Tôhoku Math. J.
42 (1990), 319–332

EQUIVARIANT INDEX OF DIRAC OPERATORS

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(Received April 27, 1989, revised January 11, 1990)

Abstract. Atiyah-Hirzebruch's rigidity theorem is derived from the index theorem for families of elliptic operators and vanishing theorem of Lichnerowicz type. Using this method we show the relative version of the rigidity of the Dirac operators.

Introduction. It is well-known that the Dirac operator plays an important role in the geometry of spin manifolds. For example, we can deduce the $\hat{A}$-genus vanishing theorem from the vanishing of the index of Dirac operators. Using the Bochner technique Lichnerowicz [L] proved that the $\hat{A}$-genus of a closed spin manifold M vanishes if M carries a metric of positive scalar curvature. On the other hand, Atiyah and Hirzebruch [A-H] proved that the $\hat{A}$-genus of a closed spin manifold M vanishes if M admits a non-trivial S^1-action. In fact, Atiyah and Hirzebruch proved that the equivariant index of the Dirac operator is identically zero as a virtual character of S^1 using the Atiyah-Singer equivariant index theorem. There seem to be no direct relations between these two theorems. But there is a result of Lawson and Yau [L-Y] which says that a closed manifold M carries a metric of positive scalar curvature if M admits an effective action of a compact non-abelian Lie group.

In this paper we observe that the rigidity of the Dirac operators on spin manifolds can be obtained from the index theorem for families of elliptic operators and the vanishing theorem of Lichnerowicz type. Here "rigidity" means the constancy of the equivariant index as virtual characters. Recently Witten [W-1], Bott and Taubes [B-T] discussed the rigidity of Dirac operators twisted by certain vector bundles. Our method does not seem to deduce such a result. But it can be applied to twisted Dirac operators on spin^c manifolds (§4) and families of Dirac operators (§5).

The author would like to express deep appreciation to Professor Akio Hattori for encouragement. After finishing the first manuscript of this paper, Witten kindly informed him of the existence of the paper [W-2].

1. Borel Construction. $ES^1 \to BS^1$ denotes the universal S^1-bundle. For an S^1-manifold X, we define the Borel construction to be $X_{S^1} = ES^1 \times_{S^1} X$. We have the following diagram.

Supported in part by the Grants-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science and Culture, Japan.

$$\begin{array}{ccc} ES^1 \times X & \longrightarrow & X_{S^1} = ES^1 \times_{S^1} X \\ \downarrow & & \downarrow \\ BS^1 \times X & \longrightarrow & BS^1 \end{array}$$

The cohomology group of the space X_{S^1} is called the equivariant cohomology group of the S^1-manifold X and the de Rham model of this cohomology theory is discussed by Atiyah and Bott [A-B] and Berline and Vergne [B-V]. The universal S^1-bundle $ES^1 \to BS^1$ is the inductive limit of the Hopf fibrations $S^{2k+1} \to CP^k$. Hence X_{S^1} is the inductive limit of $X_{S^1}^{(k)} = S^{2k+1} \times_{S^1} X$.

$$\begin{array}{ccc} S^{2k+1} \times X & \xrightarrow{p=p^{(k)}} & X_{S^1}^{(k)} = S^{2k+1} \times_{S^1} X \\ \downarrow & & \downarrow \pi = \pi^{(k)} \\ CP^k \times X & \longrightarrow & CP^k \end{array}$$

Fixing a connection on the principal S^1-bundle $S^{2k+1} \to CP^k$ with a connection form η, we can define a degree preserving algebra homomorphism

$$\psi = \psi_X^{(k)} : \Omega_{\mathrm{inv}}^*(X) \to \Omega^*(X_{S^1}^{(k)}) ,$$

where

$$\Omega_{\mathrm{inv}}^*(X) = \{S^1\text{-invariant differential forms on } X\} \text{ and}$$
$$\Omega^*(X_{S^1}^{(k)}) = \{\text{differential forms on } X_{S^1}^{(k)}\} .$$

Let $\mathcal{H}or(S^{2k+1})$ denote the horizontal distribution with respect to the connection η. Then the tangent space at $[t, x] \in S^{2k+1} \times_{S^1} X$ decomposes as

$$T_{[t,x]} X_{S^1}^{(k)} = p_* \mathcal{H}or_t(S^{2k+1}) \oplus p_* T_x X , \tag{1.1}$$

which induces a decomposition for the tangent bundle

$$TX_{S^1}^{(k)} = \mathcal{H}or(X_{S^1}^{(k)}) \oplus T_f X_{S^1}^{(k)} , \tag{1.2}$$

where $T_f X_{S^1}^{(k)}$ is the tangent bundle along the fibres of $\pi : X_{S^1}^{(k)} \to CP^k$.

DEFINITION (1.3). For $\alpha \in \Omega_{\mathrm{inv}}^r(X)$ and $u_1, \cdots, u_r \in T_{[t,x]} X_{S^1}^{(k)}$, we define $\psi\alpha = \psi_X^{(k)}\alpha$ by

$$(\psi\alpha)(u_1, \cdots, u_r) = \alpha(w_1, \cdots, w_r) ,$$

where w_j is the second component of u_j in the decomposition (1.1).

