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NOTES ON CANONICAL SURFACES

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1. Surfaces with $c_1^2=3p_g-7$. As Ashikaga and Konno are going to publish their paper on surfaces of general type [2], I would like to take this opportunity to state several results on these surfaces and related problems.

A minimal algebraic surface S is called *a canonical surface* if the map $\Phi_K: S\to \boldsymbol{P}^n$, $n=p_g-1$, associated to the canonical system $|K|$ induces a birational map of S onto its image. Let Quad(S) denote the intersection of all the quadrics through the image $\Phi_K(S)$. If S is a canonical surface, then $c_1^2\geq 3p_g-7$ (see [10, Part II, Lemma 1.1]). If the equality sign holds here, S has rather simple structure and its construction can be completely described as in [2]. These are all essentially due to Castelnuovo [4], and I obtained my proof in 1976, which is mostly similar to [2, §§1–4]. Moreover, I noticed that some of the canonical surfaces with $p_g=7$, $c_1^2=14$ (such that Quad(S) is a cone over the Veronese surface) have obstructed deformations. For such S, $|K|$ is not ample, and the canonical system $|K_t|$ remains non-ample for any small deformation S_t of S. So, by [3], S has generically non-reduced moduli. This was insinuated in [10, Part III, Remark on p. 229], but with an erroneous citation $p_g=6$, $c_1^2=11$. (I planned to write a paper entitled "On certain canonical surfaces" to discuss surfaces with $c_1^2=3p_g-7$ and $3p_g-6$, but it was never completed.)

This surface was independently found recently by Miranda [15]. But he missed one point: If $\{S_t\colon t\in M\}$ is a flat family over a parameter space M, then does $\{\mathrm{Quad}(S_t)\colon t\in M\}$ form a flat family? This is not true in general, because the dimension of Quad(S_t) may jump in some case.

LEMMA 1. *In the present case,* Quad(S_t) *form a flat family provided that the parameter space M is reduced.*

PROOF. Let $\{\phi_i\}$ be a basis of $H^0(S, \mathcal{O}(K))$. Then the products $\phi_i\phi_j$ generate $H^0(S, \mathcal{O}(2K))$. Hence, for some set of indices Λ, $\phi_i\phi_j$, $(i,j)\in\Lambda$, form a basis of $H^0(S, \mathcal{O}(2K))$. Since the irregularity q vanishes these ϕ_i are extended to the sections $\phi_i(t)$ of $H^0(S_t, \mathcal{O}(K_t))$. Hence the products $\phi_i(t)\phi_j(t)$, $(i,j)\in\Lambda$ form a basis of $H^0(S_t, \mathcal{O}(2K_t))$. Therefore, the other products are linear combinations of these products. This implies that any quadratic relation among the ϕ_i's can be extended to that of the $\phi_i(t)$'s. This proves that $\bigcup_t\mathrm{Quad}(S_t)$ is an analytic subset of $\boldsymbol{P}^6\times M$. It is well-known that, for all possible candidates for Quad(S_t), $\dim H^0(\mathrm{Quad}(S_t), \mathcal{O}(m))$ are the same. Since M is reduced, this proves the normal-flatness and hence the flatness of Quad(S_t).

By this lemma, Quad(S_t) remains the cone over the Veronese surface, because its vertex is a rigid singularity [17]. Analogous example of nonreduced moduli for surfaces of general type was previously found in [10, Part III]. Note that these two are put together in a recent work of Catanese [5].

2. Canonical surfaces with $c_1^2=3p_g-6$, $q=0$. As to these surfaces, the following lemmas will take care for large values of p_g.

LEMMA 2. (i) *If* $p_g\geq 5$ *then* Quad(S) *is of dimension* ≥ 3.

(ii) *If* $p_g\geq 12$, *then* Quad(S) *is a threefold of degree* $n-2$ *in* $\boldsymbol{P}^n$, *and* S *has a pencil of curves of genus* 3 *of non-hyperelliptic type* (*i.e., the general fibres are non-hyperelliptic*).

