyset, i = 1, \dots, m_1\}$$

and

$$\mathcal{Q}_{j_1, \dots, j_t}^{i_1, \dots, i_s} = \{Q \in \mathcal{S}_3; Q \cap B^i \neq \emptyset, i = i_1, \dots, i_s, Q \cap B^i = \emptyset, i \neq i_1, \dots, i_s, Q \cap B_j \neq \emptyset, j = j_1, \dots, j_t, Q \cap B_j = \emptyset, j \neq j_1, \dots, j_t\}.$$

Then we get a disjoint decomposition of \mathcal{S}_3 :

$$(5.9) \quad \mathcal{S}_3 = \left(\bigcup_{i_1 < \dots < i_s} \mathcal{Q}^{i_1, \dots, i_s}\right) \cup \left(\bigcup_{j_1 < \dots < j_t} \mathcal{Q}_{j_1, \dots, j_t}\right) \cup \left(\bigcup_{\substack{i_1 < \dots < i_s \\ j_1 < \dots < j_t}} \mathcal{Q}_{j_1, \dots, j_t}^{i_1, \dots, i_s}\right).$$

Now we show that

$$(5.10) \quad (\#\mathcal{Q}^{i_1, \dots, i_s}) \log \lambda = o\left(\int_A (\lambda - V)^{n/2} dx dy\right)$$

for any $i_1 < \dots < i_s$ in $\{1, \dots, m_1\}$.

Fix $i_1 < \dots < i_s$ and simply denote \mathcal{Q} instead of $\mathcal{Q}^{i_1, \dots, i_s}$.

First suppose $s < m_1$. Let \mathcal{Q}' be the set of Q in \mathcal{Q} which are contained in the first orthant. Let R be the set of all points $(x, y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ such that

$$\begin{aligned} 0 \leq x_i \leq l, & \quad i = i_1, \dots, i_s, \\ l \leq x_i, & \quad i \neq i_1, \dots, i_s, \\ l \leq y_j, & \quad j = 1, \dots, m_2, \\ |x^*| \leq K_1 \lambda^{\mu_1}, & \quad |y| \leq K_2 \lambda^{\mu_2} \end{aligned}$$

and

$$C \prod_{i=1}^p f_i(|x^* - le_1|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y - le_2|)^{\beta_j} \cdot |x^* - le_1|^{\gamma} |y - le_2|^{\delta} \leq \lambda,$$

where $x^* = (x_{\tau_1}, \dots, x_{\tau_{m_1-s}})$, $\tau_1 < \dots < \tau_{m_1-s}$, $\{\tau_1, \dots, \tau_{m_1-s}\} = \{1, \dots, m_1\} \setminus \{i_1, \dots, i_s\}$, $e_1 = (1, \dots, 1) \in R^{m_1-s}$, $e_2 = (1, \dots, 1) \in R^{m_2}$ and K_1, K_2, μ_1, μ_2 are constants given in the definition of \mathcal{S}_3 . Then, by the definitions of \mathcal{S}_3 and \mathcal{Q}' ,

$$\bigcup_{Q \in \mathcal{Q}'} Q \subset R.$$

Therefore

$$\# \mathcal{Q}' = l^{-n} \left| \bigcup_{Q \in \mathcal{Q}'} Q \right| \leq l^{-n} |R| \leq l^{-n+s} |R'|,$$

where R' is the set of all points (x^*, y) in $R^{m_1-s} \times R^{m_2}$ such that

$$\begin{aligned} 0 \leq x_i^*, \quad i = 1, \dots, m_1 - s, \quad x^* = (x_1^*, \dots, x_{m_1-s}^*), \\ 0 \leq y_j, \quad j = 1, \dots, m_2, \quad y = (y_1, \dots, y_{m_2}), \\ |x^*| \leq K_1 \lambda^{\mu_1}, \quad |y| \leq K_2 \lambda^{\mu_2} \end{aligned}$$

and

$$C \prod_{i=1}^p f_i(|x^*|)^{\alpha_i} \cdot \prod_{j=1}^q g_j(|y|)^{\beta_j} \cdot |x^*|^{\gamma} |y|^{\delta} \leq \lambda.$$

Therefore, since $l = \lambda^{-1/2} (\log \lambda)^{1/n}$,

$$(5.11) \quad \# \mathcal{Q}' \leq l^{-n+s} |R'| = (\log \lambda)^{s/n-1} \lambda^{(n-s)/2} |R'|.$$

By an argument similar to that in the proof of Lemma 5.2, we get

$$\lambda^{(n-s)/2} |R'| = O((\lambda^{\eta_1} + \lambda^{\eta_2}) \log \lambda + \lambda^{\eta_3} + \lambda^{\eta_4}),$$

where $\eta_1 = (n-s)/2 + (m_1-s)(\sum_{i=1}^p \alpha_i d_i + \gamma)^{-1}$, $\eta_2 = (n-s)/2 + m_2(\sum_{j=1}^q \beta_j h_j + \delta)^{-1}$, $\eta_3 = (m_1-s)/2 + 2^{-1}(2+\delta)(m_1-s)(\sum_{i=1}^p \alpha_i d_i + \gamma)^{-1}$ and $\eta_4 = m_2/2 + 2^{-1}(2+\gamma)m_2 \times (\sum_{j=1}^q \beta_j h_j + \delta)^{-1}$. Therefore, by (5.11), we get

$$(5.12) \quad (\# \mathcal{Q}) \log \lambda = 2^n (\# \mathcal{Q}') \log \lambda = O((\lambda^{\eta_1} + \lambda^{\eta_2})(\log \lambda)^{1+s/n} + (\lambda^{\eta_3} + \lambda^{\eta_4})(\log \lambda)^{s/n}).$$