Note that $\psi\alpha$ is well defined because α is S^1-invariant.

PROPOSITION (1.4).

$$d(\psi\alpha)=\psi(d\alpha)-\psi(i(v_X)\alpha)\wedge\pi^*\omega$$

where v_X is the vector field on X determined by the S^1-action and ω is the curvature form for the connection η, where $i(v_X)$ denotes the interior product by v_X.

PROOF. Since $p: S^{2k+1}\times X\to X^{(k)}_{S^1}$ is a submersion it suffices to prove that

$$p^*d(\psi\alpha)=p^*\{\psi(d\alpha)-\psi(i(v_X)\alpha\wedge\pi^*\omega\} . \tag{1.5}$$

It is easy to see that

$$p^*\psi\zeta=pr_2^*\zeta-pr_1^*\eta\wedge pr_2^*(i(v_X)\zeta) , \tag{1.6}$$

where pr_i is the i-th projection of $S^{2k+1}\times X$. Thus we have

$$\begin{aligned} d(p^*\psi\alpha)&=pr_2^*d\alpha-pr_1^*d\eta\wedge pr_2^*(i(v_X)\alpha)+pr_1^*\eta\wedge pr_2^*(di(v_X)\alpha)\\ &=pr_2^*d\alpha-pr_1^*\eta\wedge pr_2^*(i(v_X)d\alpha)-pr_1^*d\eta\wedge pr_2^*(i(v_X)\alpha)\\ &=p^*(\psi d\alpha)-pr_1^*d\eta\wedge pr_2^*(i(v_X)\alpha)\end{aligned}$$

with the second equality by the S^1-invariance of α, and

$$p^*\psi(i(v_X)\alpha)=pr_2^*(i(v_X)\alpha)-pr_1^*\eta\wedge pr_2^*(i(v_X)i(v_X)\alpha)=pr_2^*(i(v_X)\alpha) ,$$

the first equality being a consequence of (1.6). Therefore we get

$$p^*(d\psi\alpha)=p^*\{\psi d\alpha-\pi^*\omega\wedge i(v_X)\alpha\} .$$

Since ω is a 2-form, we obtain (1.5). ■

If we choose a connection η on $S^{2k+1}\to CP^k$ as the one determined by the horizontal distribution orthogonal to the vertical distribution with respect to the standard Riemannian metric on the unit sphere, the curvature form ω is -2 times the Kähler form of the Fubini-Study metric, i.e.

$$\omega(e_i, e_j)=\begin{cases}-2 & \text{if } \{i,j\}=\{2l-1,2l\} \text{ for some } l\\ 0 & \text{otherwise} ,\end{cases} \tag{1.7}$$

where $\{e_i\}$ is an orthonormal basis of TCP^k such that

$$Je_{2l-1}=e_{2l}\,, \text{ for the complex structure } J \text{ on } CP^k .$$

From now on, we assume that η is this canonical connection.

REMARK (1.8). [A-B] and [B-V] consider the operators

$$d_X=d+i(v_X)\otimes u \quad\text{and}\quad d_X=d-i(v_X) ,$$

respectively, for constructing the de Rham model of the equivariant cohomology theory.

In particular, we can treat the Borel construction of invariant connections on S^1-equivariant bundles.

Let $P \to X$ be an S^1-equivariant principal G-bundle with an invariant connection and with a connection form θ. Then the 1-form $\psi(\theta)$ defines a connection on the principal G-bundle $P^{(k)}_{S^1} = S^{2k+1} \times_{S^1} P \to X^{(k)}_{S^1} = S^{2k+1} \times_{S^1} X$. The curvature form for the connection $\psi(\theta)$ is

$$d(\psi(\theta)) + \frac{1}{2} \cdot \psi(\theta) \wedge \psi(\theta) = \psi(d\theta + \frac{1}{2} \cdot \theta \wedge \theta) - \psi(i(v_P)\theta) \wedge \pi^* \omega .$$

Thus we get:

LEMMA (1.9). *The curvature form of the connection $\psi(\theta)$ is given by $\psi(\Omega) - \psi(i(v_P)\theta) \wedge \pi^* \omega$, where Ω is the curvature form for the connection θ on $P \to X$.*

REMARK (1.10). The horizontal distribution determined by the connection $\psi(\theta)$ is given by

$$q_*(\mathcal{H}or(S^{2k+1}) \oplus \mathcal{H}or(P)) ,$$

where $q: S^{2k+1} \times P \to P^{(k)}_{S^1}$ is the projection and $\mathcal{H}or(P)$ is the horizontal distribution on P determined by the connection θ.

Now we recall the definition of the moment maps for equivariant bundles [B-V].

DEFINITION (1.11). Let $P \to X$ be an S^1-equivariant principal G-bundle with an invariant connection θ. The map $\theta(v_P): P \to \mathfrak{g}$ is called the moment map. For an S^1-equivariant vector bundle $E \to X$ with an invariant connection, the moment map of the frame bundle $U(E)$ of E with the induced connection is also called the moment map of E. When P is a principal S^1-bundle (or E is a complex line bundle), its moment map can be regarded as a real valued function on X.