LEMMA 3. *If* S *has a pencil of curves of genus* 3 *of non-hyperelliptic type, then it has one degenerate fibre which is given by*

$$q(x, y, z)^2+t^2f(x, y, z, t)=0 \tag{1}$$

in $\boldsymbol{P}^2\times\Delta$, *where* $\Delta=\{t\in\boldsymbol{C}\colon |t|<\varepsilon\}$ *is a parameter space and* q *and* f *are homogeneous polynomials in* (x, y, z) *of degrees* 2 *and* 4, *respectively.*

My proof of Lemma 2 is a mimic of Petri's analysis on canonical curves as presented by Saint-Donat [16], or one can apply Harris' result [8, Theorem 3.15].

The equation (1) only gives a singular model with a double curve along the conic Q defined by $q=t=0$. This can be (partially) resolved by introducing a new variable w and considering the following equations:

$$\begin{cases} w^2+f(x, y, z, t)=0\,, \\ q-wt=0\,. \end{cases} \tag{2}$$

If the coefficients of f are sufficiently general, (2) determines over $t=0$ a hyperelliptic curve C of genus 3 which is a double covering of the conic Q branched at 8 points defined by $f(x, y, z, 0)=0$ (w can be regarded as an inhomogeneous fibre coordinate on the $\boldsymbol{P}^1$-bundle associated to $\mathcal{O}(2)$).

This type of degenerate fibre contributes $+1$ to the value $c_1^2-3p_g$. Conversely, we start with a $\boldsymbol{P}^2$-bundle W over $\boldsymbol{P}^1$, and take a hypersurface S' which cuts a quartic on each fibre and which has one double conic like (1). Then take its minimal resolution essentially given by (2) (S may be described as a complete intersection in a $\boldsymbol{P}^1$-bundle over a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$).

For small values of p_g (the cases ≤ 4 are studied in [10]), Quad(S) can be either $\boldsymbol{P}^4$ or a threefold of degree $n-1$ or $n-2$, i.e. $\Delta=1$ or 0 in Fujita's sense. In the first case, S is a complete intersection of two cubics. In the second case one can apply Fujita's classification of varieties with $\Delta=1$[6a, 6b]. But I did not fully investigate the cases in which Quad(S) is singular. The third case is similar to the above with a few exceptional cases.

It may be interesting to investigate deformations of these surfaces, including the non-canonical surfaces. As a special case, this includes the class of sextic surfaces in $\boldsymbol{P}^3$ ($p_g=10$, $K^2=24$, see §4 below).

3. Pencil of curves of genus 3. The above Lemma 3 fascinated me with the idea that all degenerate fibres of pencil of genus 3 of non-hyperelliptic type may be described as degenerations of quartics like (1), extending the results for the case of pencil of genus 2 [11]. This was almost worked out around 1981. The aim is to make a complete list of all possible degenerations of plane quartics up to Cremona transfomations, and calculate the non-negative contribution of each of them to the value of $c_1^2-3p_g$. As a consequence, for such a pencil without exceptional curves in a fibre, we can prove

$$c_1^2 \geq 3\chi + 10(\pi - 1)$$

where $\chi = p_g - q + 1$ is the Euler-Poincaré characteristic and π denotes the genus of the base curve. Since we have $c_1^2 \geq (8/3)(\chi + 4\pi - 4)$ for hyperelliptic pencil of genus 3 [10, V, Theorem 2.1], it follows that, in the range $3\chi + 10(\pi - 1) > c_1^2 \geq (8/3)(\chi + 4\pi - 4)$, any hyperelliptic pencil of genus 3 is never deformed to non-hyperelliptic type. Examples are in [10, IV, Theorems 3.1, 3.2].

I found it rather difficult to write down the results in a concise, and still readable form. As of now I am not sure if (1) is only the "essential" degeneration, in a sense analogous to the case of genus 2 in [14].

I also worked out the case of hyperelliptic pencil of curves of genus 3. As one may learn from (1), this should not be studied as degeneration of double coverings of $\boldsymbol{P}^1$, but as those over the conic Q. Then some degeneration comes from that of the branch locus, and others from that of the conic Q, and from both in many cases.

