Tôhoku Math. J.
42 (1990), 333–338

INFINITESIMAL TORELLI THEOREM FOR COMPLETE INTERSECTIONS IN CERTAIN HOMOGENEOUS KÄHLER MANIFOLDS, II

KAZUHIRO KONNO

(Received May 1, 1989)

Introduction. In this paper, we continue to study the infinitesimal Torelli problem for complete intersections in Kähler C-spaces with $b_2=1$, which we began in [Ko.1] and which is referred to as Part I.

Recall that a Kähler C-space with $b_2=1$ is determined by a certain pair $(\mathfrak{g}, \alpha_r)$ of a complex simple Lie algebra $\mathfrak{g}$ and a simple root α_r (cf. Part I, §1). Let $Y=(\mathfrak{g}, \alpha_r)$ be an N-dimensional Kähler C-space with $b_2(Y)=1$ and denote by $\mathcal{O}_Y(1)$ the ample generator of $\mathrm{Pic}(Y)$. If a section of the vector bundle

$$E=\bigoplus_{i=1}^{N-n} \mathcal{O}_Y(d_i)\,, \qquad d_i>0$$

defines an irreducible nonsingular subvariety X, we call it a nonsingular complete intersection of type $(d_1, d_2, \cdots, d_{N-n})$.

In Part I, we showed that the infinitesimal Torelli theorem holds for X with the ample canonical budle if Y is an irreducible Hermitian symmetric space of compact type or a certain non-symmetric Kähler C-space with $b_2=1$. Between Part I and the present article, a big progress was made by Flenner: He developed a powerful criterion [F, Theorem (1.1)] and completely answered the infinitesimal Torelli problem for non-singular complete intersections in a projective space $\boldsymbol{P}^N$ [F, Theorem (3.1)]. The purpose of this article is to give another application of Flenner's criterion. Namely, we show the following:

MAIN THEOREM. *Let X be a nonsingular complete intersection of type $(d_1, d_2, \cdots d_{N-n})$ in a Kähler C-space Y with $b_2(Y)=1$. Assume that Y is neither a projective space nor a complex quadric. Then the infinitesimal Torelli theorem holds for X provided that*

(1) *the canonical bundle K_X of X is non-negative, or*
(2) *$d_i \geq 2$ for any i and X is neither*
  (a) *a hypersurface of degree 2 in (A_4, α_2), (D_5, α_4), (E_6, α_2), (E_7, α_1), (E_8, α_8), (F_4, α_1) or (F_4, α_3), nor*
  (b) *a complete intersection of type $(2, 2)$ in (B_l, α_2), (D_l, α_2), (E_6, α_2), (E_7, α_1), (E_8, α_8), (F_4, α_1) or (F_4, α_3).*

For the proof, we use the vanishing theorems on Kähler C-spaces developed in [Ko.2], instead of Bott's vanishing theorem on $\boldsymbol{P}^N$ which played an essential role in the proof of [F, Theorem (3.1)]. We think that most of the exceptions in (2) are inessential, since they seem to come from the weakness of our vanishing theorems. A part of the result in the case (1) was independently obtained by Kasparian [Ka].

We freely use the notation in Part I throughout the paper.

1. Known results. In this section, we recall known results which we need later. Let Y be an N-dimensional Kähler C-space with $b_2(Y)=1$. We denote by $k(Y)$ the integer satisfying $K_Y=\mathcal{O}_Y(-k(Y))$.

The proof of the following lemmas can be found in [ST, Lemma 2.1] and [Ki, I, Theorem 6 and the remark after it], respectively.

1.1. LEMMA. *For each positive integer a, the line bundle $\mathcal{O}_Y(a)$ is normally generated. In particular, the multiplication map*

$$H^0(Y, \mathcal{O}_Y(b))\otimes H^0(Y, \mathcal{O}_Y(c))\to H^0(Y, \mathcal{O}_Y(b+c))$$

is surjective for any non-negative integers b, c.