If we compare the orders in Lemma 5.4 with the one in (5.12), then we get

$$(\# \mathcal{Q}) \log \lambda = o\left(\int_A (\lambda - V)^{n/2} dx dy\right).$$

Suppose $s = m_1$. Then, by the definition of \mathcal{S}_3 and \mathcal{Q} , we get

$$\bigcup_{Q \in \mathcal{Q}} Q \subset \{(x, y) \in R^{m_1} \times R^{m_2}; |x_i| \leq l, i = 1, \dots, m_1, |y| \leq K_2 \lambda^{\mu_2}\}.$$

Therefore, by Lemma 5.4,

$$\begin{aligned}
 (\# \mathcal{Q}) \log \lambda &= l^{-n} |\mathcal{Q}| \log \lambda \leq C l^{-n+m_1} \lambda^{m_2 \mu_2} \log \lambda \\
 &= C \lambda^{-m_2/2+m_2 \mu_2} (\log \lambda)^{m_1/n} = o\left(\int_A (\lambda - V)^{n/2} dx dy\right),
 \end{aligned}$$

where C is a constnt independent of λ . Therefore (5.10) holds.

Similarly, we can show that

$$(\# \mathcal{Q}_{j_1, \dots, j_i}) \log \lambda = o\left(\int_A (\lambda - V)^{n/2} dx dy\right)$$

and

$$(\# \mathcal{Q}_{j_1^i, \dots, j_i^i}) \log \lambda = o\left(\int_A (\lambda - V)^{n/2} dx dy\right).$$

Therefore, Lemma 5.3 follows from (5.9).

q.e.d.

Thus we proved Lemma 3.3. If we set $\gamma=0$ and replace A, \mathcal{S}_3, K_1 in the proof of Lemma 3.3 by A', \mathcal{S}'_3, K'_1 , respectively, then we get the proof of Lemma 3.3' after simple modification. If we set $\gamma=\delta=0$ and replace A, \mathcal{S}_3 in the proof of Lemma 3.3 by A'', \mathcal{S}''_3 , respectively, then we get the proof of Lemma 3.3''. The differences caused by these modifications are inessential.

REMARK 5.1. The above method does not give an asymptotic estimate for $N(\lambda)$ when $\gamma m_2 > (\sum_{j=1}^q \beta_j h_j + \delta) m_1$ or $\delta m_1 > (\sum_{i=1}^p \alpha_i d_i + \gamma) m_2$. Indeed, we cannot get good estimates for error terms in that case.

REMARK 5.2. We also have the asymptotic formula for the potential

$$V(x, y) = |x|^\alpha |y|^\beta |y-1|^\gamma$$

where $(x, y) \in \mathbf{R} \times \mathbf{R}$, $\alpha, \beta, \gamma > 0$, $\beta \leq \alpha$, $\gamma \leq \alpha$ and $\alpha \leq \beta + \gamma$. Let $\mu_1 = \max\{(2 + \beta)(2\alpha)^{-1}, (2 + \gamma)(2\alpha)^{-1}\}$ and $\mu_2 = (2 + \alpha)2^{-1}(\beta + \gamma)^{-1}$. Then

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int_A (\lambda - V) dx dy \quad \text{as } \lambda \rightarrow \infty,$$

where

$$A = \{(x, y) \in \mathbf{R} \times \mathbf{R}; V(x, y) \leq \lambda, |x| \leq C_1 \lambda^{\mu_1}, |y| \leq C_2 \lambda^{\mu_2}\}$$

and C_1, C_2 are positive constants depending only on α, β and γ . The proof of this result is a modification of the proof of the Theorem.

REFERENCES

- [1] R. R. COIFMAN AND C. FEFFERMAN, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* 51 (1974), 241–250.
- [2] D. E. EDMUNDS AND W. D. EVANS, On the distribution of eigenvalues of Schrödinger operators, *Arch. Rational Mech. Anal.* 89 (1985), 135–167.
- [3] C. FEFFERMAN, The uncertainty principle, *Bull. Amer. Math. Soc.* 9 (1983), 129–206.
- [4] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.
- [5] T. KATO, Schrödinger operators with singular potentials, *Israel J. Math.* 13 (1972), 135–148.
- [6] Y. MORIMOTO, The uncertainty principle and hypoelliptic operators, *Publ. Res. Inst. Math. Sci.* 23 (1987), 955–964.
- [7] M. REED AND B. SIMON, *Method of modern mathematical physics*, vol. IV, Academic Press, New York, 1978.
- [8] D. ROBERT, Comportement asymptotique des valeurs propres d'opérateurs du type Schrödinger à potential «dégénéré», *J. Math. Pures Appl.* 61 (1982), 275–300.
- [9] G. V. ROZENBLJUM, Asymptotics of the eigenvalues of the Schrödinger operator, *Math. USSR-Sb.* 22 (1974), 349–371.
- [10] B. SIMON, Nonclassical eigenvalue asymptotics, *J. Funct. Anal.* 53 (1983), 84–98.
- [11] B. SIMON, Some quantum operators with discrete spectrum but classically continuous spectrum, *Ann. Physics.* 146 (1983), 209–220.
- [12] M. Z. SOLOMYAK, Asymptotics of the spectrum of the Schrödinger operator with nonregular homogeneous potential, *Math. USSR-Sb.* 55 (1986), 19–37.

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