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ALGEBRAIC SURFACES OF GENERAL TYPE WITH $c_1^2=3p_g-7$

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Introduction. Let S be a minimal algebraic surface of general type defined over the complex number field $\boldsymbol{C}$. Castelnuovo's second inequality states that if the canonical map of S is birational, then $c_1^2(S)\geq 3p_g(S)-7$ (see [4], [10, II, §1], [1]).

In the present paper, we study minimal algebraic surfaces of general type with $c_1^2=3p_g-7$. These surfaces are classified into two types according to the nature of their canonical map Φ_K:

Type I: Φ_K is a birational holomorphic map onto its image.

Type II: Φ_K birationally induces a double covering of a ruled surface.

Historically, surfaces of type I were already known to Castelnuovo [4]. He showed that the canonical image of a type I surface is always contained in a threefold of minimal degree and he determined its divisor class. For a modern treatment of his argument, see Harris [6]. On the other hand, Horikawa [9], [10, IV] has studied, among others, surfaces of types I and II in detail when $(p_g, c_1^2)=(4, 5), (5, 8)$. Especially he completely determined their deformation types. Surfaces of type I with $p_g=7$ and $c_1^2=14$ were recently studied by Miranda [12].

The paper consists of two parts: §§1–4 and §§5–6. The former part is devoted to surfaces of type I. In §1, we show that surfaces with $c_1^2=3p_g-7$ are divided into two types mentioned above and review Castelnuovo's argument to classify surfaces of type I according to the threefold W on which the canonical image lies. We remark that, in most cases, W is a rational normal scroll (see, [6] and [5]). We prove that the canonical image has only rational double points and that almost all type I surfaces have a pencil of nonhyperelliptic curves of genus three (Theorem 1.5). Proof of some Claims needed in §1, concerning the liftability of the canonical map to a nonsingular model of W, is postponed to §2. The technique employed here is essentially due to Horikawa [10]. In §3 and § 4, we study deformations of type I surfaces and compute the number of moduli (Theorem 3.2 and Proposition 4.3). Though we try to determine their deformation types, many cases are left unsettled. In §4, we construct a family of surfaces in which the central fiber is of type II and a general fiber is of type I.

The latter part, §§5–6, is devoted to surfaces of type II. In view of the vanishing of irregularity of a type I surface (see, §1), we restrict ourselves to regular surfaces of type II. Our concerns here are pencils of hyperelliptic curves. From a remarkable result of Xiao [16], we know that a surface of type II has such a pencil of genus less than five provided $p_g \geq 46$. In §5, we construct minimal surfaces with pencils of hyperelliptic curves of genus 3 whose invariants (p_g, c_1^2) cover a certain area in the zone of existence, which of course contains the line $c_1^2 = 3p_g - 7$ (Theorem 5.7). By the same method, we can show the existence of type II surfaces with pencils of hyperelliptic curves of genus 2, 3 or 4 (Proposition 6.3).

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1. Canonical map and surfaces of type I. Let S be a minimal algebraic surface of general type defined over the complex number field $\boldsymbol{C}$ for which the geometric genus $p_g(S)$ and the Chern number $c_1^2(S)$ satisfy the conditions $c_1^2 = 3p_g - 7$ and $p_g \geq 3$. We let $\Phi_K : S \to \boldsymbol{P}^{p_g - 1}$ denote the rational map defined by the canonical linear system $|K|$. We put $S' = \Phi_K(S)$ and call it the canonical image of S. We denote by $\phi_K : S \to S'$ the natural map induced by Φ_K.

LEMMA 1.1. *Let S be as above. Then we have the following two possibilities*:
(1) *$|K|$ is free from base points and ϕ_K is a birational holomorphic map.*
(2) *ϕ_K is a rational map of degree 2 and S' is birationally equivalent to a ruled surface.*

PROOF. We remark that ϕ_K is generically finite, since $|K|$ is not composite with a pencil by [1, Lemma 5.3]. Since S' is irreducible and nondegenerate (i.e., is not contained in any hyperplane in $\boldsymbol{P}^{p_g - 1}$), we have the inequality

$$c_1^2 \geq (\deg \phi_K)(\deg S') \geq (\deg \phi_K)(p_g - 2) .$$

Thus we have $\deg \phi_K \leq 2$. If $\deg \phi_K = 1$, then $|K|$ has no base points by [10, II, Lemma (1.1)] and [9, Lemma 2]. If $\deg \phi_K = 2$, then we get $\deg S' < 2p_g - 4$. Therefore it follows from [1, Lemma 1.4] that S' is birationally equivalent to a ruled surface. q.e.d.

We say that S is of type I or of type II according as whether the degree of ϕ_K is 1 or 2.

1.2. Surfaces of type I were essentially known to Castelnuovo [4]. Here we recall his argument. Our reference is [7] and [6].

We recall fundamental properties of the Hilbert function h_X defined for any projective variety $X \subset \boldsymbol{P}^r$ by

$$h_X(n) = \dim_{\boldsymbol{C}} \mathrm{Im}\{\rho : H^0(\boldsymbol{P}^r, \mathcal{O}(n)) \to H^0(X, \mathcal{O}(n))\} ,$$

where ρ is the restriction map and n is a nonnegative integer. If Y is a general hyperplane section of X, then we have for any $n > 0$

$$\delta h_X(n) := h_X(n) - h_X(n-1) \geq h_Y(n) . \qquad (1)$$

We remark that X is projectively normal if $\delta h_X(n) = h_Y(n)$ holds for any n.

Now let S be a surface of type I and put $r=p_g-2$. Since $|K|$ has no base point, a general member $C\in|K|$ is irreducible nonsingular and has genus $g(C)=3r$. If we put $C'=\Phi_K(C)$, then it is an irreducible nondegenerate curve in $\boldsymbol{P}^r\subset\boldsymbol{P}^{r+1}$ and $\deg C'=K^2=3r-1$. We let Γ denote a general hyperplane section of C'. Since it is a nondegenerate set of $3r-1$ distinct points in uniform position, we have

(2) $$h_\Gamma(n+1)\geq\min\{3r-1,\, h_\Gamma(n)+r-1\}\,.$$

Since $2K|_C$ is the canonical divisor of C and $h_{C'}(1)=r+1$, it follows from (1) that

$$3r=h^0(C,\mathcal{O}(2K|_C))\geq h_{C'}(2)\geq r+1+h_\Gamma(2)\,.$$

This and (2) show $h_\Gamma(2)=2r-1$ and $h_{C'}(2)=3r$. By a similar calculation, one gets $h^0(C,\mathcal{O}(nK|_C))=h_{C'}(n)$ and $\delta h_{C'}(n)=h_\Gamma(n)$ for any $n>0$. This implies that C' is projectively normal.

We turn our attention to the canonical image S'. By the well-known formula for pluri-genera of minimal surfaces of general type combined with (1), we get

$$4r+2-q(S)=h^0(S,\mathcal{O}(2K))\geq h_{S'}(2)\geq h_{S'}(1)+h_{C'}(2)=4r+2\,.$$

From this, we have $q(S)=0$, $h^0(2K)=h_{S'}(2)$ and $\delta h_{S'}(2)=h_{C'}(2)$. By a similar calculation, one can show $h_{S'}(n)=h^0(S,\mathcal{O}(nK))$, $\delta h_{S'}(n)=h_{C'}(n)$ for any $n>0$. Therefore, S' is also projectively normal and the multiplication map $\mathrm{Sym}^n H^0(S,\mathcal{O}(K))\to H^0(S,\mathcal{O}(nK))$ is surjective for any $n\geq 0$. This implies that the canonical ring of S is generated in degree 1 and therefore S' is isomorphic to the canonical model of S. In particular, S' has only rational double points (RDP's, for short) as its singularity.

We show that S' is contained in an irreducible threefold W of minimal degree $r-1$ in $\boldsymbol{P}^{r+1}$, cut out by all quadrics through S'. Since $h_\Gamma(2)=2r-1$, Castelnuovo's Lemma (see, e.g., [7]) shows that Γ lies on a rational normal curve R of degree $r-1$ in $\boldsymbol{P}^{r-1}$ cut out by all quadrics containing Γ. From this, we get $h^0(\boldsymbol{P}^{r-1},\mathcal{I}_\Gamma(2))=h^0(\boldsymbol{P}^{r-1},\mathcal{I}_R(2))=(r-1)(r-2)/2$, where $\mathcal{I}_X$ is the ideal sheaf of X. On the other hand, we have $h^0(\boldsymbol{P}^{r+1},\mathcal{I}_{S'}(2))=h^0(\boldsymbol{P}^{r+1},\mathcal{O}(2))-h^0(S,\mathcal{O}(2K))=(r-1)(r-2)/2$. Therefore, the linear system $|\mathcal{I}_{S'}(2)|$ of quadrics through S' is restricted onto $|\mathcal{I}_\Gamma(2)|$ isomorphically, and its base locus W is an irreducible threefold of minimal degree.

1.3. To describe W, we introduce some notation. Let $\mathcal{E}$ be a locally free sheaf of rank p on $\boldsymbol{P}^q$ and let $\varpi\colon\boldsymbol{P}(\mathcal{E})\to\boldsymbol{P}^q$ be the associated projective bundle. Then the Picard group of $\boldsymbol{P}(\mathcal{E})$ is generated by the tautological divisor T such that $\varpi_*\mathcal{O}(T)=\mathcal{E}$ and the pull-back F by ϖ of a hyperplane in $\boldsymbol{P}^q$. We note that the canonical boundle of $\boldsymbol{P}(\mathcal{E})$ is given by

(3) $$K_{\boldsymbol{P}(\mathcal{E})}=\mathcal{O}(-pT+(\deg(\det\mathcal{E})-q-1)F)\,.$$

According to the classification of irreducible nondegenerate threefolds of minimal degree in $\boldsymbol{P}^{p_g-1}$ (cf. [5] or [6]), W is one of the following:

(A) P^3 ($p_g=4$).