1.2. LEMMA. *If q is an integer satisfying $0<q<N=\dim Y$, then $H^q(Y, \mathcal{O}_y(a))$ vanishes for any $a\in\boldsymbol{Z}$.*

1.3. We note that there are the following isomorphisms in addition to (1.6), Part I:

$$(A_l, \alpha_{l+1-r})\simeq(A_l, \alpha_r), \quad (A_3, \alpha_2)\simeq Q^4, \quad (C_2, \alpha_2)\simeq Q^3, \quad (D_4, \alpha_3)\simeq(D_4, \alpha_1)=Q^6,$$

where Q^N is a quadric in $\boldsymbol{P}^{N+1}$. Thus it suffices for our purpose to consider the following Kähler C-spaces:

(1) (A_l, α_r): $2\le r\le l+1-r$ and $(l, r)\ne(3, 2)$.
(2) (B_l, α_r): $2\le r\le l-1$ and $l\ge 3$.
(3) (C_l, α_r): $2\le r\le l$ and $l\ge 3$,
(4) (D_l, α_r): $2\le r\le l-2$ and $l\ge 4$, (D_l, α_{l-1}): $l\ge 5$.
(5) (E_l, α_r): $6\le l\le 8$, $1\le r\le l$, $(l, r)\ne(6, 5), (6, 6)$.
(6) (F_4, α_r): $1\le r\le 4$.
(7) (G_2, α_2).

The numerical invariants such as N, $k(Y)$ can be found in Table 1, Part I.

The following can be found in [Ko.2, §4].

1.4. PROPOSITION. *Let Y be as in 1.3. The group $H^q(Y, \Omega_Y^p(a))$ vanishes for any $q\ge 1$, if*

(1) *α_r is long or $Y=(C_l, \alpha_2), (F_4, \alpha_4)$: $a\ge p>0$,*
(2) *$Y=(C_l, \alpha_r)$, $3\le r\le l-1$: $a\ge\min(p+1, 2p-1)>0$,*
(3) *$Y=(F_4, \alpha_3)$: $a\ge\min(p+3, 2p-1)>0$.*

The following two propositions can be found in [Ko.2, §5]. See also [Ki] as for symmetric spaces.

1.5. PROPOSITION. *Let Y be an irreducible Hermitian symmetric space of compact type which is neither a projective space nor a complex quadric. Let p be any integer satisfying $2 \leq p \leq N$.*

(1) *If $p+q>N$, then $H^q(Y, \Omega_Y^p(a))$ vanishes for $a \geq 2p-2-k(Y)$ unless*

$$Y=(A_4, \alpha_2), (D_5, \alpha_4): (p, q)=(2, N-1), a=2-k(Y).$$

(2) *$H^{N-p}(Y, \Omega_Y^p(a))$ vanishes for $a \geq 2p-k(Y)$.*

1.6. PROPOSITION. *Let Y be a Kähler C-space with $b_2(Y)=1$ and is not a symmetric space. Let p be any integer satisfying $2 \leq p \leq N$.*

(1) *If $p+q>N+1$, then $H^q(Y, \Omega_Y^p(a))$ vanishes for $a \geq 2p-2-k(Y)$ except possibly in the case where $a=2p-2-k(Y)$ holds for the following Y and p:*

(a) $(B_l, \alpha_2), (D_l, \alpha_2)$: $p=3$.

(b) $(E_6, \alpha_2), (F_4, \alpha_1), (F_4, \alpha_3)$: $p=3, 4$.

(c) (E_7, α_1): $p=4, 5$.

(d) (E_8, α_8): $4 \leq p \leq 7$.

(2) *$H^{N-p+1}(Y, \Omega_Y^p(a))$ vanishes for $a \geq 2p-1-k(Y)$ except possibly in the case where $a=2p-1-k(Y)$ holds for the following Y and p:*

(a) $(E_6, \alpha_2), (F_4, \alpha_1), (F_4, \alpha_3)$: $p=3$.

(b) (E_7, α_1): $p=4$.

(c) (E_8, α_8): $4 \leq p \leq 6$.