2. Calculation of the scalar curvature. Let g be an S^1-invariant Riemannian metric on X and h the standard Riemannian metric on the unit sphere S^{2k+1}. We define a Riemannian metric $g^{(k)}$ on $X^{(k)}_{S^1}$ which makes $p: (S^{2k+1} \times X, h \oplus g) \to (X^{(k)}_{S^1}, g^{(k)})$ a Riemannian submersion. Namely, $g^{(k)}$ is induced from the isomorphism

$$p_{*(t,x)}: \mathcal{H}(p)_{(t,x)} \to T_{[t,x]} X^{(k)}_{S^1} ,$$

where $\mathcal{H}(p)_{(t,x)}$ is the orthocomplement of the tangent space along the fibre.

The goal of this section is the following:

LEMMA (2.1). *The scalar curvature κ of $(X^{(k)}_{S^1}, g^{(k)})$ satisfies*

$$\kappa \geqq 2k(2k+1) + C(g) ,$$

where $C(g)$ is a constant depending on g such that $\lim_{r \to \infty} C(r \cdot g) = 0$.

First of all, we recall O'Neill's formula. Let $Z \to Y$ be a Riemannian submersion. $K^Y(u, v)$ is the sectional curvature of the plane spanned by mutually orthogonal unit vectors u and v, while $\tilde{u}$ and $\tilde{v}$ denote local horizontal vector fields which are lifts of u and v respectively. Then the vertical component $[\tilde{u}, \tilde{v}]_{\mathrm{ver}}$ of the Lie bracket $[\tilde{u}, \tilde{v}]$ depends only on u and v.

LEMMA (2.2) (O'Neill's formula). *The sectional curvatures of Z and Y satisfy the equality*

$$K^Y(u, v) = K^Z(\tilde{u}, \tilde{v}) + \frac{3}{4} \| [\tilde{u}, \tilde{v}]_{\mathrm{ver}} \|^2 .$$

REMARK (2.3). For a principal G-bundle $P \to Y$ with a connection θ, the curvature Ω for θ is a tensorial 2-form such that

$$\Omega(\tilde{u}, \tilde{v}) = -\theta([\tilde{u}, \tilde{v}]) \qquad \text{(see [K-N] Chap 2).}$$

This formula is different from the one in [K-N], since we use the convention such as $dx \wedge dy\ (\partial/\partial x, \partial/\partial y) = 1$. Thus the formula (1.7) is a consequence of Lemma (2.2).

PROOF OF LEMMA (2.1). We first calculate the curvatures at $[t, x] \in X^{(k)}_{S^1}$ such that $v_X(x) \neq 0$.

Let $\{e_i\}$ be a local orthonormal frame field of $\mathscr{H}\!or(S^{2k+1})$ around $t \in S^{2k+1}$ such that

$$J(\tau_* e_{2l-1}) = \tau_* e_{2l} \qquad (l = 1, \cdots, k),$$

where τ is the projection $S^{2k+1} \to CP^k$ and J is the complex structure on CP^k.

Let $\{f_1, \cdots, f_{m-1}\}$ be a local orthonormal system of TX around $x \in X$ such that every f_i is orthogonal to v_X. Let

$$e = \frac{1}{s\sqrt{1+s^2}}(-s^2 v_S + v_X) \in TS^{2k+1} \times TX ,$$

where v_S is the vector field on S^{2k+1} determined by the S^1-action, and the function s on X is the norm of v_X.

Notice that $\mathscr{H}\!or(S^{2k+1})$ defined in §1 in terms of the canonical connection η is the orthocomplement of the tangent bundle along the fibres with respect to h. Thus $\{p_* e_1, \cdots, p_* e_{2k}, p_* f_1, \cdots, p_* f_{m-1}, p_* e\}$ is a local orthonormal frame field of $X^{(k)}_{S^1}$, while $\{e_1, \cdots, e_{2k}, f_1, \cdots, f_{m-1}, e\}$ is a local orthonormal frame field of $\mathscr{H}(p)$. Using O'Neill's formula and comparing our case with the case of $S^{2k+1} \to CP^k$, we have

$$K(p_* e_i, p_* e_j) = \begin{cases} 1 + \dfrac{3}{1+s^2} & \text{if } \{i, j\} = \{2l-1, 2l\} \text{ for some } l \\ 1 & \text{otherwise}, \end{cases}$$

since $\{1+s^2\}^{-1/2}$ equals the inner product of v_S and $\{1+s^2\}^{-1/2}(v_S + v_X)$,

$$K(p_*e_i, p_*e)=\left(\frac{s}{\sqrt{1+s^2}}\right)^2=\frac{s^2}{1+s^2},$$

$$K(p_*f_i, p_*f_j)=K^X(f_i, f_j),$$

$$K(p_*f_i, p_*e)=\left(\frac{1}{\sqrt{1+s^2}}\right)^2\cdot K^X\left(f_i, \frac{v_X}{s}\right)\geqq\min\left(0, K^X\left(f_i, \frac{v_X}{s}\right)\right),$$

and

$$K(p_*e_i, p_*f_j)=0.$$

Therefore, the scalar curvature κ of $X^{(k)}_{S^1}$ satisfies

$$\kappa\geqq 2k(2k-1)+\frac{6k}{1+s^2}+\frac{4ks^2}{1+s^2}+C(g)\geqq 2k(2k+1)+C(g), \tag{2.4}$$

where

$$C(g)=m(m-1)\cdot\min\{\text{sectional curvatures on } X\}.$$

It is obvious that $\lim_{r\to\infty}C(r\cdot g)=0$.