(B) a hyperquadric ($p_g=5$).

(C) a cone over the Veronese surface, i.e., the image of the P^1-bundle $\tilde{W}=P(\mathcal{O}_{P^2}\oplus\mathcal{O}_{P^2}(2))$ under the holomorphic map Φ_T induced by $|T|$ ($p_g=7$).

(D) a rational normal scroll, i.e., the image of the P^2-bundle $P_{a,b,c}=P(\mathcal{O}_{P^1}(a)\oplus\mathcal{O}_{P^1}(b)\oplus\mathcal{O}_{P^1}(c))$ on P^1 under the holomorphic map Φ_T induced by $|T|$ ($p_g\geq 6$), where a, b, c are integers satisfying

$$0\leq a\leq b\leq c\,,\qquad a+b+c=p_g-3\,. \tag{4}$$

1.4. We study S more closely in each of the above cases. Claims I-III below will be proved in the next section.

The first two may be clear:

Case (A): S' is a quintic surface in P^3.

Case (B): S' is a complete intersection of a quadric and a quartic.

These are extensively studied by Horikawa in [9], [10, IV].

Case (C): The map Φ: $\tilde{W}\to W$ is the contraction of the divisor $T_\infty\sim T-2F$, where the symbol $\sim$ means the linear equivalence.

CLAIM I. *We have a holomorphic map μ: $S\to P^2$ of degree* 3. *Let ϕ: $S\to W$ be the natural map induced by the canonical map. Then ϕ can be lifted to a holomorphic map ψ: $S\to\tilde{W}$ over μ such that $K=\psi^*T$. Further, $S''=\psi(S)$ has only* RDP's.

We show that S'' is linearly equivalent to $3T+F$. Since μ is of degree 3, S'' is linearly equivalent to $3T+\alpha F$ for some integer α. Then, since $\deg S'=14$, we have

$$14=T^2(3T+\alpha F)=12+2\alpha\,,$$

where we used the relation $T^2=2TF$ in the Chow ring of $\tilde{W}$. Therefore $S''\sim 3T+F$. We note that the linear system $|3T+F|$ is free from base points and contains an irreducible nonsingular member.

We compute the invariants of S'' for the sake of completeness. Since $\tilde{W}$ is rational, we have $H^q(\tilde{W},\mathcal{O}(K_{\tilde{W}}))=0$ for $q<3$. By the cohomology long exact sequence for

$$0\to\mathcal{O}(K_{\tilde{W}})\to\mathcal{O}(K_{\tilde{W}}+S'')\to\omega_{S''}\to 0\,,$$

we get $H^q(S'',\omega_{S''})\simeq H^q(\tilde{W},\mathcal{O}(T))\simeq H^q(P^2,\mathcal{O}\oplus\mathcal{O}(2))$ for $q<2$. This shows $p_g(S''):=h^0(\omega_{S''})=7$ and $h^1(\omega_{S''})=h^1(\mathcal{O}_{S''})=0$. Further, since $\omega_{S''}=\mathcal{O}_{S''}(T)$, we get $\omega_{S''}^2=14=3p_g(S'')-7$.

Case (D): This case is divided into three subcases

$$\text{(D.1)}: a>0\,,\quad \text{(D.2)}: a=0, b>0\,,\quad \text{(D.3)}: a=b=0\,.$$

We remark that W is singular in the cases (D.2) and (D.3).

CLAIM II. (D.3) *cannot occur. If* (D.2) *is the case, then there is a lifting ψ: $S\to P_{0,b,c}$*

of the natural map $S \to W$ *such that* $K = \psi^* T$. *Further,* $S'' = \psi(S)$ *has only* RDP's.

We let $\psi: S \to \boldsymbol{P}_{a,b,c}$ denote the map induced by Φ_K in Case (D.1) and the map in Claim II in Case (D.2). Put $S'' = \psi(S)$. It is nothing but S' in Case (D.1). We show that S'' is linearly equivalent to $4T - (p_g - 5)F$. For this purpose, put $S'' \sim \alpha T + \beta F$. Note that the fibers of $\varpi|_{S''}$ are plane curves of degree α. Since S'' is birational to the surface S of general type, we have $\alpha \geq 4$. Recall that we have $T^3 = (p_g - 3)T^2 F$ in the Chow ring of $\boldsymbol{P}_{a,b,c}$. Since $\deg S' = 3p_g - 7$, we have

$$3p_g - 7 = T^2(\alpha T + \beta F) = (p_g - 3)\alpha + \beta .$$

On the other hand, it follows from (3) that $K_{\boldsymbol{P}_{a,b,c}} + S'' \sim (\alpha - 3)T + (p_g - 5 + \beta)F$. Since T and $K_{\boldsymbol{P}_{a,b,c}} + S''$ are equivalent on S'', we get

$$0 = TS''(K_{\boldsymbol{P}_{a,b,c}} + S'' - T) = \alpha(\alpha - 4)T^3 + \beta(\alpha - 4)T^2 F = (\alpha - 4)(\alpha T^3 + \beta) .$$

From these, we get $S'' \sim 4T - (p_g - 5)F$. The numerical invariants can be computed similarly as in Case (C): for $q < 2$, we have $h^q(\omega_{S''}) = h^q(\boldsymbol{P}_{a,b,c}, \mathcal{O}(T)) = h^0(\boldsymbol{P}^1, \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ and thus $p_g(S'') = a + b + c + 3 = p_g(S)$ by (4) and $h^1(\omega_{S''}) = 0$; since $\omega_{S''} = \mathcal{O}_{S''}(T)$, we get $\omega_{S''}^2 = 3p_g - 7$.

As to the linear system $|4T - (p_g - 5)F|$, we have the following:

CLAIM III. *The linear system* $|4T - (p_g - 5)F|$ on $\boldsymbol{P}_{a,b,c}$ *contains an irreducible member with only* RDP's *if and only if*

(5) $$a + c \leq 3b + 2 , \qquad b \leq 2a + 2 .$$

Now we get the following theorem essentially due to Castelnuovo [4]:

THEOREM 1.5. *If* S *is a surface of type* I, *then the irregularity* $q(S)$ *vanishes. Its canonical image* S' *is projectively normal and has only* RDP's *as its singularity. Furthermore, it is contained in an irreducible nondegenerate threefold of minimal degree.* S' *is either*

(1) *a quintic surface in* $\boldsymbol{P}^3$ $(p_g = 4)$,

(2) *a complete intersection of a quadric and a quartic in* $\boldsymbol{P}^4$ $(p_g = 5)$,

(3) *the image in the cone over the Veronese surface of a member* $S'' \in |3T + F|$ *on* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}(2))$ *under the holomorphic map defined by* $|T|$ $(p_g = 7)$, *or*

(4) *the image in the rational normal scroll of a member* $S'' \in |4T - (p_g - 5)F|$ *on* $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(a) \oplus \mathcal{O}_{\boldsymbol{P}^1}(b) \oplus \mathcal{O}_{\boldsymbol{P}^1}(c))$ *under the holomorphic map induced by* $|T|$, *where* a, b, c *are integers satisfying* $0 \leq a \leq b \leq c$, $a + b + c = p_g - 3$, $a + c \leq 3b + 2$ *and* $b \leq 2a + 2$ $(p_g \geq 6)$.

2. Lifting of the canonical map. In this section, we prove Claims I, II and III which are assumed in 1.4. We make use of the standard fact that if a surface admits a map of degree less than three onto a ruled surface, then the canonical map cannot be birational.

Among others, we use the following notation. For any nonnegative integer e, we denote by $\Sigma_e = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(e))$ the Hirzebruch surface of degree e. We let C_0 and f denote the tautological divisor ($C_0^2 = e$) and a fiber, respectively.

2.1. In the cases (C) and (D.2), the threefold W is a cone over a nonsingular surface V. Let Λ_0 be the pull-back to S by Φ_K of the linear system of hyperplanes through the vertex of W. We can choose a basis $\{x_0, x_1, \cdots, x_{p_g - 1}\}$ of $H^0(S, \mathcal{O}(K))$ such that $x_1, \cdots, x_{p_g - 1}$ span the module of Λ_0. We let G denote the fixed part of Λ_0 and put $\Lambda_1 = \Lambda_0 - G$. Since $|K|$ is free from base points, we can assume that $\mathrm{Supp}((x_0)) \cap \mathrm{Supp}(G) = \emptyset$. In particular, we have $KG = 0$. When G is not 0, we denote by ζ the section of $\mathcal{O}([G])$ with $(\zeta) = G$.

