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INFINITESIMAL ISOMETRIES OF FRAME BUNDLES WITH NATURAL RIEMANNIAN METRIC

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1. Introduction. Let $(M, \langle\,,\,\rangle)$ be a connected orientable Riemannian manifold of dimension $n \geqq 3$ and $SO(M)$ be the bundle of all oriented orthonormal frames over M. $SO(M)$ has a Riemannian metric, also denoted by $\langle\,,\,\rangle$, defined naturally as follows: At each point u of $SO(M)$, the tangent space $SO(M)_u$ is a direct sum $Q_u + V_u$, where Q_u is the horizontal space defined by the Riemannian connection and V_u is the space of vectors tangent to the fibre through u. The right action of the special orthogonal group $SO(n)$ on the bundle $SO(M)$ gives an isomorphism f_u of the Lie algebra $\mathfrak{o}(n)$ onto V_u for each $u \in SO(M)$. We denote by A_u the image of $A \in \mathfrak{o}(n)$. On the other hand, $SO(n)$ has a bi-invariant metric denoted also by $\langle\,,\,\rangle$, which is defined by

$$\langle A, C\rangle = \sum_{i,j} A_{ij} C_{ij}\,, \qquad A, C \in \mathfrak{o}(n)\,.$$

Then, the Riemannian metric $\langle\,,\,\rangle$ of $SO(M)$ is defined by

$$\begin{aligned} &\langle A_u, C_u\rangle = \langle A, C\rangle \\ &\langle A_u, X_u\rangle = 0 \\ &\langle X_u, Y_u\rangle = \langle pX_u, pY_u\rangle \end{aligned}$$

for $X_u, Y_u \in Q_u$ and $A, C \in \mathfrak{o}(n)$, where p is the projection $SO(M) \to M$.

O'Neill [4] studied the curvature of $(SO(M), \langle\,,\,\rangle)$. In the present paper, we shall study Killing vector fields on $(SO(M), \langle\,,\,\rangle)$ and prove the following Theorems A and B. Let X be a vector field on $SO(M)$. X is said to be vertical (resp. horizontal) if $X_u \in V_u$ (resp. if $X_u \in Q_u$) for all $u \in SO(M)$. X is said to be fibre preserving if $[X, X']$ is vertical for any vertical vector field X'. Let A^* be the vertical vector field defined by $(A^*)_u = A_u = f_u(A)$. A^* is called the fundamental vector field corresponding to $A \in \mathfrak{o}(n)$. X is decomposed uniquely as $X = X^H + X^V$, with X^H horizontal and X^V vertical. X^H and X^V are called th horizontal part and the vertical part of X, respectively. Let ϕ be a 2-form on M. Then the tensor field F of type (1,1) is defined by $\langle FY, Z\rangle = \phi(Y, Z)$. Then, for each $u \in SO(M)$, $F^\sharp(\tilde{u}) \in \mathfrak{o}(n)$ is defined by

$$F^\sharp(u) = u^{-1} \circ F_{p(u)} \circ u\,,$$

where u is regarded as a linear isometry of $(\boldsymbol{R}^n, \langle\,,\,\rangle)$ onto the tangent space $M_{p(u)}$ at $p(u)$. Here $\langle\,,\,\rangle$ also denotes the standard metric of $\boldsymbol{R}^n$. Then, the vertical vector field

X is defined by

$$X_u = f_u(F^{\sharp}(u)), \qquad u \in SO(M).$$

X is called the natural lift of ϕ or F and is denoted by ϕ^L or F^L. Let Y be a Killing vector field on $(M, \langle\ ,\ \rangle)$. Then the horizontal vector field X^H is defined by

$$p(X^H)_u = Y_{p(u)}, \qquad u \in SO(M).$$

Let DY be the covariant differential of Y and X^V be the natural lift of DY defined as above. The vector field $X = X^H + X^V$ on $SO(M)$ is called the natural lift of Y and is denoted by Y^L.

THEOREM A. *Let X be a fibre preserving Killing vector field on $(SO(M), \langle\ ,\ \rangle)$. Then, X is decomposed as*

$$X = Y^L + \phi^L + A^*,$$

where Y^L is the natural lift of a Killing vector field Y on $(M, \langle\ ,\ \rangle)$, ϕ^L is the natural lift of a parallel 2-form ϕ on $(M, \langle\ ,\ \rangle)$ and A^* is the fundamental vector field.

THEOREM B. *If $(SO(M), \langle\ ,\ \rangle)$ has a horizontal Killing vector field which is not fibre preserving, then $(M, \langle\ ,\ \rangle)$ has constant curvature $1/2$, except when dim $M = 3$, 4 or 8.*

Theorems A and B seem to be related to the results of Tanno [5] who gives a decomposition of any Killing vector field on the tangent bundles with a Sasakian metric.

Thanks are due to T. Asoh and F. Uchida who taught us basic facts about Lie groups and Lie algebras.

2. Preliminaries. In this section, we give definitions, notation and lemmas needed to prove Theorems A and B.

For $\xi \in \boldsymbol{R}^n$, we define the standard horizontal vector field $B(\xi)$ on $SO(M)$ by

$$p(B_u(\xi)) = u(\xi), \qquad u \in SO(M).$$

We denoted also by D the covariant differentiation with respect to the Riemannian connection of $(SO(M), \langle\ ,\ \rangle)$.