If the S^1-action is non-trivial, then the set $\{x\in X; v_x(x)\neq 0\}$ is dense in X. Therefore we obtain (2.4) on X. If the S^1-action is trivial, then $(X^{(k)}_{S^1}, g^{(k)})$ is the Riemannian product of $(\boldsymbol{C}P^k$, Fubini-Study) and (X, g). Thus (2.4) holds. ■

3. Rigidity of Atiyah and Hirzebruch. In this section we give a new proof for the rigidity of the Dirac operators on spin manifolds. Atiyah and Hirzebruch has proved the vanishing of equivariant indices of Dirac operators on spin manifolds by means of the Atiyah-Singer equivariant index formula. Here "rigidity" means the constancy of the equivariant indices as virtual character. If the S^1-action is not lifted to the spin structure P, then we consider the double covering action, which is lifted to P. Thus we may assume that the S^1-action is lifted to the spin structure. More precisely, we prove the following:

PROPOSITION (3.1) (cf. [A-H]). *Let M be a closed spin manifold with an S^1-action. Then the S^1-equivariant index S^1-ind $D\in R(S^1)$ of the Dirac operator D on M is constant.*

Before proving this proposition, we review the definition of twisted Dirac operators on spin^c manifolds. The details can be found in [Hi], which treats the Dirac operators on spin^c manifolds, and [G-L], which treats the twisted Dirac operators.

A manifold X is called a spin^c manifold if the second Stiefel-Whitney class $w_2(X)$ is contained in the image of the Bockstein homomorphism $\beta\colon H^2(X;\boldsymbol{Z})\to H^2(X;\boldsymbol{Z}/2)$. This condition is equivalent to the following:

CONDITION (3.2). There is a $\mathrm{Spin}^c(m)$-principal bundle $Q \to X$ such that $Q \times_{\rho_1} SO(m) \to X$ is the oriented frame bundle of X, where m is the dimension of X and ρ_1 is the natural homomorphism $\mathrm{Spin}^c(m) = \mathrm{Spin}(m) \times_{Z/2} S^1 \to SO(m)$.

We call $Q \to X$ a spin^c structure on X and the line bundle $L_Q = Q \times_{\rho_2} C \to X$ the associated complex line bundle, where ρ_2 is the natural homomorphism $\mathrm{Spin}^c(m) \to S^1/\{\pm 1\} = U(1)$. It is well known that $\beta(c_1(L_Q)) = w_2(X)$. Fixing a connection on L_Q and a Riemannian metric on X, we can define a connection on Q. If $\dim X = 2n$, then $\mathrm{Spin}^c(2n)$ has $\pm$ half spinor representations V^+ and V^-. We define the Dirac operator on X by

$$D = \sum_i e_i \cdot \nabla_{e_i} : \Gamma(X; S^+) \to \Gamma(X; S^-), \tag{3.3}$$

where $S^{\pm} = Q \times_{\mathrm{Spin}^c(2n)} V^{\pm}$ is the $\pm$ half spinor bundles, $\{e_i\}$ is a local oriented orthonormal frame field on X and ∇ is the covariant differentiation with respect to the connection on Q. The Weitzenböck formula for D (cf. [Hi]) is given by

$$D^*D = \nabla^*\nabla + \frac{1}{4} \cdot \kappa + \frac{1}{2} \cdot \sum_{i<j} e_i e_j \otimes \Omega^L(e_i, e_j), \tag{3.4}$$

where Ω^L is the curvature form of $L = L_Q$.

The Dirac operator D_E twisted by a vector bundle E with a connection is defined by

$$D_E = \sum_i e_i \cdot \nabla_{e_i} : \Gamma(X; S^+ \otimes E) \to \Gamma(X; S^- \otimes E), \tag{3.5}$$

where ∇ is the covariant differentiation defined by connections on Q and E. The Weitzenböck formula for D_E (cf. [G-L]) is given by

$$D_E^* D_E = \nabla^*\nabla + \frac{1}{4} \cdot \kappa + \frac{1}{2} \cdot \sum_{i<j} e_i e_j \otimes \Omega^L(e_i, e_j) + \sum_{i<j} e_i e_j \otimes \Omega^E(e_i, e_j), \tag{3.6}$$

where Ω^E is the curvature form of E.

PROOF OF PROPOSITION (3.1). We consider a family $\mathscr{D}$ of Dirac operators along the fibres of the fibre bundle $\pi : M^{(k)}_{S^1} \to CP^k$.