4. Sextic surfaces. After I finished with quintic surfaces, I have thought, from time to time, of the next surfaces, the sextic surfaces in $\boldsymbol{P}^3$. These surfaces are embedded in $\boldsymbol{P}^3$ not by the canonical system, but by one half of it. This implies that S has an even intersection form on $H^2(S, \boldsymbol{Z})$, or in other words, the second Stiefel-Whitney class W_2 vanishes. Conversely, this topological condition assures that K is divisible by 2 as $K=2L$ for any deformation of S. So we only have to study numerical sextics with $W_2=0$.

First one proves $h^0(L)=4$ (It is easy to show $h^0(L)=4$ or 5. The case $h^0(L)=5$ must be excluded with some effort.) There are six possibilities for the map Φ_L associated to L.

(Ia) Embedding as a sextic.
(Ib) Double cover over a cubic surface.
(Ic) Triple cover over a quadric.
(IIa) Double cover over a smooth quadric.
(IIb) Double cover over a singular quadric.
(III) Composed of a pencil of genus 3 of non-hyperelliptic type.

A surface of type (Ib) is a complete interesction of two hypersurfaces in the weighted projective space $\boldsymbol{P}(3, 1, ,1, 1, 1)$ defined by

$$w^2+f(x_0, x_1, x_2, x_3)=0\,, \qquad g(x_0, x_1, x_2, x_3)=0\,,$$

where $\deg w=3$, $\deg x_i=1$, $0\leq i\leq 3$, and f and g are homogeneous polynomials in x_i of degrees 6 and 3, respectively. This is deformed to sextic surfaces if we replace the second equation by

$$tw-g(x_0, x_1, x_2, x_3)=0\,,$$

where t is a parameter ranging over a neighborhood of the origin. For $t\neq 0$ these two equations are reduced to $g^2+t^2f=0$. A surface of type (Ic) is a complete intersection of two hypersurfaces of the form

$$u^3+A_2u^2+A_4u+A_6=0\,, \qquad g=0$$

in $\boldsymbol{P}(2, 1, 1, 1, 1)$, where $\deg u=2$, and A_{2j} and g are homogeneous polynomials in (x_0, x_1, x_2, x_3) of degrees $2j$ and 2, respectively. This is similarly deformed to sextic surfaces by considering the equation $tu-g=0$.

A surface of type (IIa) is constructed as follows. Take a diagonal D on $\Sigma_0=\boldsymbol{P}^1\times\boldsymbol{P}^1$, and six points P_i, $1\leq i\leq 6$ on D. Then take a curve B_0 of bidegree (9, 9) on Σ_0 which has triple points at the six points P_i. Then the minimal resolution of the double covering with branch locus $B=D+B_0$ is a general surface of type (IIa). Surfaces of type (IIb) are constructed similarly, by using the Hirzebruch surface Σ_2 in place of Σ_0.

To construct deformations of a surface S of type (IIa), we take a quadratic equation $g=0$, a linear equation $l=0$ and a cubic equation $h=0$ in the variables (x_0, x_1, x_2, x_3). If they are sufficiently general, then $l=0$ determines a diagonal D on Σ_0 defined by $g=0$, and $h=0$ cuts out six points on D. Then we define a double covering by

$$w^2=l(h^3+A_2h^2l+A_4hl^2+A_6l^3)$$

over $g=0$, where A_{2j} is of degree $2j$. This is a singular model of a surface of type (IIa). To resolve the singularity, we introduce a new variable u of degree 2 which satisfies $ul=h$ (this has the same effect as the blowing up of $h=l=0$) and set $\tilde{w}=w/l^2$. Consequently, S is defined by the following three equations:

$$\begin{aligned}&\tilde{w}^2=u^3+A_2u^2+A_4u+A_6\,,\\ &ul=h\,,\\ &g=0\,.\end{aligned}$$

We take two parameters t and s, and consider the following equations:

$$\begin{aligned}&\tilde{w}^2=u^3+A_2u^2+A_4u+A_6\,,\\ &s\tilde{w}=ul-h\,,\end{aligned}$$

$$tu = g\ .$$

Then, for $t=0$, $s \neq 0$, these equations define a surface of type (Ic) and, for $t \neq 0$, $s \neq 0$, they define a surface of type (Ia). The same construction works for surfaces of type (IIb).