The proof of the following lemma can be found in [2] and [4].

LEMMA 1. *Let $A, C \in \mathfrak{o}(n)$, $\xi, \eta, \zeta \in \boldsymbol{R}^n$ and let Ω be the curvature form of the Riemannian connection of $(M, \langle\ ,\ \rangle)$. Then,*

$$[A^*, C^*] = [A, C]^*$$
$$[A^*, B(\xi)] = B(A\xi)$$
$$\langle [B(\xi), B(\eta)], B(\zeta)\rangle = 0$$
$$\langle [B(\xi), B(\eta)], A^*\rangle = -2\langle \Omega(B(\xi), B(\eta)), A\rangle$$

$$\langle D_{B(\xi)}B(\eta), B(\zeta)\rangle=0$$
$$\langle D_{B(\xi)}B(\eta), A^*\rangle=-\langle \Omega(B(\xi), B(\eta)), A\rangle$$
$$\langle D_{B(\xi)}A^*, B(\eta)\rangle=\langle \Omega(B(\xi), B(\eta)), A\rangle$$
$$\langle D_{B(\xi)}A^*, C^*\rangle=0$$
$$\langle D_{A^*}B(\xi), B(\eta)\rangle=\langle \Omega(B(\xi), B(\eta)), A\rangle+\langle B(A\xi), B(\eta)\rangle$$
$$\langle D_{A^*}B(\xi), C^*\rangle=0$$
$$D_{A^*}C^*=(1/2)[A, C]^* .$$

Let X be a vector field on $SO(M)$. Then, X is defined by

$$x(\xi)=\langle X, B(\xi)\rangle=\langle X^H, B(\xi)\rangle , \qquad \xi\in \boldsymbol{R}^n$$
$$x(A)=\langle X, A^*\rangle=\langle X^V, A^*\rangle , \qquad A\in \mathfrak{o}(n) .$$

$x(\xi)$ and $x(A)$ are called the ξ-component and the A-component of X, respectively. X is horizontal if and only if $x(A)=0$ for all $A\in\mathfrak{o}(n)$, while X is vertical if and only if $x(\xi)=0$ for all $\xi\in\boldsymbol{R}^n$.

LEMMA 2. *Let X be a vector field on $SO(M)$. Then X is a Killing vector field if and only if*

$$B(\xi)(x(\eta))+B(\eta)(x(\xi))=0$$
$$A^*(x(\xi))-x(A\xi)+B(\xi)(x(A))-2\langle \Omega(B(\xi), X^H), A\rangle=0$$
$$A^*(x(C))+C^*(x(A))=0$$

for all ξ, $\eta\in\boldsymbol{R}^n$ and A, $C\in\mathfrak{o}(n)$.

RROOF. X is a Killing vector field if and only if

$$\langle D_{B(\xi)}X, B(\eta)\rangle+\langle D_{B(\eta)}X, B(\xi)\rangle=0$$
$$\langle D_{B(\xi)}X, A^*\rangle+\langle D_{A^*}X, B(\xi)\rangle=0$$
$$\langle D_{A^*}X, C^*\rangle+\langle D_{C^*}X, A^*\rangle=0$$

for all ξ, $\eta\in\boldsymbol{R}^n$ and A, $C\in\mathfrak{o}(n)$. Then, the assertion follows from Lemma 1 and the fact that

$$D_{B(\xi)}B(\eta)+D_{B(\eta)}B(\xi)=0$$
$$\langle D_{B(\xi)}A^*, X\rangle=\langle \Omega(B(\xi), X^H), A\rangle$$
$$\langle D_{A^*}B(\xi), X\rangle=\langle \Omega(B(\xi), X^H), A\rangle+\langle B(A\xi), X\rangle .$$

By virtue of Lemma 2, it is easy to see that a fundamental vector field is a Killing vector field.

LEMMA 3. *Let X be a vector field on $SO(M)$. Then,*

$$\langle [B(\xi), X], B(\eta)\rangle + \langle [B(\eta), X], B(\xi)\rangle = B(\xi)(x(\eta)) + B(\eta)(x(\xi))$$
$$\langle [A^*, X], B(\xi)\rangle = A^*(x(\xi)) - x(A\xi)$$
$$\langle [B(\xi), X], A^*\rangle = B(\xi)(x(A)) - 2\langle \Omega(B(\xi), X^H), A\rangle$$
$$\langle [A^*, X], C^*\rangle = A^*(x(C)) - x([A, C])$$
$$\langle [B(\xi), X], B(\eta)\rangle_u = B_u(\xi)(x(\eta)) - \langle f_u{}^{-1}((X^V)_u)\xi, \eta\rangle$$

for all $\xi, \eta \in \boldsymbol{R}^n$, $A, C \in \mathfrak{o}(n)$ and $u \in SO(M)$.

PROOF. The assertion follows from Lemma 1 and the fact that A^* is a Killing vector field and $\langle B(\xi), B(\eta)\rangle = \langle \xi, \eta\rangle$.

3. Proof of Theorem A.

LEMMA 4. *Let X be a vertical Killing vector field. Then X is decomposed uniquely as*

$$X = \phi^L + A^*\,,$$

where ϕ is a parallel 2-form on $(M, \langle\ ,\ \rangle)$ and A^* is the fundamental vector field.