Claim 1. $\mathrm{ind}\,\mathscr{D} = \gamma^{(k)}(S^1\text{-}\mathrm{ind}\,D)$ holds in $K(CP^k)$, where $\gamma^{(k)} : R(S^1) \to K(CP^k)$ is defined by $\gamma^{(k)}(V) = S^{2k+1} \times_{S^1} V$ for an S^1-module V.

$\mathrm{ind}\,(p^*\mathscr{D}) \in K_{S^1}(S^{2k+1})$ for the family $p^*\mathscr{D}$ of Dirac operators on the fibre bundle $S^{2k+1} \times M \to S^{2k+1}$ is represented by $S^{2k+1} \times \ker D - S^{2k+1} \times \mathrm{coker}\,D$. Since $\mathrm{ind}\,\mathscr{D}$ is the image of $\mathrm{ind}(p^*\mathscr{D})$ by the isomorphism $K_{S^1}(S^{2k+1}) \xrightarrow{\simeq} K(CP^k)$, we have $\mathrm{ind}\,\mathscr{D} = \gamma^{(k)}(S^1\text{-}\mathrm{ind}\,D)$.

As we mentioned above, a Dirac operator on a spin^c manifold is determined by a complex line bundle L with a connection which satisfies $\beta(c_1(L)) = w_2(M)$. On CP^k with

k even, we use the Dirac operator D_{CP^k} with the hyperplane section bundle L with the canonical connection. On CP^k with k odd, we use the usual Dirac operator. We choose a spin^c structure on $M_{S^1}^{(k)}$ as follows: Let $\bar{Q}=\Delta^*(Q_{S^1}^{(k)}\times\pi^*P_{CP^k})$, where $\Delta: M_{S^1}^{(k)}\to M_{S^1}^{(k)}\times M_{S^1}^{(k)}$ is the diagonal inclusion map, $Q\to M$ is the spin structure on M, and P_{CP^k} is the spin^c structure mentioned above. $\bar{Q}$ is a principal $\mathrm{Spin}(m)\times\mathrm{Spin}^c(2k)$-bundle. The spin^c structure on $M_{S^1}^{(k)}$ which we use is $\bar{Q}\times_\iota\mathrm{Spin}^c(m+2k)$, where ι is the homomorphism $\mathrm{Spin}(m)\times\mathrm{Spin}^c(2k)=\mathrm{Spin}(m)\times(\mathrm{Spin}(2k)\times_{Z/2}S^1)\to\mathrm{Spin}^c(m+2k)$. Then the associated complex line bundle of this spin^c structure is π^*L. The Dirac operator $D_{M_{S^1}^{(k)}}$ is defined by the pull-back of the canonical connection on the hyperplane section bundle. When k is odd, $M_{S^1}^{(k)}$ is a spin manifold and we use the usual Dirac operator. We also use the notation $D_{CP^k}\otimes\mathrm{ind}\,\mathscr{D}$ for the Dirac operator twisted by $\mathrm{ind}\,\mathscr{D}$.

Claim 2. $$\mathrm{ind}(D_{CP^k}\otimes\mathrm{ind}\,\mathscr{D})=\mathrm{ind}\,D_{M_{S^1}^{(k)}}\,.$$

This is a special case of the multiplicative axiom of the index in [A-S I]. Claim 2 is also derived from the index formula for a family of elliptic operators in [A-S IV] and the index formula in [A-S III].

Claim 3. $$\mathrm{ind}\,D_{M_{S^1}^{(k)}}=0 \qquad \text{for every} \quad k\geqq 1\,.$$

Indeed, we use the vanishing theorem of Lichnerowicz [L] and Hitchin [Hi] (cf. (3.4)). For odd k, Lichnerowicz's vanishing theorem and Lemma (2.1) insures that $\mathrm{ind}\,D_{M_{S^1}^{(k)}}=0$. Then k is even, let $\{p_*e_1,\cdots,p_*e_{2k},p_*f_1,\cdots,p_*f_{m-1},p_*e\}$ denote an orthonormal basis as in the proof of Lemma (2.1). Since the curvature form Ω^{π^*L} equals $-\pi^*\omega$, only the terms $\Omega^{\pi^*L}(p_*e_i,p_*e_j)$ do not vanish. Lemma (3.1), Formula (1.7), and Hitchin's vanishing theorem insures that $\mathrm{ind}\,D_{M_{S^1}^{(k)}}=0$.

Claim 4.

$$\mathrm{ch}^{(2l)}(\mathrm{ind}\,\mathscr{D})=0 \quad \text{in} \quad H^{2l}(CP^k;\,Q) \quad \text{for} \quad l\geqq 1 \tag{3.7}$$

where $\mathrm{ch}^{(2l)}$ denotes the $2l$ degree component of ch.

We show this by induction on l.