A surface S of type (III) is constructed as follows. Let $V = \boldsymbol{P}(\mathcal{O} \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-6))$ be a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$, and let (z_0, z_1) be a system of homogeneous coordinates on $\boldsymbol{P}^1$. We take homogeneous coordinates (Z_0, Z_1, Z_2) on the fibres. We can consider that z_i has weight $(1, 0)$ and the Z_j, $j=0, 1, 2$ have weights $(0, 1)$, $(5, 1)$, $(6, 1)$, respectively. Then, S is birationally equivalent to a hypersurface S' in V defined by

$$A^2 = z_0^2 Z_0 B\,, \tag{3}$$

where A is of weight $(10, 2)$ and B is of weight $(18, 3)$. This defines a singular fibre of the form (1) in §2 over $z_0=0$. Since A cannot contain the term Z_2^2, it follows that S' contains the line G: $Z_0 = Z_1 = 0$. Moreover, Z_0, restricited on S', vanishes to the fourth order on G. From this fact it follows that the minimal resolution S of S' is even (Note that there is an ordinary double point on G). To construct deformations of a surface S of type (III) to sextic surfaces, we use a construction which is analogous to what Griffin [7] has done for quintic surfaces. The graded ring

$$R = \bigoplus_{m \geq 0} H^0(S, \mathcal{O}(mL))$$

is generated by four elements x_0, x_1, x_2, x_3 of degree 1, three elements y_1, y_2, z of degree 2 and one element w of degree 3 (x_i, y_j and z generate the coordinate ring of V, and w corresponds to $A/z_0 Z_0^{1/2}$ in (3)). There are three relations of degree 2 and three of degree 3 among x_i and y_j, three relations of degree 4 involving w linearly, and one relation of degree 6 involving w^2. All the syzygies among these relations can be also written down. After these preparations we construct a family R_t of deformations of the ring R as in [7]. The computation is rather long and cannot be reproduced here.

The above list essentially exausts the surfaces which are the deformations of sextic surfaces (As usual, some mild degenerations are allowed. For example, the diagonal may decompose into two intersecting lines in (IIa), and the branch locus may have some mild singularities which do not affect the canonical ring.) In particular, there is no such surface with hyperelliptic pencil of genus 3. The proof of this fact seems to require the knowledge of most degenerate fibres of hyperelliptic pencils of genus 3.

It eventually turns out that all these surfaces (Ia)–(III) together form an irreducible family. Since $W^2 = 0$, any complex structure on the underlying differentiable manifold X is automatically minimal. Therefore the above list exhausts all possible complex structures on X.

5. Even algebraic surfaces, or semi-canonical surfaces. It may be worthwhile to study surfaces of general type with $W_2 = 0$. In this case, $K = 2L$, and we can prove that

L is either composed of a pencil, or satisfies $L^2 \geq 2h^0(L) - 4$([10, Part I, Lemma 7.6]).

In the exceptional case $L^2 < 2h^0(L) - 4$, S has a pencil of curves of genus 2 and $K^2 = 2p_g - 4$, $p_g \equiv 2 \bmod 4$. All of these surfaces already appeared in [10, Part I].

If the equality $L^2 = 2h^0(L) - 4$ holds, then Φ_L is a map of degree 2 onto $\boldsymbol{P}^2$, some Hirzebruch surface, or a cone over a rational curve as in [10, Part I], but this time, with two exceptions, S has a pencil of curves of genus 3 of hyperelliptic type of some simple kind. More precisely, the branch locus has at most simple triple points.

6. Regular threefolds with trivial canonical bundle. If V is a smooth threefold with trivial canonical bundle and if V is embedded in a projective space, then its hyperplane section H is a smooth surface S and the restriction H_S is the canonical bundle of S. So we have $H_S^2 \geq 2h^0(H_S) - 4$, which implies $H^3 \geq 2h^0(H) - 6$. If one starts with V and an ample line bundle H on it, then this inequality is not necessarily true. But in the exceptional case, it can be shown that V has a structure of elliptic threefold with a rational section, which should be manageable through the Weierstrass model. So excluding this case, I studied the extreme case $H^3 = 2h^0(H) - 6$, and determined their structures. They are mostly pencils of $K3$ surfaces of degree 2 which doubly cover a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$.