PROOF. We first show that X is decomposed uniquely as

$$X = X_1 + X_2\,,$$

where X_1 and X_2 are smooth vertical vector fields on $SO(M)$ such that

$$[A^*, X_1] = 0\,, \qquad A^*(x_2(C)) = 0$$

for all $A, C \in \mathfrak{o}(n)$. Here $x_2(C)$ denotes the C-component of X_2. It should be noted that each fibre is totally geodesic and is isometric to the Riemannian symmetric space $(SO(n), \langle\ ,\ \rangle)$. For each $u \in SO(M)$, we define an isometry $g_u\colon p^{-1}(p(u)) \to SO(n)$ by $g_u(ua) = a$. Then $g_u(X)$ is a Killing vector field on $(SO(n), \langle\ ,\ \rangle)$. By a standard theory of symmetric space (cf. [1]), $g_u(X)$ is decomposed uniquely as

$$g_u(X) = W_1(u) + W_2(u)\,,$$

where $W_1(u)$ is a right invariant vector field and $W_2(u)$ a left invariant vector field on $SO(n)$. Define the vector fields X_1 and X_2 on $p^{-1}(p(u))$ by

$$X_1 = g_u{}^{-1}(W_1(u))\,, \qquad X_2 = g_u{}^{-1}(W_2(u))\,.$$

It is easy to check that the definition of X_1 and X_2 is independent of the choice of g_v for $v \in p^{-1}(p(u))$, since $g_u = L_b \circ g_v$ when $v = ub$ for $b \in SO(n)$. The smoothness of X_1 and X_2 follows from the local triviality of this bundle. The properties $[A^*, X_1] = 0$ and $A^*(x_2(C)) = 0$ follow from

$$[A^*, X_1] = g_u{}^{-1}([A, W_1(u)])$$

$$x_2(C)=\langle X_2, C^*\rangle=\langle g_u{}^{-1}(W_2(u)), g_u{}^{-1}(C)\rangle=\langle W_2(u), C\rangle .$$

Next, we show that X_2 is a fundamental vector field. Since X is vertical, it follows from Lemma 2 that

$$B(\xi)(x(C))=B(\xi)(x_1(C)+x_2(C))=0$$

for all $C\in\mathfrak{o}(n)$. On the other hand, $[A^*, X]=[A^*, X_2]$ is a vertical Killing vector field and, by Lemma 3,

$$\langle[A^*, X_2], C^*\rangle=A^*(x_2(C))-x_2([A, C])=-x_2([A, C])$$

for all A, $C\in\mathfrak{o}(n)$. Thus, by Lemma 2, we have $B(\xi)$ $(x_2([A, C]))=0$ for all $\xi\in\boldsymbol{R}^n$ and A, $C\in\mathfrak{o}(n)$. Hence, the semisimplicity of the Lie algebra $\mathfrak{o}(n)$ implies

$$B(\xi)(x_2(A))=0$$

for all $\xi\in\boldsymbol{R}^n$ and $A\in\mathfrak{o}(n)$. These conditions on $x_2(A)$ imply that $x_2(A)$ is constant on $SO(M)$ for all $A\in\mathfrak{o}(n)$, hence X_2 is a fundamental vector field.

X_1 is a vertical Killing vector field satisfying

$$B(\xi)(x_1(C))=0 , \qquad [C^*, X_1]=0$$

for all $\xi\in\boldsymbol{R}^n$ and $C\in\mathfrak{o}(n)$. Now, it suffices to show that X_1 is the lift of a parallel 2-form on $(M, \langle\,,\rangle)$. Let $F^\#$ be the $\mathfrak{o}(n)$-valued function defined by

$$F^\#(u)=f_u{}^{-1}((X_1)_u) , \qquad u\in SO(M).$$

Then, we have $F^\#(ua)=a^{-1}\circ F^\#(u)\circ a$ for all $a\in SO(n)$. This follows from the fact that the condition $[C^*, X_1]=0$ for all $C\in\mathfrak{o}(n)$ is equivalent to the condition $R_aX_1=X_1$ for all $a\in SO(n)$ and the fact $f_{ua}=R_a\circ f_u\circ \mathrm{ad}(a)$. Hence, the tensor field of type (1, 1) on M is well-defined by

$$F_{p(u)}=u\circ F^\#(u)\circ u^{-1} , \qquad p(u)\in M .$$

Let ϕ be a 2-form corresponding to F. Then, $X_1=\phi^L$ and ϕ is parallel. The last assertion follows from

$$\begin{aligned}(x_1(C))(u)=\langle(C^*)_u, (X_1)_u\rangle=\langle C, f_u{}^{-1}(X_1)_u\rangle=\langle C, F^\#(u)\rangle\\ B(\xi)(x_1(C))=\langle C, B(\xi)F^\#\rangle\\ D_{u(\xi)}F=u\circ(B(\xi)F^\#)\circ u^{-1} .\end{aligned}$$

For the proof of the last equality, see lemma of section 1 of chapter III of [2]. Note that any vertical vector field is fibre preserving, since each fibre is totally geodesic.

