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FINITENESS OF A COHOMOLOGY ASSOCIATED WITH CERTAIN JACKSON INTEGRALS

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Abstract. A structure theorem on q-analogues of b-functions is stated. Basic properties for Jackson integrals of associated q-multiplicative functions are given. Finiteness of cohomology group attached to them is proved for arrangement of A-type root system. Some problems about the derived q-difference systems are posed. An example of basic hypergeometric functions are given.

1. Let $E_n := E^n$ be the direct product of n copies of an elliptic curve E of modulus $q=e^{2\pi\sqrt{-1}\tau}$ for $\mathrm{Im}\,\tau>0$. The first cohomology group $H^1(E_n, \boldsymbol{C})$ has the Hodge decomposition $H^1(E_n, \boldsymbol{C})=H^{1,0}(E_n)+H^{0,1}(E_n)$, where $H^{1,0}(E_n)$ is isomorphic to the direct sum of n copies of $H^{1,0}(E)$, the space of holomorphic 1-forms on E. Let $\{\mathfrak{z}, \cdots, \mathfrak{z}_n; \mathfrak{z}_{n+1}, \cdots, \mathfrak{z}_{2n}\}$ be a basis of the first homology group $H_1(E_n, \boldsymbol{Z})$ such that each pair $\{\mathfrak{z}_j, \mathfrak{z}_{n+j}\}$ represents a pair of canonical loops in E. There exists a system of holomorphic 1-forms $\theta_1, \cdots, \theta_n$ on E_n such that

$$(1.1)\qquad \int_{\mathfrak{z}_j} \theta_k = 2\pi\sqrt{-1}\,\delta_{j,k}$$

$$\int_{\mathfrak{z}_{n+j}} \theta_k = 2\pi\sqrt{-1}\,\tau\delta_{j,k}\,, \qquad \mathrm{Im}\quad \tau>0\,.$$

We denote by $\bar{X}$ the factor space of the dual $H^{1,0}(E_n)^*$ of $H^{1,0}(E_n)$ with respect to the abelian subgroup $A=\langle \mathfrak{z}_1, \cdots, \mathfrak{z}_n\rangle$ of $H_1(E_n, \boldsymbol{Z})$ generated by $\mathfrak{z}_j$, $1\leq j\leq n$. This is possible because $H_1(E_n, \boldsymbol{Z})$ can be contained in $H^{1,0}(E_n, \boldsymbol{C})^*$. In the same way we denote by X the factor space $H_1(E_n, \boldsymbol{Z})/A$. X can be assumed to be a submodule of $\bar{X}$ and has a basis $\chi_j=\mathfrak{z}_{n+j} \bmod A$. An arbitrary $\chi\in X$ is written uniquely as

$$(1.2)\qquad \chi=\sum_{j=1}^{n} \nu_j\chi_j \qquad \text{for} \quad \nu_j\in \boldsymbol{Z}\,.$$

The quotient $\bar{X}/X$ is canonically isomorphic to E_n. By the map

$$(1.3)\qquad \bar{X}\ni\omega\mapsto x=(x_1=\exp((\theta_1, \omega)), \cdots, x_n=\exp((\theta_n, \omega)))\in(\boldsymbol{C}^*)^n$$

for $\omega\in\bar{X}$, $\bar{X}$ is isomorphic to the algebraic torus $q^{\bar{X}}=(\boldsymbol{C}^*)^n$ and X is isomorphic to the discrete subgroup q^X generated by $q^{\chi_1}=(q, 1, \cdots, 1), \cdots, q^{\chi_n}=(1, 1, \cdots, q)$. Here (θ, ω) denotes the canonical bilinear form on $H^{1,0}(E_n, \boldsymbol{C})$ and its dual.

We denote by $R(\bar{X})$ the field of rational functions on $q^{\bar{X}}$ and by $R^\times(\bar{X})$ the multiplicative group $R(\bar{X})-\{0\}$. Then X acts on $\bar{X}$ and also on $R(\bar{X})$ or $R^{\times}(\bar{X})$ in a natural manner. We denote these operations by $\hat{Q}_j$ and Q_j as follows:

$$\hat{Q}_j(x_1, \cdots, x_j, \cdots, x_n) \mapsto (x_1, \cdots, x_{j-1}, qx_j, x_{j+1}, \cdots, x_n) \tag{1.4}$$

$$Q_j\varphi(x)=\varphi(\hat{Q}_j(x)), \tag{1.5}$$

for $x=(x_1, \cdots, x_n) \in q^{\bar{X}}$ and $\varphi \in R(\bar{X})$, respectively.

A cocycle $b_\chi(\omega)$ on X with values in $R^{\times}(\bar{X})$ is defined by the cocycle condition

$$b_{\chi+\chi'}(\omega)=b_\chi(\omega)\cdot b_{\chi'}(\omega+\chi) \tag{1.6}$$

for any $\chi, \chi' \in X$ and $\omega \in \bar{X}$. A coboundary $b_\chi(\omega)$ is defined as $\varphi(\omega+\chi)/\varphi(\omega)$ for a certain $\varphi \in R^{\times}(\bar{X})$. The quotient space of the space $Z^1(X, R^{\times}(\bar{X}))$ of all cocycles with respect to the space $B^1(X, R^{\times}(\bar{X}))$ of all coboundaries defines the first cohomology group of X with values in $R^{\times}(\bar{X})$:

$$H^1(X, R^{\times}(\bar{X})) \simeq Z^1(X, R^{\times}(\bar{X}))/B^1(X, R^{\times}(\bar{X})). \tag{1.7}$$

$H^1(X, R^{\times}(\bar{X}))$ has a multiplicative group structure.