2.2. PROOF OF CLAIM I. Since V is the Veronese surface, we have a net Λ such that $2H \in \Lambda_1$ for $H \in \Lambda$ and $K \sim 2H + G$. Since $K^2 = 14$ and $KG = 0$, we have $7 = KH = 2H^2 + HG$. Since $KH + H^2$ is even, we get $H^2 = 1$ or 3. Let $\mu: S \rightarrow \boldsymbol{P}^2$ denote the rational map induced by Λ. If $H^2 = 1$, then μ is birational. This contradicts the assumption that S is of general type. Therefore, we get $H^2 = 3$, $HG = 1$ and $G^2 = -2$. We claim that μ is holomorphic. Indeed, if μ is not holomorphic, then blow S up at any base point of Λ and let $\tilde{H}$ be the proper transform of H. Then we have $\tilde{H}^2 < H^2 = 3$. This means that μ is of degree < 3 onto $\boldsymbol{P}^2$, contradicting the fact that S is of type I. Therefore, μ is holomorphic and $\deg \mu = 3$. The pair (ζ, x_0) defines a homomorphism $\mathcal{O}_S \rightarrow \mathcal{O}_S(G) \oplus \mathcal{O}_S(K)$, which in turn gives a section $\sigma: S \rightarrow S \times_V \tilde{W}$ because $\mathrm{Supp}((x_0)) \cap \mathrm{Supp}(G) = \emptyset$. We get a holomorphic map $\psi: S \rightarrow \tilde{W}$ by setting $\psi = pr_2 \circ \sigma$, where pr_2 is the projection of $S \times_V \tilde{W}$ on the second factor. It is clear from the construction that $\psi^* T_\infty = G$. Therefore $K \sim 2H + G \sim 2\psi^* F + \psi^*(T - 2F) \sim \psi^* T$. Note that $\tilde{W}$ is obtained by blowing up the vertex of W, and S'' is the proper transform of S'. Since S' has only RDP's, we see that S'' has only RDP's.

2.3. PROOF OF CLAIM II. We separately treat (D.2) and (D.3).

(D.2) $a = 0$, $b > 0$, $b + c \geq 3$: W is a cone over $V = \Sigma_{c-b}$ embedded into $\boldsymbol{P}^{b+c+1}$ by $|C_0 + bf|$. Let X be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(\mathcal{O}_{\Sigma_{c-b}} \oplus \mathcal{O}_{\Sigma_{c-b}}(C_0 + bf))$ on Σ_{c-b}. We denote by π and L_0 the projection map and the tautological divisor, respectively. Then W is the image of X under the holomorphic map Φ_{L_0} defined by $|L_0|$. Let L_∞ be the divisor on X which is linearly equivalent to $L_0 - \pi^*(C_0 + bf)$. Then we have the holomorphic map $\nu: X \rightarrow \boldsymbol{P}_{0,b,c}$ which contracts L_∞ to a nonsingular rational curve Z and satisfies $\Phi_{L_0} = \Phi_T \circ \nu$, $\nu^* T = L_0$.

We first show that $\phi: S \rightarrow W$ can be lifted to a holomorphic map $\tilde{\phi}: S \rightarrow X$. Λ_1 induces a rational map $\mu: S \rightarrow \boldsymbol{P}^{b+c+1}$ whose image is V. We let $\rho: \tilde{S} \rightarrow S$ denote a composite of blowing-ups such that the proper transform Λ of Λ_1 is free from base points. We can assume that ρ is the shortest among those which enjoy the property mentioned above. Let E be the exceptional divisor of ρ. Then the canonical divisor $\tilde{K}$ of $\tilde{S}$ is linearly equivalent to $\rho^* K + E$. Further, we have $\rho^* K \sim \tilde{\mu}^*(C_0 + bf) + \tilde{E} + \rho^* G$, where $\tilde{\mu}: \tilde{S}\to\Sigma_{c-b}$ is the holomorphic map induced by Λ and $\tilde{E}$ is a sum of exceptional curves satisfying $\tilde{E}\geq E$. We put $L=\tilde{\mu}^*(C_0+bf)$. Then

$$3(b+c)+2=(\rho^*K)^2=L^2+L(\tilde{E}+\rho^*G)\geq L^2=(\deg\tilde{\mu})(b+c)\,.$$

Since $\deg\tilde{\mu}$ is at least 3, we have $\deg\tilde{\mu}=3$ and $L(\tilde{E}+\rho^*G)=2$. We also remark that

$$0=(\rho^*K)(\rho^*G)=L(\rho^*G)+G^2\,,\quad 0=(\rho^*K)\tilde{E}=L\tilde{E}+\tilde{E}^2\,.$$

We have the following three possibilities:

(1) $L\tilde{E}=0$, $L(\rho^*G)=2$.

(2) $L\tilde{E}=1$, $L(\rho^*G)=1$.

(3) $L\tilde{E}=2$, $L(\rho^*G)=0$.

If (1) is the case, then we have $L\tilde{E}=\tilde{E}^2=0$. By the Hodge index theorem, we get $\tilde{E}=0$. This means that ρ is the identity map. Further we have $G^2=-2$. If (2) is the case, then we get $G^2=-1$ which contradicts the fact that $KG+G^2$ is even. If (3) is the case, then we have $G=0$ and $\tilde{E}^2=-2$. Since $\tilde{K}L+L^2=6(b+c)+2+LE$, we see that LE is even. Since ρ is the shortest, $\tilde{E}\neq 0$ implies the existence of a (-1)-curve E_0 with $LE_0>0$ which is contained in both $\tilde{E}$ and E. Thus LE is positive. From this and $LE\leq L\tilde{E}$, we conclude $LE=2$. We see that $\tilde{\mu}(\tilde{E}-E)$ cannot be a curve, because $L(\tilde{E}-E)=0$ and L is the pull-back of the ample divisor C_0+bf. This in particular implies $(\tilde{\mu}^*f)(\tilde{E}-E)=0$. Then we get a contradiction, because $\tilde{K}(\tilde{\mu}^*f)+(\tilde{\mu}^*f)^2=3f(C_0+bf)+(\tilde{E}+E)(\tilde{\mu}^*f)=3+2E(\tilde{\mu}^*f)$ is odd.

In summary, ρ is the identity map and μ is holomorphic. Then, as in 2.2, we get a lifting $\tilde{\phi}: S\to X$ such that $\tilde{\phi}^*L_\infty=G$. We remark that $K\sim(\tilde{\phi}\circ\pi)^*(C_0+bf)+\tilde{\phi}^*L_\infty\sim\tilde{\phi}^*L_0$. Thus we get the desired map ψ by putting $\psi=\nu\circ\tilde{\phi}$.

By the same reasoning as in the proof of Claim I, we see that $S^*=\tilde{\phi}(S)$ has only RDP's. Since $KG=0$, G consists of (-2)-curves. Therefore, we obtain S'' from S^* by contracting some (-2)-curves. This implies that S'' has only RDP's.

(D.3) $a=b=0$, $c\geq 3$: W is a generalized cone over a rational normal curve of degree $c+1$ in $\boldsymbol{P}^{c+2}$ and the ridge of W is a line. We let Λ be the pull-back to S of the linear system of hyperplanes containing the rigde. Then it is composite with a pencil $|D|$ and we have $K\sim cD+G$, where G is the fixed part of Λ (see, [10, I, §1]). Since $3c+2=K^2=cKD+KG$, we get $KD=1$, 2 or 3. Since $KD+D^2$ is even and $KD=cD^2+DG$, we have the following possibilities:

(1) $KD=2$, $D^2=0$, $DG=2$.

(2) $KD=3$, $D^2=1$, $DG=0$ (in this case $c=3$).

If (1) is the case, then S has a pencil of curves of genus two, a contradiction. If (2) is the case, then we get $G^2=2$ by $11=K^2=9D^2+6DG+G^2$. Since $DG=0$, this contradicts the Hodge index theorem. Therefore the case (D.3) cannot occur.

2.4. Proof of Claim III. We choose sections X_0, X_1 and X_2 of $T-aF$, $T-bF$ and $T-cF$, respectively, in such a way that they form a system of homogeneous fiber coordinates on each fiber of $\boldsymbol{P}_{a,b,c}$. Then any $\Psi \in H^0(\boldsymbol{P}_{a,b,c}, \mathcal{O}(4T-(p_g-5)F)) \simeq H^0(\boldsymbol{P}^1, \mathrm{Sym}^4(\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c))\otimes\mathcal{O}(-p_g+5))$ can be written as

$$\Psi = \sum_{i,j\geq 0, i+j\leq 4} \psi_{ij} X_0^{4-i-j} X_1^i X_2^j\,, \tag{6}$$

where ψ_{ij} is a homogeneous form of degree $(4-i-j)a+ib+jc-(p_g-5)$ on $\boldsymbol{P}^1$. If $4b<p_g-5$, then we can divide Ψ by X_2 and, therefore, the divisor (Ψ) is reducible. If $3a+c<p_g-5$, then (Ψ) is singular along the curve Z defined by $X_1=X_2=0$. Thus the condition (5) is necessary.

Conversely, assume that (5) holds. If $4a\geq p_g-5$, then the linear system $|4T-(p_g-5)F|$ has no base locus and contains an irreducible nonsingular member. So we assume $4a<p_g-5$. Then (Ψ) contains Z, and $|4T-(p_g-5)F|$ has no base locus outside it by (5). Thus it suffices to consider the singularity of (Ψ) in a neighborhood of Z. We shall identify Z with the base curve $\boldsymbol{P}^1$ of $\boldsymbol{P}_{a,b,c}$. If $3a+b\geq p_g-5$, then we can assume that ψ_{10} and ψ_{01} have no common zero. Then (Ψ) is nonsingular in a neighborhood of Z. We next assume $3a+b<p_g-5$. If $3a+c=p_g-5$, then ψ_{01} is constant. Unless it is identically zero, (Ψ) is nonsingular along Z. If $3a+c>p_g-5$, that is, ψ_{01} is of positive degree, then we can assume that it has only simple zeros. Then in a neighborhood of a zero P of ψ_{01} on Z, Ψ can be expressed locally as

$$\Psi = tx_2 + \psi_{20}(t)x_1^2 + \psi_{11}(t)x_1x_2 + \psi_{02}(t)x_2^2 + \cdots\,,$$

where $x_i = X_i/X_0$ and t is a local parameter of Z at P. Thus (Ψ) is defined locally by

$$x_2(t+\psi_{11}(t)x_1+\cdots)+\psi_{20}(t)x_1^2+\psi_{30}(t)x_1^3+\psi_{40}(t)x_1^4=0\,.$$

This shows that P is an RDP if Ψ is general. Thus (5) is also sufficient.