LEMMA 5. *Let X be a fibre preserving Killing vector field on $(SO(M), \langle\,,\rangle)$. Then There exists a Killing vector field Y on $(M, \langle\,,\rangle)$ such that $X^H=(Y^L)^H$ and Y^L is a Killing vector field.*

PROOF. Since X^V is fibre preserving, so is X^H and, by Lemma 3,

$$\langle [A^*, X^H], B(\xi)\rangle = 0\,, \qquad \langle [A^*, X^H], C^*\rangle = 0$$

for all $A, C \in \mathfrak{o}(n)$ and $\xi \in \boldsymbol{R}^n$. It means that $[A^*, X^H] = 0$ for all $A \in \mathfrak{o}(n)$, that is, $R_a X^H = X^H$ for all $a \in SO(n)$. Hence, there exists a unique vector field Y on M satisfying $pX^H = Y$.

We first show that Y is a Killing vector field. Let h be an $\boldsymbol{R}^n$-valued function on $SO(M)$ defined by $h(u) = u^{-1} Y_{p(u)}$. Then we have $D_{u(\xi)} Y = u(B(\xi)h)$ for all $\xi \in \boldsymbol{R}^n$ and $u \in SO(M)$. For the proof, see lemma of section 1 of chapter III of [2]. It follows that

$$\begin{aligned}(x(\xi))(u) &= \langle X_u, B_u(\xi)\rangle = \langle (X^H)_u, B_u(\xi)\rangle \\ &= \langle p(X^H)_u, pB_u(\xi)\rangle = \langle Y_{p(u)}, u(\xi)\rangle \\ &= \langle u^{-1} Y_{p(u)}, u^{-1} \circ u(\xi)\rangle = \langle h(u), \xi\rangle\,.\end{aligned}$$

Moreover,

$$\begin{aligned}B_u(\eta)(x(\xi)) &= \langle B_u(\eta)h, \xi\rangle = \langle D_{u(\eta)} Y, u(\xi)\rangle \\ B_u(\xi)(x(\eta)) &= \langle B_u(\xi)h, \eta\rangle = \langle D_{u(\xi)} Y, u(\eta)\rangle\,.\end{aligned}$$

Then, by Lemma 2, for all $\xi, \eta \in \boldsymbol{R}^n$ and $u \in SO(M)$,

$$\langle D_{u(\eta)} Y, u(\xi)\rangle + \langle D_{u(\xi)} Y, u(\eta)\rangle = 0\,,$$

which shows that Y is a Killing vector field.

Now, we show that $Y^L = X^H + X_1$ is a Killing vector field, where $X_1 = (DY)^L$. Let $F = DY$ and let $F^\sharp$ be the $\mathfrak{o}(n)$-valued function on $SO(M)$ defined by $F^\sharp(u) = u^{-1} \circ F_{p(u)} \circ u$. Then $F^\sharp(ua) = a^{-1} \circ F^\sharp(u) \circ a$ for all $a \in SO(n)$, which implies that $R_a(X_1)_u = (X_1)_{ua}$ for all $a \in SO(n)$, that is, $[A^*, X_1] = 0$ for all $A \in \mathfrak{o}(n)$. The proof is the same as that of Lemma 4. Then, by Lemma 3, it follows that

$$A^*(x_1(C)) - x_1([A, C]) = 0$$

and hence

$$A^*(x_1(C)) + C^*(x_1(A)) = 0\,.$$

On the other hand, in the same way as in the proof of Lemma 4,

$$\begin{aligned}B_u(\xi)(x_1(C)) &= \langle C, B_u(\xi)F^\sharp\rangle = \langle C, u^{-1} \circ (D_{u(\xi)}F) \circ u\rangle \\ &= \langle C, u^{-1} \circ (D_{u(\xi)} DY) \circ u\rangle = -\langle C, u^{-1} \circ R(Y, u(\xi)) \circ u\rangle \\ &= -2\langle C, \Omega_u(X^H, B(\xi))\rangle\end{aligned}$$

for all $C \in \mathfrak{o}(n)$ and $\xi \in \boldsymbol{R}^n$, where $R(Y, u(\xi))$ denotes the curvature operator of $(M, \langle\ ,\ \rangle)$. The last two equalities are well-known (see [2]). By $\langle Y^L, B(\xi)\rangle = x(\xi)$ and $\langle Y^L, A^*\rangle = x_1(A)$, these results show that Y^L is a Killing vector field.

Now, let X be a fibre preserving Killing vector field and let Y be the vector field given in Lemma 5. Then, $X-Y^L$ is a vertical Killing vector field which can be written as $X-Y^L=\phi^L+A^*$ by Lemma 4. This completes the proof of Theorem A.

4. Proof of Theorem B. Throughout this section, we assume dim $M\geqq 5$. Let U be the space of all horizontal Killing vector fields on $(SO(M), \langle\,,\,\rangle)$ and let U_u be the subspace of Q_u obtained as the restriction of U to $u\in SO(M)$. Let $\mathfrak{o}(U_u)$ be the algebra of all skew symmetric linear transformations of U_u.