(3) *$H^{N-p}(Y, \Omega_Y^p(a))$ vanishes for $a \geq 2p-k(Y)$ except possibly in the case where $Y=(E_8, \alpha_8)$, $4 \leq p \leq 5$ and $a=2p-k(Y)$.*

As a special case of a more general result due to Flenner [F, Theorem (1.1)], we have the following:

1.7. THEOREM. *Let X be a nonsingular complete intersection of type $(d_1, d_2, \cdots, d_{N-n})$ in a Kähler C-space Y with $b_2(Y)=1$. Denote by N_X and N_X^* the normal and the conormal bundles of X in Y respectively and by $S^m N_X$ and $S^m N_X^*$ their m-th symmetric tensor products. Assume that the following conditions are satisfied:*

(1) *$H^{i+1}(X, S^i N_X^* \otimes \Omega_Y^{n-i-1} \otimes K_X^{-1})=0$ for $0 \leq i \leq n-2$.*

(2) *The multiplication map*

$$H^0(X, S^{n-p} N_X \otimes K_X) \otimes H^0(X, S^{p-1} N_X \otimes K_X) \to H^0(X, S^{n-1} N_X \otimes K_X^2)$$

is surjective for some $p \in \{1, \cdots, n\}$.

Then the infinitesimal period map

$$v_p: H^1(X, T_X) \to \mathrm{Hom}_C(H^{n-p}(X, \Omega_X^p), H^{n+1-p}(X, \Omega_X^{p-1}))$$

is injective.

2. Proof of the main theorem.

2.1. Let X be a complete intersection in Y defined by a section $x \in H^0(Y, E)$ with $E = \bigoplus_{i=1}^{N-n} \mathcal{O}_Y(d_i)$. Let $d := \sum_{i=1}^{N-n} d_i$. Then the canonical bundle K_X of X is $\mathcal{O}_X(d - k(Y))$. The section x gives the Koszul resolution,

$$0 \to \bigwedge^{N-n} E^* \to \bigwedge^{N-n-1} E^* \to \cdots \to E^* \to \mathcal{O}_Y \to \mathcal{O}_X \to 0 .$$

which in turn defines a spectral sequence

$$E_1^{-p,q} = H^q(Y, \bigwedge^p E^* \otimes V) \Longrightarrow H^{q-p}(X, V \otimes \mathcal{O}_X)$$

for any locally free sheaf V on Y.

2.2. LEMMA. *Let X be as in* 2.1. *The multiplication map*

$$H^0(\mathcal{O}_X(a)) \otimes H^0(\mathcal{O}_X(b)) \longrightarrow H^0(\mathcal{O}_X(a+b))$$

is surjective for $a, b \geq 0$.

PROOF. For any $c \in \boldsymbol{Z}$, we have a surjection $H^0(\mathcal{O}_Y(c)) \to H^0(\mathcal{O}_X(c))$ by Lemma 1.2 and the spectral sequence in 2.1 with $V = \mathcal{O}_Y(c)$. Consider the commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_X(a)) \otimes H^0(\mathcal{O}_X(b)) & \longrightarrow & H^0(\mathcal{O}_X(a+b)) \\ \uparrow \quad\quad\quad \uparrow & & \uparrow \\ H^0(\mathcal{O}_Y(a)) \otimes H^0(\mathcal{O}_Y(b)) & \longrightarrow & H^0(\mathcal{O}_Y(a+b)) . \end{array}$$

Since the map on the bottom row is surjective by Lemma 1.1, so is the map on the top row. Q.E.D.