We close this section with the following:

PROPOSITION 2.5. *Let S be a type* I *surface with $p_g=4$ and S' its canonical image. S has a pencil of nonhyperelliptic curves of genus* 3 *if and only if S' contains a line.*

PROOF. Assume that S' contains a line l. We blow $\boldsymbol{P}^3$ up along l to get $\boldsymbol{P}_{0,0,1}$. Then the proper transform S'' of S' is linearly equivalent to $4T+F$ and has a pencil of nonhyperelliptic curves of genus 3 induced by the projection map of $\boldsymbol{P}_{0,0,1}$.

Conversely, assume that S has a pencil $|D|$ as in the statement. Then we have $KD=4$, $D^2=0$. We choose a general $D\in|D|$ and consider the exact sequence

$$0\to\mathcal{O}(K-(i+1)D)\to\mathcal{O}(K-iD)\to\mathcal{O}_D(K_D)\to 0\,,$$

for $i=0$, 1. Since Φ_K is birational, $H^0(K)\to H^0(K_D)$ is surjective. Thus $h^0(K-D)=1$. We show $H^0(K-2D)=0$. For this purpose, we take a general $C\in|K|$ and consider

$$0\to\mathcal{O}(-2D)\to\mathcal{O}(K-2D)\to\mathcal{O}_C(K-2D)\to 0\,.$$

We have $H^0(-2D)=0$. Further, since $C(K-2D)=-3$, we have $H^0(C, \mathcal{O}_C(K-2D))=0$.

Thus $H^0(K-2D)=0$. We can take $w_0 \in H^0(K-D)$ and $w_1, w_2 \in H^0(K)$ so that they span $H^0(K_D)$. Then, by using the triple (w_0, w_1, w_2), we can lift the canonical map to $\psi : S \to \boldsymbol{P}_{0,0,1}$ and have $K=\psi^*T$. Then $S'' := \psi(S)$ is linearly equivalent to $4T+F$, since $\psi(D)$ is a plane curve of degree 4 (cf. §1). Φ_K is the composite of ψ and the map Φ_T induced by $|T|$. Since $H^0(\boldsymbol{P}_{0,0,1}, \mathcal{O}(T)) \simeq H^0(\boldsymbol{P}^1, \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$, we can take $\{X_0, X_1, z_0X_2, z_1X_2\}$ as a basis, where (X_0, X_1, X_2) is the same as that in 2.4 and (z_0, z_1) is a homogeneous coordinate system of $\boldsymbol{P}^1$. Φ_T contracts the rational curve $X_1 = X_2 = 0$. If $(\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3)$ is a homogeneous coordinate system on $\boldsymbol{P}^3$ and if Φ_T is given by

$$\zeta_0 = X_0\,, \quad \zeta_1 = X_1\,, \quad \zeta_2 = z_0X_2\,, \quad \zeta_3 = z_1X_2\,,$$

then, by substituting these to (6), we find that the equation of S' can be written as

$$\alpha_1\zeta_0^4 + \alpha_2\zeta_0^3\zeta_1 + \alpha_3\zeta_0^2\zeta_1^2 + \alpha_4\zeta_0\zeta_1^3 + \alpha_5\zeta_1^4 + \beta_1\zeta_0^3 + \beta_2\zeta_0^2\zeta_1 + \beta_3\zeta_0\zeta_1^2 + \beta_4\zeta_1^3 + \gamma_1\zeta_0^2$$
$$+ \gamma_2\zeta_0\zeta_1 + \gamma_3\zeta_1^2 + \delta_1\zeta_0 + \delta_2\zeta_1 + \varepsilon = 0\,,$$

where α, β, γ, δ and ε are homogeneous forms of respective degrees 1, 2, 3, 4 and 5 in ζ_2, ζ_3. Therefore S' contains a line l defined by $\zeta_2 = \zeta_3 = 0$. q.e.d.

3. Number of moduli. In this and the next sections, we study deformations of surfaces of type I. Since we have Horikawa's works [9] and [10, IV] for $p_g \leq 5$, we assume $p_g \geq 6$ throughout. Further, we restrict ourselves to the case (D) in §1, because the case (C) can be found in [12]. Our main result here is Theorem 3.2 below. For a complex manifold M, we denote by Θ_M the tangent sheaf of M.

3.1. We say that S is a Castelnuovo surface of type (a, b, c) if W (or its nonsingular model) is $\boldsymbol{P}_{a,b,c}$, where the integers a, b, c satisfy the conditions (4) and (5). For the sake of simplicity, we put $W = \boldsymbol{P}_{a,b,c}$ even if $a=0$. We say S to be generic if it is the minimal resolution of a general member of $|4T-(p_g-5)F|$.

THEOREM 3.2. *Let S be a generic Castelnuovo surface of type (a, b, c) with $c \leq 2a+2$. Then*

$$h^1(S, \Theta_S) = \begin{cases} 5p_g + 18\,, & \text{if } a > 0\,, \\ 5p_g + 19\,, & \text{if } a = 0\,. \end{cases}$$

Further, the Kuranishi space is nonsingular of dimension $h^1(\Theta_S) = 5p_g + 18$ if $a > 0$.

For the proof, we need some lemmas.

LEMMA 3.3. *Let S be a Castelnuovo surface of type (a, b, c) and assume that $p_g(S) \geq 6$. Let $|D|$ be the pencil of curves of genus 3 on S induced by the projection map of $W = \boldsymbol{P}_{a,b,c}$.*

(1) *If $a>0$, then $h^0(2D)=3$, $h^1(2D)=0$ and $h^2(2D)=p_g-6$.*

(2) *If $a=0$, then $h^0(2D)=3$, $h^1(2D)=1$ and $h^2(2D)=p_g-5$.*

PROOF. Let S'' be the image of S in $\boldsymbol{P}_{a,b,c}$ described in §1. Since it has only RDP's, we have $\psi_* \mathcal{O}_S \simeq \mathcal{O}_{S''}$ and $R^q \psi_* \mathcal{O}_S = 0$ for $q > 0$, where $\psi: S \to S''$ is the natural map. Thus $H^p(S, \mathcal{O}(2D)) \simeq H^p(S'', \mathcal{O}(2F|_{S''}))$ for any p. We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_W(2F - S'') \to \mathcal{O}_W(2F) \to \mathcal{O}_{S''}(2F) \to 0 .$$

We have $H^p(W, \mathcal{O}_W(2F)) \simeq H^p(\boldsymbol{P}^1, \mathcal{O}(2))$ and $H^p(2F - S'')) \simeq H^{3-p}(\mathcal{O}_W(K_W + S'' - 2F))^*$ by the Serre duality. Since $S'' \sim 4T - (p_g - 5)F$ and $a + b + c = p_g - 3$, we have $K_W + S'' - 2F \sim T - 2F$. Thus $H^{3-p}(\mathcal{O}_W(K_W + S'' - 2F)) \simeq H^{3-p}(\boldsymbol{P}^1, \mathcal{O}(a-2) \oplus \mathcal{O}(b-2) \oplus \mathcal{O}(c-2))$. From these, Lemma 3.3 follows. q.e.d.

LEMMA 3.4. *If $W = \boldsymbol{P}_{a,b,c}$, then*

$$h^q(W, \Theta_W) = \begin{cases} 2(c-a) + 8 + (a-b+1)^+ + (a-c+1)^+ + (b-c+1)^+ , & (q=0) , \\ (b-a-1)^+ + (c-a-1)^+ + (c-b-1)^+ , & (q=1) , \\ 0 , & (q \geq 2) , \end{cases}$$

where $m^+ = \max(m, 0)$.

PROOF. We recall the fundamental exact sequences

$$0 \to \Theta_{W/\boldsymbol{P}^1} \to \Theta_W \to \varpi^* \Theta_{\boldsymbol{P}^1} \to 0 \tag{7}$$

and

$$0 \to \mathcal{O} \to \mathcal{O}(T - aF) \oplus \mathcal{O}(T - bF) \oplus \mathcal{O}(T - cF) \to \Theta_{W/\boldsymbol{P}^1} \to 0 , \tag{8}$$

where $\Theta_{W/\boldsymbol{P}^1}$ is the relative tangent sheaf. Since any automorphism of $\boldsymbol{P}^1$ preserves $\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$, the natural map $\mathrm{Aut}(W) \to \mathrm{Aut}(\boldsymbol{P}^1)$ is surjective, hence so is the map $H^0(\Theta_W) \to H^0(\varpi^* \Theta_{\boldsymbol{P}^1})$. By (7) and the isomorphism $H^q(\varpi^* \Theta_{\boldsymbol{P}^1}) \simeq H^q(\Theta_{\boldsymbol{P}^1})$, we have $h^0(\Theta_W) = h^0(\Theta_{W/\boldsymbol{P}^1}) + 3$ and $h^q(\Theta_W) = h^q(\Theta_{W/\boldsymbol{P}^1})$ for $q > 0$. Then a calculation using (8) shows Lemma 3.4. q.e.d.