LEMMA 6. (i) *For each $A\in\mathfrak{o}(n)$, the linear map $r_u(A): U_u\to U_u$ is well-defined by*

$$r_u(A)(X_u)=[A^*, X]_u\,, \qquad X\in U\,.$$

(ii) *The linear map $r_u: \mathfrak{o}(n)\to\mathfrak{o}(U_u)$ is a Lie algebra homomorphism.*

PROOF. If $X\in U$, then, by Lemmas 2 and 3,

$$\langle[A^*, X], B(\xi)\rangle=2\langle\Omega(B(\xi),X), A\rangle\,, \qquad \langle[A^*, X], C^*\rangle=0$$

for all A, C $\in\mathfrak{o}(n)$ and $\xi\in\boldsymbol{R}^n$. These equalities mean that $[A^*, X]$ is a horizontal Killing vector field and that $[A^*, X]_u$ depends only on X_u. (ii) follows from the Jacobi identity

$$[[A^*, C^*], X]=[A^*, [C^*, X]]-[C^*, [A^*, X]]$$

and

$$\langle r_u(A)(X_u), (X')_u\rangle=2\langle\Omega((X')_u, X_u), A\rangle$$

for $X'\in U$.

Now, we assume that there exists a horizontal Killing vector field which is not fibre preserving. Then, at each point u of a certain open dense subset of $SO(M)$, the dimensions of both U_u and $r_u(\mathfrak{o}(n))$ are greater than 0. However, this is possible only when $U_u=Q_u$ and $r_u(\mathfrak{o}(n))=\mathfrak{o}(U_u)$. Otherwise, r_u has a non-zero kernel which contradicts the simplicity of the Lie algebra $\mathfrak{o}(n)$. We can define the automorphism s_u of $\mathfrak{o}(n)$ by

$$s_u(A)\circ u^{-1}\circ p=u^{-1}\circ p\circ r_u(A)$$

for each point u of the subset. We note that any fibre is contained in the subset or has no intersection with it. This follows from Lemma 2 which shows that, if a horizontal Killing vector field X attains zero at a point u, then X is zero along the fibre through u.

LEMMA 7. (i) *s_u is an involutive automorphism.*

(ii) $\mathrm{ad}(a)\circ s_{ua}=s_u\circ\mathrm{ad}(a)$ *for* $a\in SO(n)$.

PROOF. (i) First, we note

$$s_u(A)\xi=(u^{-1}\circ p\circ r_u(A))B_u(\xi) \qquad \text{for all} \quad \xi\in\boldsymbol{R}^n\,.$$

Hence we have

$$\begin{aligned}\langle s_u(A)\xi, \eta\rangle &= \langle (u^{-1}\circ p\circ r_u(A))B_u(\xi), \eta\rangle = \langle (p\circ r_u(A))B_u(\xi), u(\eta)\rangle \\ &= \langle r_u(A)B_u(\xi), B_u(\eta)\rangle = 2\langle \Omega(B_u(\eta), B_u(\xi)), A\rangle \\ &= \langle u^{-1}\circ R(u(\eta), u(\xi))\circ u, A\rangle\end{aligned}$$

for all $A\in\mathfrak{o}(n)$ and $\xi, \eta\in \boldsymbol{R}^n$, where $R(u(\xi), u(\eta))$ denotes the curvature operator. By the symmetry of the curvature tensor R and the fact that the metric $\langle\ ,\ \rangle$ of $\mathfrak{o}(n)$ is a scalar multiple of the Killing form, we have

$$\langle s_u(A), C\rangle = \langle A, s_u(C)\rangle\,, \qquad \langle s_u(A), s_u(C)\rangle = \langle A, C\rangle$$

for all A, $C\in\mathfrak{o}(n)$. Thus

$$\langle A, C\rangle = \langle s_u(A), s_u(C)\rangle = \langle s_u^2(A), C\rangle\,,$$

which implies $s_u^2 = 1$.

(ii) follows from

$$\begin{aligned}\langle s_{ua}(A)\xi, \eta\rangle &= \langle (ua)^{-1}\circ R((ua)(\eta), (ua)(\xi))\circ(ua), A\rangle \\ &= \langle a^{-1}\circ u^{-1}\circ R(u(a\eta), u(a\xi))\circ u\circ a, A\rangle \\ &= \langle u^{-1}\circ R(u(a\eta), u(a\xi))\circ u, aAa^{-1}\rangle \\ &= \langle (s_u(\mathrm{ad}(a)A))a\xi, a\eta\rangle = \langle a^{-1}(s_u(\mathrm{ad}(a)A))a\xi, \eta\rangle\,.\end{aligned}$$

Let h be one of the matrices

$$I_n\,, \qquad I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

where I_n denotes the identity matrix of degree n, $p+q=n$, $1\leqq p\leqq q\leqq n$ and $2m=n$. Then, by the classification theory of symmetric spaces of type $SO(n)/K$, any involutive automorphism of $\mathfrak{o}(n)$ is conjugate to $\mathrm{ad}(h)$ in the group $\mathrm{Aut}(\mathfrak{o}(n))$ of all automorphisms of $\mathfrak{o}(n)$. Furthermore, it is well-known that $\mathrm{Aut}(\mathfrak{o}(n))$ for n odd is isomorphic to the group $\mathrm{Int}(\mathfrak{o}(n))$ of all inner automorphisms of $\mathfrak{o}(n)$, while the quotient group $\mathrm{Aut}(\mathfrak{o}(n))/\mathrm{Int}(\mathfrak{o}(n))$ for n even and $n\neq 8$ is isomorphic to $\boldsymbol{Z}_2$ (see [1], [3]). Here, we note that, if n is even, $\pm I_{1,n-1}$ is an element of $O(n)$ but not $SO(n)$ and hence $\mathrm{ad}(I_{1,n-1})$ is not an element of $\mathrm{Int}(\mathfrak{o}(n))$. These facts show that any element of $\mathrm{Aut}(\mathfrak{o}(n))$ is of the form $\mathrm{ad}(a)$ for some a $a\in O(n)$, unless $n=8$.