THEOREM 1. *Let V be a threefold with $K_V = 0$ which is not elliptic, and let H be an ample line bundle on V. Suppose $H^3 = 2h^0(H) - 6$. Then V is one of the following*:

(1) *A double covering of $\boldsymbol{P}^3$ with branch locus of degree* 8 ($h^0 = 4$).

(2) *A double covering of a smooth quadric in $\boldsymbol{P}^4$ whose branch locus is cut out by a hypersurface of degree* 6 ($h^0 = 5$).

(3) *A double covering of a $\boldsymbol{P}^2$-bundle $W = \boldsymbol{P}(\mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O}(\gamma))$ over $\boldsymbol{P}^1$ whose branch locus is in $|-2K_W|$* ($h^0 = \alpha + \beta + \gamma + 3$).

(4) *A double covering of the cone over the Veronese embedding of $\boldsymbol{P}^2$ branched along an intersection with a quintic hypersurface and the vertex* ($h^0 = 7$).

The third case occurs with

$$(\alpha, \beta, \gamma) = (k, k, k), (k, k, k+1), (k, k+1, k+1), (k, k, k+2), (k, k+1, k+2) \qquad (k \geq 1)$$

And these five types appear periodically with various polarizations. They have the Picard number 2 and the other ones have the Picard number 1.

The above list is quite similar to [10, Part I]. Note that the fourth case corresponds to surfaces with $p_g = 2$, $K^2 = 1$ embedded by $|2K|$.

THEOREM 2. *Let (V, H) be as in Theorem* 1, *but we suppose $H^3 = 2h^0(H) - 5$. Then V is one of the following*:

(1) *The same as* (4) *in Theorem* 1 (*equipped with one-half of H*, $h^0 = 3$).

(2) *A triple covering of $\boldsymbol{P}^3$ realized in the line bundle of degree* 2 ($h^0 = 4$).

(3) *Smooth model of a double covering of $\boldsymbol{P}^3$ whose branch locus consists of a plane*

L and a surface B_0 of degree 9 *which has a triple curve on a cubic on L* ($h^0=4$, *cf.* [10, II, *Theorem* 2.3]).

(4) *A smooth quintic in $\boldsymbol{P}^4$* ($h^0=5$).

(5) *Smooth model of a certain double covering of the $\boldsymbol{P}^2$-bundle $W=\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(2))$ over $\boldsymbol{P}^1$ whose branch locus is of degree* 8 *on each fibre* ($h^0=6$, *cf.* [10, II, *Theorem* 1.3, B_1)]).

(6) *Smooth model of a double covering of $W=\boldsymbol{P}(\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta)\oplus\mathcal{O}(\gamma))$ over $\boldsymbol{P}^1$ whose branch locus consists of a fibre Γ and a divisor $B_0\in|-2K_W+\Gamma|$ which has a triple curve along a conic on Γ* ($h^0=\alpha+\beta+\gamma+3$, *cf.* [10, II, *Theorem* 1.3, A)]).

For large h^0 only the sixth class appears. Such threefolds exist for

$$(\alpha, \beta, \gamma)=(k, k, k), (k, k, k+1), (k, k+1, k+1) \qquad (k\geq 1)\,.$$

Each of them has a pencil of $K3$ surfaces of degree 2 which degenerate into an elliptic $K3$ surface at one fibre (see [12, §8]). Needless to say, this list is parallel to [10, Part II].

I asked Masahisa Inoue if one can say something about the upper bound of H^3 in terms of $h^0(H)$. Then he obtained the inequality $H^3\leq 6h^0(H)$, and also proved that, if the equality sign holds, then V is unramifiedly covered by an abelian threefold. The proof of these facts is based on Yau's solution of Calabi's conjecture [18].

All these results were announced in a note [13] in Japanese.

I think these should be studied in the category of minimal models. Also, threefolds of general type may be investigated in this way if one can sufficiently develop the investigation in §5.

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Added in proof. I tacitly assumed $q=0$ in §5.