LEMMA 3.5. *Let S be as in Lemma 3.3 and consider the linear map $\psi_p^*: H^p(W, \Theta_W) \to H^p(S, \psi^* \Theta_W)$.*

(1) *If $a > 0$, then ψ_p^* is bijective for $p \leq 1$ and $h^2(\psi^* \Theta_W) = p_g - 6$.*

(2) *If $a = 0$, then ψ_p^* is bijective for $p = 0$ and is injective for $p = 1$. Furthermore, $h^1(\psi^* \Theta_W) = h^1(\Theta_W) + 1$, $h^2(\psi^* \Theta_W) = p_g - 5$.*

PROOF. We use the commutative diagram

$$\begin{array}{ccccc} H^p(\psi^* \Theta_{W/\boldsymbol{P}^1}) & \to & H^p(\psi^* \Theta_W) & \to & H^p(2D) \\ \uparrow & & \uparrow & & \uparrow \\ H^p(\Theta_{W/\boldsymbol{P}^1}) & \to & H^p(\Theta_W) & \to & H^p(2F) , \end{array}$$

where the bottom row comes from the exact sequence (7). By Lemmas 3.3 and 3.4, it suffices to show that $H^p(\Theta_{W/\boldsymbol{P}^1}) \to H^p(\psi^* \Theta_{W/\boldsymbol{P}^1})$ is bijective for any p. Since we have

$H^p(\psi^*\Theta_{W/P^1}) \simeq H^p(S'', \Theta_{W/P^1})$, we only have to show $H^p(W, \Theta_{W/P^1}(-S''))=0$ for any p in view of the exact sequence

$$0 \to \Theta_{W/P^1}(-S'') \to \Theta_{W/P^1} \to \Theta_{W/P^1}|_{S''} \to 0 .$$

If Ω_{W/P^1} is the relative cotangent sheaf, we get $H^p(\Theta_{W/P^1}(-S''))^* \simeq H^{3-p}(\Omega_{W/P^1}(T))$ by the Serre duality. We recall that $H^q(P^2, \Omega^1(1))$ vanishes for any q. Thus we get $R^q\varpi_*\Omega_{W/P^1}(T)=0$ for any q. Then it follows from the Leray spectral sequence that $H^{3-p}(\Omega_{W/P^1}(T))=0$ for any p. q.e.d.

LEMMA 3.6. *Let S be as in Theorem 3.2 and denote by $\mathcal{T}_{S/W}$ the cokernel of the natural map $\Theta_S \to \psi^*\Theta_W$. Then $H^2(S, \mathcal{T}_{S/W})=0$. Further, the composite $P \circ \psi_1^*$ of $\psi_1^*: H^1(W, \Theta_W) \to H^1(S, \psi^*\Theta_W)$ and $P: H^1(S, \psi^*\Theta_W) \to H^1(S, \mathcal{T}_{S/W})$ is surjective.*

PROOF. We first assume that $4a \geq p_g - 5$. As we have seen in 2.4, a general member of $|4T-(p_g-5)F|$ is irreducible and nonsingular. Thus we can assume $S \in |4T-(p_g-5)F|$. Then $\mathcal{T}_{S/W}$ is nothing but the normal sheaf $N_{S/W}$. Consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_W \to \mathcal{O}_W(4T-(p_g-5)F) \to N_{S/W} \to 0 .$$

We see that $H^q(S, N_{S/W})=0$ for $q>0$, because we have $H^q(W, \mathcal{O}(4T-(p_g-5)F))=0$ for $q>0$ by the assumption $4a \geq p_g-5$.

We next consider the case $4a < p_g - 5$. As we have seen in 2.4, $S''=\psi(S)$ contains a rational curve Z defined by $X_1=X_2=0$. We denote by $\nu: X \to P_{a,b,c}$ the blowing-up along Z. It is easy to see that X is the total space of the P^1-bundle $\pi: P(\mathcal{O} \oplus \mathcal{O}(C_0+(b-a)f)) \to \Sigma_{c-b}$. We denote by L_0 the tautological divisor of X. If we let L_∞ be the unique divisor linearly equivalent to $L_0 - \pi^*(C_0+(b-a)f)$, then $L_\infty = \nu^{-1}(Z)$. The proper transform of S'' is in $|3L_0 + \pi^*(C_0+(2a-c+2)f)|$. Since $c \leq 2a+2$, this linear system has no base points. Thus we can assume $S \in |3L_0 + \pi^*(C_0+(2a-c+2)f)|$.

By a simple calculation, we have $H^q(X, \mathcal{O}(S))=0$ for $q>0$. This implies $H^q(S, N_{S/X})=0$ for $q>0$. Then by the exact sequence

$$0 \to N_{S/X} \to \mathcal{T}_{S/W} \to \mathcal{T}_{X/W}|_S \to 0 ,$$

we have $H^q(S, \mathcal{T}_{S/W}) \simeq H^q(S, \mathcal{T}_{X/W}|_S)$ for $q>0$. Since $\mathcal{T}_{X/W}|_S$ is supported on a curve, we have $h^2(\mathcal{T}_{S/W}) = h^2(\mathcal{T}_{X/W}|_S)=0$. By [10, III, p. 235], the following sequence is exact:

$$0 \to N_{L_\infty/X} \to \nu^* N_{Z/W} \to \mathcal{T}_{X/W} \to 0 .$$

We identify Z and L_∞ with P^1 and Σ_{c-b}, respectively. Then we have $N_{Z/W} \simeq \mathcal{O}(a-b) \oplus \mathcal{O}(a-c)$ and $N_{L_\infty/X} \simeq \mathcal{O}(-C_0-(b-a)f)$. Thus $\mathcal{T}_{X/W} \simeq \nu^*(\det N_{Z/X}) \otimes N^*_{L_\infty/X} \simeq \mathcal{O}(C_0-(c-a)f)$.

To show the surjectivity of $P \circ \psi_1^*$, it suffices to show that the map $H^1(X, \nu^*\Theta_W) \to H^1(S, \mathcal{T}_{X/W}|_S)$ is surjective, since ψ_1^* is injective by Lemma 3.5. Note that we have $H^2(X, \Theta_X)=0$ by the exact sequence

$$0\to\mathcal{O}(2L_0-\pi^*(C_0+(b-a)f))\to\Theta_X\to\pi^*\Theta_{\Sigma_{c-b}}\to 0 .$$

Thus $H^1(X, \nu^*\Theta_W)\to H^1(X, \mathcal{T}_{X/W})$ is surjective. Consider the exact sequence

$$0\to\mathcal{T}_{X/W}(-S)\to\mathcal{T}_{X/W}\to\mathcal{T}_{X/W}|_S\to 0 .$$

Since $S|_{L_\infty}\sim C_0+(2a-c+2)f$, we have $\mathcal{T}_{X/W}(-S)\simeq\mathcal{O}(-(a+2)f)$. Then $H^2(X, \mathcal{T}_{X/W}(-S))=0$ and thus $H^1(X, \mathcal{T}_{X/W})\to H^1(S, \mathcal{T}_{X/W}|_S)$ is surjective. q.e.d.

3.7. Proof of Theorem 3.2. By Lemmas 3.5 and 3.6, we have

$$h^2(S, \Theta_S)=\begin{cases} p_g-6, & \text{if } a>0, \\ p_g-5, & \text{if } a=0 . \end{cases}$$

Since S is of general type, we have $H^0(S, \Theta_S)=0$. Thus the formula for $h^1(\Theta_S)$ follows from the Riemann-Roch theorem.

In order to show the second assertion, we use Horikawa's deformation theory of holomorphic maps [8]. By Lemma 3.3, we have $H^1(2D)=0$. Thus it follows from [8, II, Theoren 4.4] that there is a family $p: \mathcal{S}\to M$ of deformations of $S=p^{-1}(o)$, $o\in M$, such that the characteristic map $\tau: T_oM\to D_{S/\boldsymbol{P}^1}$ is bijective. Further, we see from [8, II, Lemma 4.2] that the Kodaira-Spencer map $\rho: T_oM\to H^1(\Theta_S)$ is surjective. Note that the parameter space M is nonsingular. Thus we can choose a submanifold N of M passing through o such that the Kodaira-Spencer map $\rho: T_oN\to H^1(\Theta_S)$ is bijective. This completes the proof.

Corollary 3.8. *Let S be as in Theorem 3.2. Then the infinitesimal Torelli theorem holds for S.*

Proof. By the criterion of Kii [11], we only have to show $h^0(\Omega_S^1(K))\le p_g-2$. Since $h^0(\Omega_S^1(K))=h^2(\Theta_S)\le p_g-5$, we are done. q.e.d.

4. A remark on deformations.

4.1. We construct a family of deformations of $\boldsymbol{P}_{a,b,c}$ (for a geometric treatment of deformations of scrolls, see [6]). We denote by d the greatest integer not exceeding $(a+b+c)/3$. By Lemma 3.3, we can assume $(a, b, c)\neq(d, d, d)$, $(d, d, d+1)$, $(d, d+1, d+1)$, since in these cases $\boldsymbol{P}_{a,b,c}$ is rigid.