Consequently, any involutive automorphism s of $\mathfrak{o}(n)$ is written as $s=\mathrm{ad}(aha^{-1})$ for some $a\in O(n)$ except when $n=8$. By Lemmas 6 and 7, we have:

LEMMA 8. *Assume that* $\dim M\neq 8$ *and that* $(SO(M), \langle\ ,\ \rangle)$ *has a horizontal Killing vector field which is not fibre preserving. Then, for each point u of a certain open dense subset of $SO(M)$, there exists an automorphism s_u of $\mathfrak{o}(n)$ such that*

(i) $\qquad s_u = \mathrm{ad}(aha^{-1}) \qquad$ *for some* $\quad a\in O(n)\,,$

(ii) $$\langle s_u(A)\xi, \eta\rangle = \langle u^{-1}\circ R(u(\eta), u(\xi))\circ u, A\rangle$$

for all $A\in\mathfrak{o}(n)$ *and* $\xi, \eta\in \boldsymbol{R}^n$,

(iii) $$s_{ub} = \mathrm{ad}(b^{-1}aha^{-1}b) \qquad \textit{for} \quad b\in SO(n)\,,$$

where h is one of the matrices I_n, $I_{p,q}$ *and* J.

Let V be a vector space with an inner product $\langle\ ,\ \rangle$. For each $\xi, \eta\in V$, we define a skew-symmetric linear transformation $\xi\wedge\eta$ of V by

$$(\xi\wedge\eta)(\zeta) = \langle\eta, \zeta\rangle\xi - \langle\xi, \zeta\rangle\eta\,.$$

Let H be a linear transformation of a tangent space of M such that, with respect to a certain orthonormal basis, the representation matrix of H is one of I_n, $I_{p,q}$ and J.

LEMMA 9. *Under the assumption of Lemma 8, the curvature operator* $R(X,Y)$ *of* $(M, \langle\ ,\ \rangle)$ *is expressed as*

$$R(X, Y) = (1/2)HX\wedge HY$$

at each point of a certain open dense subset of M.

PROOF. First we note that, if $A\in\mathfrak{o}(n)$, $\xi, \eta\in\boldsymbol{R}^n$ and $b\in O(n)$, then

$$\langle\xi\wedge\eta, A\rangle = -2\langle A\xi, \eta\rangle\,, \qquad \mathrm{ad}(b)(\xi\wedge\eta) = b\xi\wedge b\eta\,.$$

Let k be aha^{-1} appearing in (i) of Lemma 8. Then, $k^{-1} = \pm k$ since $h^2 = \pm 1$. Taking into account the fact that the metric $\langle\ ,\ \rangle$ of $SO(n)$ is adjoint-invariant, it follows that

$$\begin{aligned}\langle(\mathrm{ad}(k)A)\xi, \eta\rangle &= -(1/2)\langle\xi\wedge\eta, \mathrm{ad}(k)A\rangle = -(1/2)\langle\mathrm{ad}(k^{-1})(\xi\wedge\eta), A\rangle\\ &= -(1/2)\langle k\xi\wedge k\eta, A\rangle\,.\end{aligned}$$

Then, by (ii) of Lemma 8,

$$u^{-1}\circ R(u(\eta), u(\xi))\circ u = -(1/2)k\xi\wedge k\eta = (1/2)k\eta\wedge k\xi$$

and hence

$$R(u(\eta), u(\xi)) = (1/2)u(k\eta)\wedge u(k\xi)\,.$$

Put $H = u\circ k\circ u^{-1}$. Then

$$R(u(\eta), u(\xi)) = (1/2)H(u(\eta))\wedge H(u(\xi))\,,$$

which completes the proof.

Now, we note that H of Lemma 9 cannot have the representation matrix J, since, as is easily to checked, if H has the representation matrix J, then the tensor R defined by $R(X, Y) = (1/2)HX\wedge HY$ does not satisfy the first Bianchi identity.

Next, we show that H has the same representation matrix over the set of points of M, where $R(X, Y)$ is expressed as $(1/2)HX \wedge HY$. Indeed, if $R(X, Y)=(1/2)X \wedge Y$ at a point of M, then the scalar curvature is equal to $n(n-1)/2$ at the point. If $R(X, Y)=(1/2)HX \wedge HY$ for H having the representation matrix $I_{p,q}$ at a point of M, then the scalar curvature is equal to $\{(q-p)^2-n\}/2$ at the point. But, $n(n-1)-\{(q-p)^2-n\} \geqq 4$ and, if $q>q'$, then $\{(q-p)^2-n\}-\{(q'-p')^2-n\} \geqq 4$. Hence, the connectivity of M and the continuity of the scalar curvature imply the assertion.

Next, we show that, if $R(X, Y)=(1/2)HX \wedge HY$ for some H having the representation matrix $I_{p,q}$, then such an H can be chosen smoothly in a neighbourhood of each point. This is clear for the case $p<q$, because the Ricci transformation S is written as $S=(q-p)H-I$. We need the following lemma to prove it for the case $p=q$.