Suppose $l=1$. By Claims 1, 2 and 3, we have

$$\mathrm{ind}(D_{CP^1}\otimes\gamma^{(1)}(S^1\text{-ind}\,D))=0\,.$$

The Atiyah-Singer index formula says

$$\mathrm{ch}(\gamma^{(1)}(S^1\text{-ind}\,D))\cdot\hat{\mathfrak{A}}(CP^1)[CP^1]=0\,.$$

Since $\hat{\mathfrak{A}}(CP^1)=1$, we get $\mathrm{ch}^{(2)}(\gamma^{(1)}(S^1\text{-ind}\,D))=0$. Let $\iota_1^k: CP^1\to CP^k$ be the inclusion map. Then $\iota_1^k\mathrm{ch}^{(2)}(\gamma^{(k)}(S^1\text{-ind}\,D))=\mathrm{ch}^{(2)}(\gamma^{(1)}(S^1\text{-ind}\,D))$ holds. Since $\iota_1^{k*}: H^2(CP^k)\to H^2(CP^1)$ is injective, we get

$$\mathrm{ch}^{(2)}(\gamma^{(k)}(S^1\text{-ind}\,D))=0 \qquad \text{for} \quad k\geqq 1\,.$$

Suppose that (3.7) holds for $l \leqq l_0$. Since $\operatorname{ind} D_{CP^{l_0+1}} \otimes \gamma^{(l_0+1)}(S^1\text{-ind}\, D)=0$, the Atiyah-Singer index formula says

$$\operatorname{ch}(\gamma^{(l_0+1)}(S^1\text{-ind}\, D)) \cdot \operatorname{ch}(L) \cdot \hat{\mathfrak{A}}(CP^{l_0+1})[CP^{l_0+1}]=0 \qquad (l_0\text{: odd})$$

$$\operatorname{ch}(\gamma^{(l_0+1)}(S^1\text{-ind}\, D)) \cdot \hat{\mathfrak{A}}(CP^{l_0+1})[CP^{l_0+1}]=0 \qquad (l_0\text{: even})\,.$$

By the induction assumption,

$$\operatorname{ch}(\gamma^{(l_0+1)}(S^1\text{-ind}\, D)) = \operatorname{ind} D + \operatorname{ch}^{(2l_0+2)}(\gamma^{(l_0+1)}(S^1\text{-ind}\, D))\,.$$

It is well-known that

$$\operatorname{ch}(L) \cdot \hat{\mathfrak{A}}(CP^{l_0+1})[CP^{l_0+1}]=0 \qquad (l_0\text{: odd})$$

$$\hat{\mathfrak{A}}(CP^{l_0+1})[CP^{l_0+1}]=0 \qquad (l_0\text{: even})\,,$$

for instance, as a consequence of the vanishing theorem of Lichnerowicz and Hitchin. Thus we have $\operatorname{ch}^{(2l_0+2)}(\gamma^{(l_0+1)}(S^1\text{-ind}\, D)=0$, which implies $\operatorname{ch}^{(2l_0+2)}(\gamma^{(k)}(S^1\text{-ind}\, D))=0$ for $k \geqq 1$.

Claim 5. $\qquad S^1\text{-ind}\, D \in R(S^1)$ is constant.

Claim 5 is derived from Claim 4 and the following:

LEMMA (3.8). *Let ξ be a virtual S^1-module, i.e., a formal difference of S^1-modules. If $\operatorname{ch}(\gamma^{(k)}(\xi))$ has only the zero degree component, the virtual character of ξ is constant in $R(S^1)$, i.e., ξ is a virtual vector space with a trivial S^1-action.*

PROOF OF LEMMA (3.8). $\{\gamma^{(k)}\}$ induces a map $\gamma\colon R(S^1) \to \operatorname{proj\,lim}_l K(CP^l)$ and the following commutative diagram.

$$\begin{array}{ccccc} R(S^1) & \xrightarrow{\ \gamma\ } & \operatorname{proj\,lim}_l K(CP^l) & \xrightarrow{\ \operatorname{ch}\ } & \operatorname{proj\,lim}_l H(CP^l; Q) \\ & \searrow^{\gamma^{(k)}} & \downarrow & & \downarrow p_k \\ & & K(CP^k) & \xrightarrow{\ \operatorname{ch}\ } & H(CP^k; Q) \end{array}$$

It is easy to see that $\operatorname{ch} \circ \gamma$ is injective. By the assumption of Lemma (3.8), $\operatorname{ch} \circ \gamma(\xi)$ has only the zero component, which implies the conclusion. ∎

Thus Claim 5 is proved and the proof of Proposition (3.1) is completed. ∎

REMARK (3.9). For even k, we can also use complex quadrics instead of complex projective spaces, which are spin, since BS^1 is also the inductive limit of complex quadrics. Then we need not consider Dirac operators on spin^c manifolds.

4. Equivariant index of twisted Dirac operators. Let M be a closed manifold with an S^1-equivariant spin^c structure $Q \to M$, $H \to M$ the complex line bundle associated to

Q, i.e. $H=Q\times_{\rho_2}C$, and $E\to M$ an S^1-equivariant vector bundle with an invariant unitary connection. We consider the Dirac operator $D_M\otimes E$ twisted by E. Then S^1-ind$(D_M\otimes E)$ is a virtual character of S^1. Identifying the character ring $R(S^1)$ with $\mathbf{Z}[t, t^{-1}]$, we can write $S^1\text{-ind}(D_M\otimes E)=\sum_j m_j t^j$. We shall give a restriction to possible weights appearing in S^1-ind$(D_M\otimes E)$.