Let U_1 and U_2 be two copies of $\boldsymbol{C}\times\boldsymbol{P}^2$. We denote by $(z; X_0, X_1, X_2)$ and $(\hat{z}; \hat{X}_0, \hat{X}_1, \hat{X}_2)$ the coordinates of U_1 and U_2, respectively. Then we can construct $\boldsymbol{P}_{a,b,c}$ from $U_1\cup U_2$ by identifying $(z; X_0, X_1, X_2)$ with $(\hat{z}; \hat{X}_0, \hat{X}_1, \hat{X}_2)$ if and only if

(9) $$X_0=\hat{z}^a\hat{X}_0,\ X_1=\hat{z}^b\hat{X}_1,\ X_2=\hat{z}^c\hat{X}_2,\ z\hat{z}=1 .$$

We let t be a complex parameter and identify $(z; X_0, X_1, X_2)$ with $(\hat{z}; \hat{X}_0, \hat{X}_1, \hat{X}_2)$ if and only if

(10) $$X_0=\hat{z}^a\hat{X}_0\,,\quad X_1=\hat{z}^b\hat{X}_1\,,\quad X_2=\hat{z}^c\hat{X}_2+t\hat{z}^{a+k}\hat{X}_0\,,\quad z\hat{z}=1\,,$$

where k is an integer satisfying $0\leq k\leq c-a$. Then we get a family $\{W_t\}$ of $\boldsymbol{P}^2$-bundles on $\boldsymbol{P}^1$. For $t\neq 0$, we put

$$Y_0=tX_2\,,\quad Y_1=X_1\,,\quad Y_2=z^kX_2-tX_0\,,$$
$$\hat{Y}_0=t^2\hat{X}_0+t\hat{z}^{c-a-k}\hat{X}_2\,,\quad \hat{Y}_1=\hat{X}_1\,,\quad \hat{Y}_2=\hat{X}_2\,.$$

Then we get $Y_0=\hat{z}^{a+k}\hat{Y}_0$, $Y_1=\hat{z}^b\hat{Y}_1$, $Y_2=\hat{z}^{c-k}\hat{Y}_2$. Thus $W_t\simeq\boldsymbol{P}_{a+k,b,c-k}$ if $t\neq 0$. Similarly, if we consider the families

(11) $$X_0=\hat{z}^a\hat{X}_0\,,\quad X_1=\hat{z}^b\hat{X}_1+t\hat{z}^{a+k}\hat{X}_0\,,\quad X_2=\hat{z}^c\hat{X}_2\,,\quad z\hat{z}=1$$

and

(12) $$X_0=\hat{z}^a\hat{X}_0\,,\quad X_1=\hat{z}^b\hat{X}_1\,,\quad X_2=\hat{z}^c\hat{X}_2+t\hat{z}^{b+k}\hat{X}_1\,,\quad z\hat{z}=1$$

for a suitable k, then we see that $\boldsymbol{P}_{a,b,c}$ is a deformation of $\boldsymbol{P}_{a+k,b-k,c}$ and $\boldsymbol{P}_{a,b+k,c-k}$, respectively. Thus we get:

PROPOSITION 4.2. *The* $\boldsymbol{P}^2$*-bundle* $\boldsymbol{P}_{a,b,c}$ *is a deformation of* $\boldsymbol{P}_{d,d,d}$, $\mathbf{P}_{d,d,d+1}$ *or* $\boldsymbol{P}_{d,d+1,d+1}$ *according as* $a+b+c$ *is* 0, 1 *or* 2 *modulo* 3.

PROPOSITION 4.3. *Let S be a Castelnuovo surface of type* (a, b, c) *and assume that* $c\leq 2a+3$. *Then S is a deformation of a Castelnuovo surface of type* (d, d, d), $(d, d, d+1)$ *or* $(d, d+1, d+1)$.

PROOF. We showed in §1 that S is a minimal resolution of a surface $S''\sim 4T-(p_g-5)F$ on $\boldsymbol{P}_{a,b,c}$.

If $3a+1\geq b+c$, then $H^1(\boldsymbol{P}_{a,b,c}, \mathcal{O}(4T-(p_g-5)F))=0$. Thus if $s\in H^0(4T-(p_g-5)F)$ defines S'', then it can be extended to any family of deformations of $\boldsymbol{P}_{a,b,c}$. Thus we get a family $\{S''_t\}$ with $S''=S''_0$ from the family (10) for example. Since S'' has only RDP's, so does S''_t provided that t is sufficiently small. We simultaneously resolve RDP's (cf. [2] and [3]) and get a family $\{S_t\}$ of deformations of $S=S_0$. This family shows that S is a specialization of a Castelnuovo surface of type $(a+k, b, c-k)$. Continuing this procedure using (10), (11) or (12), we get the desired result.

If $3a+1<b+c$ but $c\leq 2a+3$, we consider the $\boldsymbol{P}^1$-bundle X in the proof of Lemma 3.6 instead of $\boldsymbol{P}_{a,b,c}$. Since X is a monoidal transform of $\boldsymbol{P}_{a,b,c}$, a sufficiently small deformation of the former is a monoidal transform of a deformation of the latter (see, [8, III]). Indeed, by blowing up the rational curve defined by $X_1=X_2=0$ in the family (12) simultaneously, we get a family $\{X_t\}$ of deformations of $X=X_0$. We remark that $H^1(X, \mathcal{O}(3L_0+\pi^*(C_0+(2a-c+2)f)))=0$ if $c\leq 2a+3$. Thus similar arguments also work. q.e.d.

REMARK 4.4. The moduli space of Castelnuovo surfaces has several components in general. To see this, let $\mu=\dim|4T-(p_g-5)F|-h^0(\boldsymbol{P}_{a,b,c}, \Theta)$ be the number of parameters on which Castelnuovo surfaces of type (a, b, c) depend. Then μ is sometimes strictly greater than the number $5p_g+18$ of moduli of a generic Castelnuovo surface as in Theorem 3.2. For example, if $p_g=12$, we have

$$(a, b, c; \mu)=(3, 3, 3; 78), \quad (2, 3, 4; 77), \quad (2, 2, 5; 74), \quad (1, 2, 6; 74),$$
$$(1, 3, 5; 75), \quad (1, 4, 4; 76), \quad (0, 2, 7; 79).$$

4.5. Here we give a family of surfaces with $c_1^2=3p_g-7$ such that the central fiber is of type II while a general fiber is of type I.

We let W be the $\boldsymbol{P}^2$-bundle $\boldsymbol{P}_{a,b,c}$, where a, b, c are integers satisfying (4) and (5). We assume that p_g is odd and put $2k=p_g-5$. The linear system $|L|$, $L=2T-kF$, is free from base points if $2a-k\geq 0$, i.e., $3a+2\geq b+c$. We assume this condition for the sake of simplicity. We choose $\eta\in H^0(W, \mathcal{O}(L))$ which defines an irreducible nonsingular divisor Y. Let $\hat{W}$ be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(\mathcal{O}_W\oplus\mathcal{O}_W(L))$ on W. We put $\Delta_\varepsilon=\{t\in\boldsymbol{C}; |t|<\varepsilon\}$, where ε is a sufficiently small positive number. Consider a family $\{S_t\}$, $t\in\Delta_\varepsilon$, of subvarieties of $\hat{W}$ given by the equation

$$\text{(13)} \qquad S_t:\begin{cases} Z_0^2+\alpha_1 Z_0 Z_1+\alpha_2 Z_1^2=0 \\ tZ_0=\eta X_1 \end{cases}, \quad t\in\Delta_\varepsilon,$$

where (Z_0, Z_1) is a system of homogeneous fiber coordinates on $\hat{W}$ and $\alpha_i\in H^0(W, \mathcal{O}(iL))$, $1\leq i\leq 2$. We assume that α's are general.

If $t\neq 0$, then S_t is biholomorphically equivalent to a surface in W defined by the equation $\eta^2+t\alpha_1\eta+t^2\alpha_2=0$. Thus $S_t\in|4T-(p_g-5)F|$ and it is of type I.

On the other hand, S_0 is a double covering of Y via the projection map of $\hat{W}$. Since Y is a conic bundle on $\boldsymbol{P}^1$, S_0 has a pencil of hyperelliptic curves of genus three. Thus it is of type II.

5. Surfaces with hyperelliptic pencils of genus 3.

We call a pencil on a surface *a hyperelliptic pencil of genus g* if its general member is a nonsingular hyperelliptic curve of genus g. In this section, we study the geography of surfaces with hyperelliptic linear pencils of genus 3.

5.1. Let V be a normal Gorenstein surface and denote by $\sigma: V'\to V$ the minimal resolution of an isolated singularity ξ of V. Then there exists an effective divisor Z_ξ on V' supported on $\sigma^{-1}(\xi)$ such that $\omega_{V'}\simeq\sigma^*\omega_V\otimes\mathcal{O}(-Z_\xi)$ (e.g., [14]). Then we have $\omega_{V'}^2=\omega_V^2+Z_\xi^2$. On the other hand, the spectral sequence $H^p(V, R^q\sigma_*\mathcal{O}_{V'})\Rightarrow H^{p+q}(V', \mathcal{O}_{V'})$ implies that $\chi(\mathcal{O}_{V'})=\chi(\mathcal{O}_V)-p_g(\xi)$, where $p_g(\xi)=h^0(V, R^1\sigma_*\mathcal{O}_{V'})$ is the geometric genus of ξ. We call $(p_g(\xi): -Z_\xi^2)$ *the type of singularity* ξ. If ξ is a double point, then its type can be easily calculated by Horikawa's canonical resolution ([9, §2] or [13, §1]).

Let $\{\xi_i\}_{1\leq i\leq s}$ be a set of isolated singularities of V and assume that ξ_i is of type

(m_i, n_i). If $V^* \to V$ is the minimal resolution of these singularities, then we have

(14) $$\chi(\mathcal{O}_{V^*}) = \chi(\mathcal{O}_V) - \sum_{i=1}^{s} m_i,\ \omega_{V^*}^2 = \omega_V^2 - \sum_{i=1}^{s} n_i .$$

We list the types of the singularities which we shall need later:

(i) If ξ is an RDP, then $Z_\xi = 0$. Thus ξ is of type $(0:0)$.

(ii) If ξ is a simple elliptic singularity of type $\tilde{E}_8$ or $\tilde{E}_7$ (see [15]), then $Z_\xi = E$, where E is the exceptional elliptic curve. Thus ξ is of type $(1:1)$ or $(1:2)$ according as whether it is $\tilde{E}_8$ or $\tilde{E}_7$.