LEMMA 10. *Assume that H has the representation matrix $I_{p,p}$* and $R(X, Y)=(1/2)HX \wedge HY$. *Then we have the following*:
(i) *At each point, such an H is determined uniquely without distinction of the signs.*
(ii) *Such an H can be taken smoothly in a neighbourhood of each point.*

PROOF. (i) Suppose K also has the representation matrix $I_{p,p}$ and $HX \wedge HY= KX \wedge KY$. It suffices to show $K=\pm H$. By the definition of $X \wedge Y$, if $\{X, Y\}$ is linearly independent, then the plane spanned by KX and KY coincides with the one spanned by HX and HY. Let $\{X_i\}$ be an orthonormal basis such that $HX_i=a_iX_i$ for $a_i=\pm 1$. Then, KX_1 and KX_2 are linear combinations of X_1 and X_2. KX_1 and KX_3 are also linear combinations of X_1 and X_3. It follows that $KX_1=b_1X_1$ for some $b_1 \in \boldsymbol{R}$. Similarly, $KX_i=b_iX_i$ for some $b_i \in \boldsymbol{R}$ $(1 \leqq i \leqq n=2p)$, which implies $b_i=\pm 1$, as $K^2=I$. Then, the equality $(HX_i \wedge HX_k)X_k=(KX_i \wedge KX_k)X_k$ implies $a_ia_k=b_ib_k$ for $i \neq k$. Hence, if $b_1=\pm a_1$, then, $b_k=\pm a_k$ for $k \geqq 2$.

(ii) Let us consider the following system of quadratic equations with unknown variables H_{ji}:

$$(*) \qquad H_{kh}H_{ji}-H_{ki}H_{jh}=2R_{kjih}\,,$$

where R_{kjih} are the components of the curvature tensor with respect to a smooth field of orthonormal basis $\{X_i\}$ defined in a neighbourhood of a point $m \in M$, that is, we put $R_{kjih}=\langle R(X_k, X_j)X_i, X_h\rangle$. We assume that $(*)$ has two solutions $\pm(H_{ij})$ at each point and that the solution matrix (H_{ij}) is diagonalizable to $I_{p,p}$ by a certain orthogonal matrix at each point. We first show that there exist smooth functions H_{ii} $(1 \leqq i \leqq n)$ of variables H_{ij} $(1 \leqq i<j \leqq n)$ and $R_{hiih}(i \neq h)$ such that the components of one of the solution matrices must satisfy these relations. By $(*)$, the two solutions satisfy the equations

$$(1) \qquad H_{hh}H_{ii}-(H_{hi})^2=2R_{hiih} \qquad (i \neq h)$$

$$(2) \qquad H_{hh}H_{ij}-H_{hi}H_{hj}=2R_{hijh} \qquad (i \neq j)\,.$$

Here, we may assume $(H_{ij})=I_{p,p}$ at $m \in M$. By assumption, it follows that, at m,

$$2R_{hiih}=1 \qquad (1 \leqq h<i \leqq p \quad \text{or} \quad p+1 \leqq h<i \leqq n)$$

$$(H_{12})^2+2R_{1221}=(H_{2k})^2+2R_{2kk2}=(H_{1k})^2+2R_{1kk1}=1 \qquad (3\leqq k\leqq p).$$

Note that R_{kjih} is a smooth function and hence we way assume $R_{hiih}>0$ for $1\leqq h<i\leqq p$ in the neighbourhood. Then, by (1), we have

$$\begin{aligned} H_{11}H_{22}&=(H_{12})^2+2R_{1221}>0 \\ H_{22}H_{kk}&=(H_{2k})^2+2R_{2kk2}>0 \qquad (3\leqq k\leqq p) \\ H_{kk}H_{11}&=(H_{1k})^2+2R_{1kk1}>0 \end{aligned}$$

in the neighbourhood. The components of one of the solution matrices must satisfy

$$\begin{aligned} H_{11}&=-\{((H_{12})^2+2R_{1221})((H_{1k})^2+2R_{1kk1})/((H_{2k})^2+2R_{2kk2})\}^{1/2} \\ H_{22}&=-\{((H_{12})^2+2R_{1221})((H_{2k})^2+2R_{2kk2})/((H_{1k})^2+2R_{1kk1})\}^{1/2} \\ H_{kk}&=-\{((H_{2k})^2+2R_{2kk2})((H_{1k})^2+2R_{1kk1})/((H_{12})^2+2R_{1221})\}^{1/2}, \end{aligned}$$

$(3\leqq k\leqq p)$. We have similar results for $1^*=p+1$, $2^*=p+2$, $k^*=p+k$ $(3\leqq k\leqq p)$, that is, the components of one of the solution matrices must satisfy

$$H_{1^*1^*}=\{\dots\}^{1/2}, \quad H_{2^*2^*}=\{\dots\}^{1/2}, \quad H_{k^*k^*}=\{\dots\}^{1/2},$$

where three brackets $\{\dots\}$ are obtained from the above three by exchanging subscript indices 1, 2, k for 1^*, 2^*, k^*, respectively.