THEOREM (4.1).

$$m_j=0 \quad \text{if} \quad |j|\geqq \max\left(\frac{1}{2}\cdot|\mu^H|+\|\mu^E\|\right)+\frac{3}{2}, \tag{4.2}$$

where $\mu^E\colon U(E)\to \mathrm{u}(r)$, $r=\operatorname{rank} E$, *and* $\mu^H\colon M\to \mathbf{R}$ *are the moment maps of* E *and* H, *respectively, and* $\|\ \ \|$ *denotes the operator norm on* $\mathrm{u}(r)$.

LEMMA (4.3). *Let* $\xi=\sum_j n_j t^j$ *be a virtual* S^1 *module, and assume that*

$$\operatorname{ind}(D_{\mathbf{CP}^k}\otimes\gamma^{(k)}(\xi))=0 \qquad \text{for} \quad k\geqq l.$$

Then we have

$$n_j\neq 0 \qquad \text{only if} \begin{cases} -\dfrac{l-1}{2}\leqq j\leqq \dfrac{l-1}{2} & \text{and } l \text{ is odd},\\[2mm] -\dfrac{l-2}{2}\leqq j\leqq \dfrac{l}{2} & \text{and } l \text{ is even}.\end{cases} \tag{4.4}$$

PROOF OF LEMMA (4.3). We shall prove (4.4) by induction on l. We have nothing to prove in the case $l=1$ (see Lemma (3.8) and the proof of Claim 4 in §3). We recall the following fact.

LEMMA (4.5) (Hattori [Ha], Kawakubo [K]). *Let* $F\to\mathbf{CP}^k$ *be the tautological line bundle. Then the following hold:*

$$\operatorname{ind} D_{\mathbf{CP}^k}\otimes F^j=0 \qquad \text{if} \begin{cases} -\dfrac{k-1}{2}\leqq j\leqq \dfrac{k-1}{2} & \text{and } k \text{ is odd}\\[2mm] -\dfrac{k-2}{2}\leqq j\leqq \dfrac{k}{2} & \text{and } k \text{ is even}\end{cases}$$

$$\operatorname{ind} D_{\mathbf{CP}^k}\otimes F^j=1 \qquad \text{if} \begin{cases} j=-\dfrac{k+1}{2} & \text{and } k \text{ is odd}\\[2mm] j=-\dfrac{k}{2} & \text{and } k \text{ is even}\end{cases}$$

$$\operatorname{ind} D_{\mathbf{CP}^k}\otimes F^j=(-1)^k \qquad \text{if} \begin{cases} j=\dfrac{k+1}{2} & \text{and } k \text{ is odd}\\[2mm] j=\dfrac{k+2}{2} & \text{and } k \text{ is even}.\end{cases}$$

Listed in Table are values of $\operatorname{ind} D_{CP^k} \otimes F^j$ for several pairs of j and k.

TABLE.

j	k				
	1	2	3	4	5
3	−3	3	−4	1	−1
2	−2	1	−1	0	0
1	−1	0	0	0	0
0	0	0	0	0	0
−1	1	1	0	0	0
−2	2	3	1	1	0
−3	3	6	4	5	1

REMARK (4.6). (1) The Hopf fibration $S^{2k+1} \to CP^k$ can be regarded as a principal S^1-bundle, i.e., $S^1 \subset C$ acts on $S^{2k+1} \subset C^{k+1}$ from the right by complex multiplication. Thus the line bundle $\gamma^{(k)}(t)$ is the tautological line bundle $F \to CP^k$.

(2) Lemma (4.5) can be seen to be a consequence of the relation between the Dirac operators and the Dolbeault operators on Kähler manifolds (cf. [Hi]).

Next we show (4.4) for $l = l_0 + 1$ under the assumption that (4.4) holds for $l \leqq l_0$. Set $d = \operatorname{ind} D_{CP^{l_0}} \otimes \gamma^{(k)}(\xi)$.

Suppose l_0 is odd. Set $\zeta = d \cdot t^{(l_0+1)/2}$ and $\xi' = \xi - \zeta$. Lemma (4.5) says $\operatorname{ind} D_{CP^k} \otimes \gamma^{(k)}(\xi') = 0$ for $k \geqq l_0$. By the induction assumption, we have an expression

$$\xi' = \sum_{j=-(l_0-1)/2}^{(l_0-1)/2} m'_j \cdot t^j$$

which implies (4.4) for ξ.