5.2. Let $\pi_0 : Y = \Sigma_e \to \boldsymbol{P}^1$ be the Hirzebruch surface of degree e. Put $L := 4C_0 + \beta f$, where β is an integer satisfying

(15) $$e + \beta \geq 2 .$$

We consider the $\boldsymbol{P}^1$-bundle

$$\pi : X = \boldsymbol{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L)) \to Y ,$$

and set $L_0 = \mathcal{O}_X(1)$, $D_0 = \pi^* C_0$ and $F = \pi^* f$.

LEMMA 5.3. *Let V be an irreducible reduced divisor on X linearly equivalent to $2L_0$. Then,*

(1) $\omega_V \simeq \pi^*(K_Y + L)|_V \simeq \pi^*(2C_0 + (\beta + e - 2)f)|_V$,

(2) $\omega_V^2 = 16e + 8\beta - 16$,

(3) $p_g(V) = 6e + 3\beta - 3$, $q(V) = 0$.

PROOF. Since $K_X \simeq -2L_0 + \pi^*(K_Y + L)$ and $\omega_V \simeq (K_X + V)|_V$, we get (1). Since $L_0 D_0^2 = e$, $L_0 D_0 F = 1$ and $L_0 F^2 = 0$, we get (2) by (1). Considering the cohomology long exact sequence for $0 \to \mathcal{O}(K_X) \to \mathcal{O}(K_X + V) \to \omega_V \to 0$, we easily obtain (3) by (15).

q.e.d.

5.4. Let V be as above. By a suitable system of homogeneous fiber coordinates $(X_0 : X_1)$ of $\pi : X \to Y$, the equation ϕ of V can be written as

(16) $$\phi = X_0^2 + \phi_{2L} X_1^2 ,$$

where $\phi_{2L} \in H^0(Y, \mathcal{O}(2L))$. We note that V is a double covering of Y via $\pi|_V$ and its branch locus is $B_V = (\phi_{2L})$. We assume that B_V is reduced. Then V is normal. Set $\lambda = \pi_0 \circ \pi|_V : V \to \boldsymbol{P}^1$. Then, by the Hurwitz formula, a general fiber of λ is a hyperelliptic curve of genus 3. Assume further that

(*) B_V has k infinitely close triple points (cf. [13, §1]) and l ordinary quadruple points, and the other singularities of B_V are at most double points.

The V is a normal Gorenstein surface with k singular points of type $\tilde{E}_8$ and l singular points of type $\tilde{E}_7$. We remark that the other singularities are at most RDP's. Let $\sigma : V^* \to V$ be the minimal resolution of all singularities of V. Then by (14), (i) and

(ii) of 5.1 and Lemma 5.3, we have

$$(17) \qquad \chi(\mathcal{O}_{V^*})=6e+3\beta-2-k-l\,, \qquad \omega_{V^*}^2=16e+8\beta-16-k-2l\,.$$

In order to construct B_V satisfying ($*$), we use the method essentially due to Persson [13]. For an integer $r\geq 0$, any $\psi\in H^0(Y, \mathcal{O}(2C_0+rf))$ can be written as

$$\psi=\psi_r Y_0^2+\psi_{r+e}Y_0Y_1+\psi_{r+2e}Y_1^2$$

where $(Y_0: Y_1)$ is a system of homogeneous fiber coordinates of $\pi_0: Y\to \boldsymbol{P}^1$ and $\psi_{r+ie}\in H^0(\boldsymbol{P}^1, \mathcal{O}(r+ie))$, $0\leq i\leq 2$. We put $C_0=(Y_0)$ and $C_\infty=(Y_1)$. We often identify them with the base curve $\boldsymbol{P}^1$ of Y.

We define a sublinear system of $|2C_0+rf|$ by

$$|2C_0+rf|_P=\{(\psi): \psi=\psi_r Y_0^2+\psi_{r+2e}Y_1^2\}$$

and call it *Persson's system*. As is easily seen, it has the following properties:

(a) Put $\psi\in|2C_0+rf|_P$. If ψ_{r+2e} and ψ_r have simple zeros only and if they have no common zero, then (ψ) is nonsingular. We regard (ψ_{r+2e}) and (ψ_r) as reduced divisors on C_0 and C_∞, respectively. Set $(\psi_{r+2e})=\sum_{i=1}^{2e+r}P_i$ and $(\psi_r)=\sum_{i=2e+r+1}^{2e+2r}P_i$. Then the tangent line $T_{P_i}((\psi))$ of (ψ) at P_i is vertical for $1\leq i\leq 2e+2r$, i.e., T_{P_i} coincides with the fiber of π_0 passing through P_i.

(b) Let k_0, k_∞ and l be nonnegative integers satisfying $k_0\leq 2e+r$, $k_\infty\leq r$ and $l\leq 2e+2r-k_0-k_\infty-1$. Let $P_1, \cdots, P_{k_0}$ and $Q_1, \cdots, Q_{k_\infty}$ be mutually distinct points on C_0 and C_∞, respectively, and let $R_1, \cdots, R_l$ be generic points on $Y\setminus(C_0\cup C_\infty)$. Let Λ be the linear subsystem of $|2C_0+rf|_P$ consisting of those elements passing through all the $k_0+k_\infty+l$ points P_i, Q_j, R_k. Then a general member of Λ is nonsingular, and we have $\dim\Lambda=2e+2r-k_0-k_\infty-l\geq 1$. Moreover, the system Λ has no base point except P_i, Q_j, R_k.

By using these properties, we can show the following:

LEMMA 5.5. *Fix a nonnegative integer l and put*

$$k_{\max}=\max_{(l_1,l_2)}\{[(2/3)(4e+\beta-2l_1-2)]+[(2/3)(\beta-2l_2)]\}\,,$$

where l_1 and l_2 run through nonnegative integers satisfying $l_1+l_2=l$, $4e+\beta-2l_1-2\geq 0$ and $\beta-2l_2\geq 0$, and $[q]$ is the greatest integer not exceeding q. Then, for any integer k with $0\leq k\leq k_{\max}$, there exists a reduced divisor B on Y such that

(1) *$B\sim 8C_0+2\beta f$,*

(2) *B has k infinitely close triple points and l ordinary quadruple points. The other singularities of B are at most double points.*

PROOF. Let k_0, k_∞ be nonnegative integers satisfying

$$(18) \qquad k=k_0+k_\infty\,, \quad k_0\leq[(2/3)(4e+\beta-2l_1-2)]\,, \quad k_\infty\leq[(2/3)(\beta-2l_2)]\,.$$

We choose mutually distinct points $P_1, \cdots, P_{k_0}$ on C_0 and $Q_1, \cdots, Q_{k_\infty}$ on C_∞. We also choose general points $R_1, \cdots, R_l$ on $Y \setminus (C_0 \cup C_\infty)$.

When β is even, we set $r_i = [\beta/2]$ for $1 \le i \le 4$. When β is odd, we set $r_1 = r_2 = [\beta/2]+1$ and $r_3 = r_4 = [\beta/2]$. By (18), we can choose nonnegative integers $k_0(i)$ and $k_\infty(i)$ for $1 \le i \le 4$ such that

(i) $k_0(i) \le 2e + r_i - l_1 - 1$, $k_\infty(i) \le r_i - l_2$,

(ii) there are subsets $\{P_1^{(i)}, \cdots, P_{k_0(i)}^{(i)}\}$ and $\{Q_1^{(i)}, \cdots, Q_{k_\infty(i)}^{(i)}\}$ of $\{P_j; 1 \le j \le k_0\}$ and $\{Q_j; 1 \le j \le k_\infty\}$, respectively, such that $\sum_{i=1}^4 (P_1^{(i)} + \cdots + P_{k_0(i)}^{(i)}) = 3\sum_{j=1}^{k_0} P_j$ and $\sum_{i=1}^4 (Q_1^{(i)} + \cdots + Q_{k_\infty(i)}^{(i)}) = 3\sum_{j=1}^{k_\infty} Q^j$.

Let Λ_i be the subsystem of $|2C_0 + r_i f|_P$ consisting of elements passing through all the $k_0(i) + k_\infty(i) + l$ points $P_1^{(i)}, \cdots, P_{k_0(i)}^{(i)}, Q_1^{(i)}, \cdots, Q_{k_\infty(i)}^{(i)}$ and $R_1, \cdots, R_l$. Let B_i be a general member of Λ_i and set $B = \sum_{i=1}^4 B_i$. Then B satisfies (1) and (2). q.e.d.

Take k, l and B as in Lemma 5.5 and let $\mu := \pi|_V: V \to Y$ be the double cover branched along B. V has k_0 (resp. k_∞) singular points of type $\tilde{E}_8$ on $\mu^{-1}(C_0)$ (resp. $\mu^{-1}(C_\infty)$) with $k = k_0 + k_\infty$ and l singular points of type $\tilde{E}_7$ on $\mu^{-1}(Y \setminus (C_0 \cup C_\infty))$. Moreover the other singularities of V are at most RDP's. We let $\sigma: V^* \to V$ be the minimal resolution.

PROPOSITION 5.6. *Assume that* $k_0 \le 3e + \beta - 2$, $k_\infty \le e + \beta - 2$ *and* $k + l \le 6e + 3\beta - 7$. *Then, we have*:

(1) $p_g(V^*) = p_g(V) - k - l$ *and* $q(V^*) = 0$.