(2) is regarded as a system of equations of $n(n-1)/2$ unknown variables H_{ij} $(i<j)$. Taking account of $(H_{ij}(m))=I_{p,p}$, it is easily seen that the coefficient of the partial derivative of the function $H_{tt}H_{ij}-H_{ti}H_{tj}$ with respect to variable H_{hk} is equal to $\pm\delta_{hi}\delta_{jk}$ at m, if $h<k$, $i<j$, $t\neq i$ and $t\neq j$. The implicit function theorem shows that $(*)$ has a unique smooth solution in a neighbourhood of m satisfying $(H_{ij}(m))=I_{p,p}$.

In the next lemma, we prove that H cannot have the representation matrix $I_{p,q}$. Then, the remaining possibility is $H=I$, that is, $(M, \langle\, ,\rangle)$ has constant curvature $1/2$, which completes the proof of Theorem B.

LEMMA 11. *There exists no Riemannian manifold* $(M, \langle\, ,\rangle)$ *such that the curvature operator* $R(X, Y)$ *is expressed as* $R(X, Y)=cHX\wedge HY$, *where* c *is a non-zero constant and* H *is a linear transformation defined pointwise and having the representation matrix* $I_{p,q}$ $(1\leqq p\leqq q\leqq n)$ *with repect to a certain orthonormal basis.*

PROOF. By Lemma 10, we may assume that H is smooth in a neighbourhood of each point. Let $\{X_i\}$ be a local orthonormal frame field such that $HX_i=a_iX_i$ satisfying $a_i=\pm 1$ $(1\leqq i\leqq n)$. Let $D_{jih}=\langle D_{X_j}X_i, X_h\rangle$ and $H_{jih}=\langle(D_{X_j}H)(X_i), X_h\rangle$, where D is the covariant differentiation of $(M, \langle\, ,\rangle)$. By definition,

$$D_{jih}+D_{jhi}=0, \quad H_{jih}=H_{jhi} \quad \text{and} \quad H_{jih}=D_{jih}(a_i-a_h).$$

Hence we have

$$\langle R(X_k, X_j)X_i, X_h\rangle = ca_i a_h(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) ,$$
$$(1/c)\langle (D_{X_m}R)(X_k, X_j)X_i, X_h\rangle$$
$$= a_i\delta_{ji}D_{mkh}(a_k - a_h) + a_h\delta_{kh}D_{mji}(a_j - a_i) - a_h\delta_{jh}D_{mki}(a_k - a_i) - a_i\delta_{ki}D_{mjh}(a_j - a_h) .$$

The second Bianchi identify is expressed as

$$\begin{aligned} &a_i\delta_{ji}\{D_{mkh}(a_k - a_h) - D_{kmh}(a_m - a_h)\} + a_h\delta_{kh}\{D_{mji}(a_j - a_i) - D_{jmi}(a_m - a_i)\} \\ &+ a_i\delta_{mi}\{D_{kjh}(a_j - a_h) - D_{jkh}(a_k - a_h)\} + a_h\delta_{jh}\{D_{kmi}(a_m - a_i) - D_{mki}(a_k - a_i)\} \\ &+ a_h\delta_{mh}\{D_{jki}(a_k - a_i) - D_{kji}(a_j - a_i)\} + a_i\delta_{ki}\{D_{jmh}(a_m - a_h) - D_{mjh}(a_j - a_h)\} = 0 . \end{aligned}$$

When $i = j \neq k = h \neq m \neq i = j$, the last identity reduces to

$$a_i D_{hhm}(a_m - a_h) + a_h D_{iim}(a_m - a_i) = 0 .$$

Furthermore, if $a_m = a_i \neq a_h$, then

$$(*) \qquad D_{hhm} = D_{hmh} = 0 .$$

When $m \neq j = i$, $k \neq j = i$, $h \neq j = i$, the second Bianchi identity reduces to

$$a_i\{D_{mkh}(a_k - a_h) - D_{kmh}(a_m - a_h)\} - a_h\delta_{kh}D_{imi}(a_m - a_i) + a_h\delta_{mh}D_{iki}(a_k - a_i) = 0 .$$

But the last two terms vanish by $(*)$. Hence

$$D_{mkh}(a_k - a_h) - D_{kmh}(a_m - a_h) = 0 .$$

Furthermore, if $a_m = a_h \neq a_k$, then

$$(**) \qquad D_{mkh} = D_{mhk} = 0 .$$

It is easy to see that $(*)$ and $(**)$ imply $DH = 0$ and hence the two complementary distributions defined by the eigenspaces of H are parallel. Hence, if $HX = -X$ and $HY = Y$, then $R(X, Y) = 0$, a contradiction.

REMARK. Let $(M, \langle\ ,\ \rangle)$ be an n-dimensional sphere of curvature 1/2. Then, $SO(M)$ is the Lie group $SO(n+1)$ and the metric $\langle\ ,\ \rangle$ of $SO(M)$ is a bi-invariant metric of $SO(n+1)$. The last assertion follows from the fact that, in this case,

$$\begin{aligned} [A^*, B(\xi)] &= B(A\xi) \\ [A^*, C^*] &= [A, C]^* \\ [B(\xi), B(\eta)] &= -(1/2)(\xi \wedge \eta)^* \end{aligned}$$

for all $A, C \in \mathfrak{o}(n)$ and $\xi, \eta \in \boldsymbol{R}^n$. Then, $B(\xi)$ is a horizontal Killing vector field for any $\xi \in \boldsymbol{R}^n$ which is not fibre preserving if $\xi \neq 0$.

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