The proof in the case l_0 even is similar with $\zeta = -dt^{l_0/2}$. ■

PROOF OF THEOREM (4.1). From Lemma (4.3), it is sufficient to prove that

$$\operatorname{ind} D_{CP^k} \otimes \gamma^{(k)}(S^1\text{-ind}(D_M \otimes E)) = 0 \tag{4.7}$$

for $k \geqq \max(|\mu^H| + 2 \cdot \|\mu^E\|) + 1$. As in the proof of Proposition (3.1), we have

$$\operatorname{ind} D_{CP^k} \otimes \gamma^{(k)}(S^1\text{-ind}(D_M \otimes E)) = \operatorname{ind} D_{CP^k} \otimes (\operatorname{ind} \mathscr{D} \otimes E^{(k)}_{S^1}) = \operatorname{ind} D_{M^{(k)}_{S^1}} \otimes E^{(k)}_{S^1}.$$

Let $\{e_1, \cdots, e_{2k}, f_1, \cdots, f_{m-1}, e\}$ be as in §2. By Lemma (1.9), the curvature form on $E^{(k)}_{S^1}$ is given by

$$\bar{\Omega} = \psi(\Omega) - \psi(i(v_P)\theta) \wedge \pi^*\omega,$$

where θ is the given connection on the frame bundle P of E and Ω is its curvature form. This implies that

$$\bar{\Omega}(p_* e_i, p_* e_j) = -i(v_P)\theta \cdot \omega(\tau_* e_i, \tau_* e_j) = -\mu(E) \cdot \omega(\tau_* e_i, \tau_* e_j)$$

$$\bar{\Omega}(p_* f_j, p_* f_j) = \Omega(f_i \cdot f_j)$$

$$\bar{\Omega}(p_* e_i, p_* f_j) = 0$$

$$\bar{\Omega}(p_* e_i, p_* e) = 0$$

and

$$\bar{\Omega}(p_* f_i, p_* e) = \frac{1}{\sqrt{1+s^2}} \cdot \Omega(f, \frac{v_M}{s}) .$$

If we change a Riemannian metric g on M with $r \cdot g$ for $r > 0$, we can easily see that

$$\lim_{r \to \infty} \bar{\Omega}(p_* f_i, p_* f_j) = 0 \quad \text{and} \quad \lim_{r \to \infty} \bar{\Omega}(p_* f_i, p_* e) = 0 .$$

Then (4.7) is derived from the standard Bochner technique and Formulas (1.7) and (3.6). ■

5. Family of Dirac operators. Let X be a compact space and $N \to X$ a Riemannian submersion whose tangent bundle along the fibres has a spin structure. We denote by $\mathscr{D}_{N \to X}$ the family of Dirac operators associated to the spin structure. If S^1 acts on N preserving the fibres and the spin structure, then the equivariant index of $S^1\text{-ind}\,\mathscr{D}_{N \to X} \in K_{S^1}(X) = K(X) \otimes R(S^1)$ of $\mathscr{D}_{N \to X}$ is defined.

We can show the following:

THEOREM (5.1). *S^1-ind $\mathscr{D}_{N \to X}$ lies in the image* $\mathrm{Im}\{K(X) \to K(X) \otimes R(S^1)\}$, *i.e. the virtual S^1-vector bundle S^1-ind $\mathscr{D}_{N \to X}$ is a virtual vector bundle with trivial S^1-action.*

The proof goes as follows:

Step 1. Let $N^{(k)}_{S^1} = S^{2k+1} \times_{S^1} N \to CP^k \times X$ be the Borel construction of the Riemannian submersion $N \to X$. Then we see that

$$\mathrm{ind}\, D_{CP^k} \otimes (\mathrm{ind}\, \mathscr{D}_{N^{(k)}_{S^1} \to CP^k \times X}) = \mathrm{ind}\, \mathscr{D}_{N^{(k)}_{S^1} \to X} .$$

Since X is compact, $\mathscr{D}_{N^{(k)}_{S^1} \to X}$ is a family of invertible operators for some Riemannian metric along the fibres of $N \to X$. Hence its index is zero.

Step 2. As in the proof of Proposition (3.1), there exists a homomorphism $\gamma^{(k)}_X \colon K(X) \otimes R(S^1) \to K(X) \otimes K(CP^k) \to K(X \times CP^k)$. It is easy to see that $\gamma^{(k)}_X(S^1\text{-ind}\, \mathscr{D}_{N \to X}) = \mathrm{ind}\, \mathscr{D}_{N^{(k)}_{S^1} \to X}$. By the Künneth formula and the fact that $K(CP^k)$ is free, the homomorphism $K(X) \otimes K(CP^k) \to K(X \times CP^k)$ is an isomorphism and there is the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(X)\otimes\tilde{K}(S^{2k}) & \longrightarrow & K(X)\otimes K(CP^k) & \longrightarrow & K(X)\otimes K(CP^{k-1}) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & K(X\times CP^k) & \longrightarrow & K(X\times CP^{k-1}) & & \end{array}$$

Since $K(X)\otimes\tilde{K}(S^{2k})\to K(X\times CP^k)\xrightarrow{p_!}K(X)$ is an isomorphism, we obtain $\gamma_X^{(k)}(S^1\text{-}\operatorname{ind}\mathscr{D}_{N\to X})\in\operatorname{Im}\{K(X)\xrightarrow{p^*}K(X\times CP^k)\}$, where p denotes the projection to the first factor and $p_!$ is the Gysin map.

Step 3. It can be seen that $\operatorname{proj\,lim}_k\gamma_X^{(k)}\colon K(X)\otimes R(S^1)\to\operatorname{proj\,lim}_k K(X\times CP^k)$ is injective, which implies the conclusion. ■

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