(2) $|K_{V^*}|$ *is free from fixed components, and has exactly k base points. Especially* V^* *is relatively minimal.*

(3) *The canonical map of* V^* *is generically* 2:1 *map onto its image.*

PROOF. Set $\tilde{\mu} = \sigma \circ \mu$, $\{\eta_1, \cdots, \eta_{k+l}\} = \{P_1, \cdots, P_{k_0}, Q_1, \cdots, Q_{k_\infty}, R_1, \cdots, R_l\}$ and $\xi_i = \mu^{-1}(\eta_i)$ for the sake of brevity. By (i) and (ii) of 5.1 and Lemma 5.3, we have

$$K_{V^*} \simeq \tilde{\mu}^*(K_Y + L) \otimes \mathcal{O}_{V^*}\left(-\sum_{i=1}^{k+l} E_i\right), \tag{19}$$

where $E_i = \sigma^{-1}(\xi_i)$ is the exceptional elliptic curve. Hence from the exact sequence

$$0 \to \mathcal{O}(K_{V^*}) \to \mathcal{O}(\tilde{\mu}^*(K_Y + L)) \to \bigoplus_i E_i \to 0\,,$$

we get the exact sequence

$$0 \to H^0(V^*, K_{V^*}) \to H^0(V^*, \tilde{\mu}^*(K_Y + L)) \xrightarrow{\rho} \bigoplus_i \mathbf{C}_{E_i}\,, \tag{20}$$

where $\mathbf{C}_{E_i}$ is the sheaf of constant functions on E_i.

Since V is normal, $\sigma_* \mathcal{O}_{V^*} \simeq \mathcal{O}_V$. Thus we have $\tilde{\mu}_* \tilde{\mu}^* \mathcal{O}_Y(K_Y + L) \simeq \mu_*(\mu^* \mathcal{O}_Y(K_Y + L) \otimes \sigma_* \mathcal{O}_{V^*}) \simeq \mathcal{O}_Y(K_Y + L) \otimes \mu_* \mathcal{O}_V \simeq \mathcal{O}_Y(K_Y + L) \otimes (\mathcal{O}_Y \oplus \mathcal{O}_Y(-L)) \simeq \mathcal{O}_Y(K_Y + L) \oplus \mathcal{O}_Y(K_Y)$. Hence $H^0(V^*, \tilde{\mu}^*(K_Y + L))$ is isomorphic to $H^0(Y, K_Y + L)$ and its dimension is $p_g(V)$.

We show that ρ in (20) is surjective. For any $\psi\in H^0(Y, K_Y+L)\simeq H^0(V^*, \tilde{\mu}^*(K_Y+L))$, the map ρ is given by

$$\rho(\psi)=(\psi(\eta_1), \cdots, \psi(\eta_{k+l}))\in \bigoplus_i \boldsymbol{C}_{E_i}$$

where $\psi(\eta_i)$ is the value of ψ at η_i.

Denote by M_i, $0\leq i\leq k+l$, the linear subsystem of $|K_Y+L|$ consisting of elements passing through $\eta_1, \cdots, \eta_i$. If η_{i+1} does not belong to the base locus of M_i, then the descending filtration

$$|K_Y+L|=M_0\supset M_1\supset\cdots\supset M_{k+l}$$

satisfies $\dim M_{i+1}=\dim M_i-1$ for any i.

On the other hand, for any $(\phi)\in|K_Y+L|=|2C_0+(e+\beta-2)f|$, ϕ can be written as

$$\phi=\phi_{e+\beta-2}Y_0^2+\phi_{2e+\beta-2}Y_0Y_1+\phi_{3e+\beta-2}Y_1^2\,,$$

where $\phi_{ie+\beta-2}\in H^0(\boldsymbol{P}^1, \mathcal{O}(ie+\beta-2))$. Thus from our construction and assumption, it is easy to see that, for each $0\leq j\leq k+l$, M_j separates points on $Y\setminus\{\eta_1, \cdots, \eta_j\}$, that is, for any points $P, Q\in Y\setminus\{\eta_1, \cdots, \eta_j\}$, there exists $(\phi)\in M_j$ such that $\phi(P)=0$ and $\phi(Q)\neq 0$. Especially the base locus of M_j coincides with $\{\eta_1, \cdots, \eta_j\}$. Thus ρ is surjective. Then, by (20), we get (1). Moreover, since the rational map of Y associated with M_{k+l} is birational onto its image, (3) follows.

It remains to prove (2). By the above argument, the base locus of $|K_{V^*}|$ is contained in $\bigcup_{i=1}^{k+l}E_i$. Since a generic member of M_{k+l} passes through η_i $(1\leq i\leq k+l)$ smoothly, E_i is not a fixed component of $|K_{V^*}|$ by (19). Thus $|K_{V^*}|$ is free from fixed components.

Let η_i be an infinitely close triple point of B. Let C be a member of M_{k+l} and C^* the proper transform of $\mu^{-1}(C)$ by σ. When C varies in M_{k+l}, C^* passes through the unique point on E_i. (This is easily observed by means of the canonical resolution.) However, it is not the case when η is an ordinary quadruple point. Thus $|K_{V^*}|$ has exactly k base points. q.e.d.

THEOREM 5.7. *Let x, y be any pair of integers satisfying one of the following two conditions*:

(a) $(8/3)x-8\leq y\leq 4x-16$, $y\neq(1/3)(8x-i)$ $(i=21, 23)$.

(b) $y=4x-i$, $x\geq 6$ *and x is equivalent modulo 5 to j, where* $(i, j)=(8, 0)$, $(9, 2)$, $(10, 2)$, $(10, 4)$, $(11, 1)$, $(11, 4)$, $(12, 1)$, $(12, 3)$, $(12, 4)$, $(13, 0)$, $(13, 1)$, $(13, 3)$, $(14, 0)$, $(14, 1)$, $(14, 3)$, $(14, 4)$, $(15, 0)$, $(15, 2)$, $(15, 3)$, $(15, 4)$.

Then there exists a minimal surface S such that

(1) $p_g(S)=x$, $q(S)=0$ *and* $c_1^2(S)=y$,

(2) *there is a fibration* $\lambda: S\to\boldsymbol{P}^1$ *whose general fiber is a hyperelliptic curve of genus* 3,

(3) $|K_S|$ *is free from fixed components and* Φ_{K_S} *is of degree* 2 *onto its image.*

PROOF. We set $0\leq l\leq 4$, $e+\beta\geq 3$ and $(e, \beta, k)\neq(0, 3, 0)$. Then the assumptions in

Proposition 5.6 are satisfied. Under these conditions, we let k vary with $0 \leq k \leq k_{\max}$. By (17), Lemma 5.5 and Proposition 5.6, a calculation shows that the invariants of our surfaces cover the area (a) and (b). q.e.d.

REMARK 5.8. If a regular surface has a hyperelliptic pencil of genus 3, then it satisfies $c_1^2 \geq (8/3)p_g - 8$. See, [10, V] or [13].

6. Surfaces of type II. In this section, we give some remarks on surfaces of type II. Let S be a minimal surface of type II in the sense of §1. We assume that the irregularity of S vanishes. Then the canonical image S' is a rational ruled surface. Thus S has a hyperelliptic pencil induced by the canonical map and the ruling of S'.

For the hyperelliptic structure of S, we have the following theorem due to Xiao [16, §1]:

THEOREM 6.1 (Xiao). *Let S be a regular minimal surface of general type with a hyperelliptic pencil. Suppose that the invariants of S satisfy*

$$p_g(S) > (2g-1)(g+1)+1,\ c_1^2(S) < (4g/(g+1))(p_g(S)-g-1)$$

for some integer $g \geq 2$. Then S has a hyperelliptic pencil of genus g. Moreover, the hyperelliptic pencil of genus less than $g+1$ is unique.

COROLLARY 6.2. *Assume that S is a regular surface of type* II *with $c_1^2 = 3p_g - 7$. If $p_g(S) \geq 46$, then it has a hyperelliptic linear pencil of genus less than* 5.

For the existence of surfaces of type II with hyperelliptic pencils of genus less than 5, we have the following:

PROPOSITION 6.3. *Let g be* 2, 3 *or* 4. *Then, for any pair of integers (x, y) satisfying $y = 3x - 7$ and $x \geq 4$, there exists a minimal surface S with a hyperelliptic linear pencil of genus g such that $p_g(S) = x$, $q(S) = 0$ any $c_1^2(S) = y$.*

PROOF. The case $g = 2$ follows from a more general result of Persson [13, §3]. The case $g = 3$ with $p_g \geq 6$ follows from Theorem 5.7. For $p_g = 4$, 5, consult [9] and [10, IV].

We consider the case $g = 4$. Set $(k, l) = (1, 3)$ or $(2, 1)$. By an argument similar to that in §5, there exists a reduced divisor B on $Y = \Sigma_e$ such that

(i) $B \sim 10C_0 + 2\beta f$ $(\beta \geq 0)$,

(ii) B has k infinitely close triple points and l ordinary quadruple points, and the other singularities are at most double points.

Let V be the double covering of Y branched along B. If S is the minimal resolution of V, then

$$p_g(S) = 10e + 4(\beta - 1) - k - l, \quad q(S) = 0,$$

$$c_1^2(S) = 30e + 12(\beta - 2) - k - 2l = 3p_g(S) - 7,$$

and S has the desired properties. q.e.d.

REMARK 6.4. For a given $g \geq 2$, there exists a regular surface of type II with a hyperelliptic pencil of genus g. Indeed, we have constructed in the proof of Proposition 6.3 a surface S using the double covering V of $Y = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ whose branch locus B is linearly equivalent to $10C_0 + 2\beta f$ for any $\beta \geq 3$. The second projection of Y induces on S another hyperelliptic pencil of genus $g' = \beta - 1$. So we cannot give an upper bound on the genus of hyperelliptic pencils.

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