2.3. LEMMA. *The multiplication map*

$$H^0(X, S^{n-p} N_X \otimes K_X) \otimes H^0(X, S^{p-1} N_X \otimes K_X) \longrightarrow H^0(X, S^{n-1} N_X \otimes K_X^2)$$

is surjective for some $p \in \{1, \cdots, n\}$ if one of the following conditions are satisfied:

(1) *K_X is ample, i.e., $d := \sum d_i > k(Y)$.*

(2) *$d_i \geq 2$ for any i, $1 \leq i \leq N-n$.*

PROOF. As for (1), we can take arbitrary p by virtue of Lemma 2.2. Consider the case (2). If n is even, put $n = 2p$. By Lemma 2.2 again, it suffices to show that each summand of $S^{p-1} N_X \otimes K_X$ has non-negative degree. Since $d_i \geq 2$, we have

$$d_{\nu_1} + d_{\nu_2} + \cdots + d_{\nu_{p-1}} + d - k(Y) \geq 2(p-1) + 2(N-n) - k(Y) = (N-n-1) + (N-k(Y)-1)$$

for any $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{p-1}$ with $\nu_i \in \{1, 2, \cdots, N-n\}$. We note that $N > k(Y)$ holds, since Y is neither a projective space nor a complex quadric (cf. Table 1, Part I). Thus the above inequality implies the assertion. If n is odd, we take p with $2p-1 = n$. Then the assertion follows from a similar argument. Q.E.D.

2.4. LEMMA. *If the condition*

$$(\mathrm{V})_{i,j}:\quad H^{N-i-j-1}(Y, \Omega_Y^{N-n+i+1}\otimes K_Y\otimes \det E\otimes \bigwedge^j E\otimes S^i E)=0$$

is satisfied for $0\leq i\leq n-2$ *and* $0\leq j\leq N-n$, *then* (1) *in Theorem* 1.7 *holds.*

PROOF. Since $N_X=E|_X$ and $K_X=(K_Y\otimes \det E)|_X$, we see from 2.1 that $H^{i+j+1}(Y, \Omega_Y^{n-i-1}\otimes S^i E^*\otimes (K_Y\otimes \det E)^{-1}\otimes \bigwedge^j E^*)=0$ for $0\leq j\leq N-n$ implies $H^{i+1}(X, \Omega_Y^{n-i-1}\otimes S^i N_X^*\otimes K_X^{-1})=0$. By the Serre duality, we get the assertion.

Q.E.D.

2.5. COROLLARY. *If K_X is ample, then the following condition is sufficient for* (1) *in Theorem* 1.7 *to hold*:

$$(\mathrm{V})_{i,N-n}:\quad H^{n-i-1}(Y, \Omega_Y^{N-n+i+1}\otimes (\det E)^2\otimes S^i E\otimes K_Y)=0$$

for $0\leq i\leq n-2$.

PROOF. Consider the condition in Lemma 2.4 and assume that $j<N-n$. Then we have $(N-i-j-1)+(N-n+i+1)>N$. Since $K_Y\otimes \det E\otimes S^i E\otimes \bigwedge^j E$ is a direct sum of ample line bundles, we see that the cohomology groups in question vanish by the vanishing theorem of Kodaira-Nakano. Q.E.D.

Now, we get our main theorem in the Introduction by the following two theorems:

2.6. THEOREM. *The infinitesimal Torelli theorem holds for a non-singular complete intersection X in a Kähler C-space Y with $b_2(Y)=1$, if K_X is ample or trivial.*

2.7. THEOREM. *Let X be a nonsingular complete intersection of type $(d_1, d_2, \cdots, d_{N-n})$, $d_i\geq 2$, in a Kähler C-space Y with $b_2(Y)=1$. Suppose that Y is neither a projective space nor a complex quadric. Then the infinitesimal Torelli theorem holds for X except possibly in the following cases*:

(1) *X is a hypersurface of degree* 2 *in* (A_4, α_2), (D_5, α_4), (E_6, α_2), (E_7, α_1), (E_8, α_8), (F_4, α_1) *or* (F_4, α_3).

(2) *X is a complete intersection of type* (2, 2) *in* (B_l, α_2), (D_l, α_2), (E_6, α_2), (E_7, α_1), (E_8, α_8), (F_4, α_1) *or* (F_4, α_3).

PROOF OF THEOREM 2.6. If K_X is trivial, then the infinitesimal Torelli theorem trivially holds. Further, in Theorem (3.11), Part I, we already dealt with the case where Y is (C_l, α_r), (F_4, α_4) or an irreducible Hermitian symmetric space of compact type. Thus we assume that Y is none of the above and K_X is ample. Let $Y=(\mathfrak{g}, \alpha_r)$ and suppose that α_r is a long root. Then we can easily check the condition $(\mathrm{V})_{i,N-n}$ in Corollary 2.5 by using Proposition 1.4, since we have

$$2d-k(Y)+d_{\nu_1}+d_{\nu_2}+\cdots+d_{\nu_i}\geq (N-n)+i+(d-k(Y))\geq N-n+i+1\,,$$

for any $\nu_1\leq \nu_2\leq\cdots\leq \nu_i$ with $\nu_j\in\{1, 2, \cdots, N-n\}$. This and Lemma 2.3 show the assertion by virtue of Theorem 1.7. Next, suppose that $Y=(F_4, \alpha_3)$. We see that (3.8), Part I, works for $n \geq 8$. If $n \leq 7$, then $N-n \geq 13$ and we have

$$2d-k(Y)+d_{\nu_1}+\cdots+d_{\nu_i} \geq N-n+i+(N-n-k(Y))>N-n+i+4.$$

Thus we are done as in the above case. Q.E.D.

PROOF OF THEOREM 2.7. We only have to check the condition $(\mathrm{V})_{i,j}$ in Lemma 2.4. Let Y be as in 1.3. Assume first that Y is symmetric. We put $p=N-n+i+1$ and $q=N-i-j-1$. Suppose first that $j<N-n$. Then we have $p+q>N$. Moreover, since $d_\nu \geq 2$ for any ν, we have

$$d-k(Y)+d_{\nu_1}+\cdots+d_{\nu_i}+d_{\mu_1}+\cdots+d_{\mu_j} \geq 2(N-n+i+j)-k(Y) \geq 2p-2-k(Y)$$

for any $\nu_1 \leq \cdots \leq \nu_i$ and $\mu_1<\cdots<\mu_j$. Thus (1) of Proposition 1.5 implies that $(\mathrm{V})_{i,j}$ holds for $j<N-n$ except when $Y=(A_4, \alpha_2)$, (D_5, α_4) and $p=2$. We note that, in the above inequality, the equality holds only if $i=j=0$. Thus, in the exceptional case, we have $n=N-1$ and $d=2$. Next put $j=N-n$. Then we have $p+q=N$ and

$$2d-k(Y)+d_{\nu_1}+\cdots+d_{\nu_i} \geq 4(N-n)+2i-k(Y) \geq 2p-k(Y).$$

Thus we can apply (2) of Proposition 1.5 to see $(\mathrm{V})_{i,N-n}$ holds. This completes the proof for the case where Y is a symmetric space. If Y is not a symmetric space, we apply Proposition 1.6 to check $(\mathrm{V})_{i,j}$. Then a similar calculation to the one above show the assertion. Q.E.D.

REFERENCES

[F] H. FLENNER, The infinitesimal Torelli problem for zero sets of sections of vector bundles, Math. Z. 193 (1986), 307–322.

[Ka] A. KASPARIAN, A letter to the author.

[Ki] Y. KIMURA, On the hypersurfaces of Hermitian symmetric spaces of compact type, Osaka J. Math. 16 (1979), 97–119, ibid. II, ibid. 17 (1980), 455–469.

[Ko.1] K. KONNO, Infinitestimal Torelli theorem for complete intersections in certain homogeneous Kähler manifolds, Tôhoku Math. J. 38 (1986), 609–624. (Referred to as Part I.)

[Ko.2] K. KONNO, Generic Torelli theorem for hypersurfaces of certain compact homogeneous Kähler manifolds, Duke Math. J. 59 (1989), 83–160.

[ST] Y. SAKANE AND M. TAKEUCHI, On defining equations of symmetric submanifolds in complex projective spaces, J. Math. Soc. Japan 33 (1981), 267–279.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
KYUSHU UNIVERSITY
ROPPONMATSU, CHÛOKU, FUKUOKA 810
JAPAN