An arbitrary element $\mu \in \mathrm{Hom}(X, \boldsymbol{Z})$ can be uniquely extended to $\bar{\mu} \in \mathrm{Hom}_X(\bar{X}, \boldsymbol{C}/(\boldsymbol{Z}(2\pi\sqrt{-1}\,\tau)^{-1}))$ and to $q^{\mu} \in \mathrm{Hom}(\bar{X}, \boldsymbol{C}^*)$ by

$$\bar{\mu}\left(\sum_{j=1}^{n} \omega_j\chi_j\right)=\sum_{j=1}^{n} \omega_j\mu(\chi_j), \qquad \omega_j \in \boldsymbol{C}. \tag{1.8}$$

Then the following important result holds.

PROPOSITION 1. *$H^1(X, R^{\times}(\bar{X}))$ is represented by cocycles of the following form*:

$$b_\chi(\omega)=a_\chi \prod_{\nu=0}^{\mu_0(\chi)-1} q^{\bar{\mu}_0(\omega)+\nu}\cdot \prod_{i=1}^{k} \{(q^{\gamma_i+\bar{\mu}_i(\omega)})_{\mu_i(\chi)}\}^{\pm 1} \tag{1.9}$$

for $\mu_0, \mu_i \in \mathrm{Hom}(X, \boldsymbol{Z})$ and $\gamma_i \in \boldsymbol{C}$. Here $(a_\chi)_{\chi\in X}$ denotes an element of $\mathrm{Hom}(X, \boldsymbol{C}^*)$. *$(a)_n$ means $\prod_{j=0}^{n-1}(1-aq^j)$ or $\prod_{j=1}^{-n}(1-aq^{-j})^{-1}$ according as $n \geq 0$ or $n<0$. The expression* (1.9) *is not unique.*

This result is a q-analogue of a result of M. Sato which was proved as early as in 1970. He called the functions $b_\chi(\omega)$ "b-functions" and made use of them for the theory of prehomogeneous spaces and classical hypergeometric functions of Mellin-Ore type (see [S1], [S2] and also the classical papers [B] and [O2]).

The proof can be carried out in a way completely parallel to his. (See [S2] for the English version recently elaborated by M. Muro from Sato-Shintani's original [S1].)

We denote by $\Theta(t)$ the theta function on $\boldsymbol{C}^*$ defined as the triple product $\Theta(t)=(t)_\infty(q/t)_\infty(q)_\infty$ where $(t)_\infty=\prod_{n=0}^{\infty}(1-tq^n)$. This is a meromorphic function on $\boldsymbol{C}^*$.

DEFINITION 1. A function φ on $\bar{X}$ is said to be quasi-meromorphic if there exist $\rho_1, \cdots, \rho_n \in \boldsymbol{C}$ such that $\varphi x_1^{-\rho_1} \cdots x_n^{-\rho_n}$ is meromorphic on $q^{\bar{X}}$.

Since

$$Q_j q^{\alpha_1\omega_1+\cdots+\alpha_n\omega_n}=q^{\alpha_j}q^{\alpha_1\omega_1+\cdots+\alpha_n\omega_n}, \tag{1.10}$$

$$Q_j(q^{\bar{\mu}_i(\omega)+\beta_i})_\infty/(q^{\bar{\mu}_i(\omega)+\beta_i})_\infty=(1-q^{\bar{\mu}_i(\omega)+\beta_i})^{-1}_{\mu_i(\chi_j)}, \tag{1.11}$$

$$Q_j(\Theta(q^{\mu_0(\omega)+\beta_0}))=(-1)^{\mu_0(\chi_j)}q^{-\mu_0(\chi_j)(\mu_0(\omega)+\beta_0)}q^{-\mu_0(\chi_j)(\mu_0(\chi_j)-1)/2}\cdot\Theta(q^{\mu_0(\omega)+\beta_0}), \tag{1.12}$$

for $\alpha_1, \cdots, \alpha_n, \beta_0, \beta_1, \cdots, \beta_n \in \boldsymbol{C}$, we can solve the functional equation

$$\Phi(\omega+\chi)=b_\chi(\omega)\Phi(\omega) \tag{1.13}$$

in the space of quasi-meromorphic functions on $\bar{X}$:

PROPOSITION 2. *There exists a quasi-meromorphic function $\Phi(\omega)$ satisfying* (1.13). *The quotient $\Phi_1(\omega)/\Phi_2(\omega)$ of any two solutions $\Phi_1(\omega)$ and $\Phi_2(\omega)$ of* (1.13) *is doubly periodic on $q^{\bar{X}}$ and hence meromorphic on E_n.*

$\Phi(\omega)$ has an expression as follows:

$$x_1^{\alpha_1}\cdots x_n^{\alpha_n}\frac{\prod_{i=1}^{k'}(v_i' x^{\mu_i'})_\infty}{\prod_{i=1}^{k}(v_i x^{\mu_i})_\infty} \tag{1.14}$$

for some $\alpha_j \in \boldsymbol{C}$, $v_i, v_i' \in \boldsymbol{C}^*$ and $\mu_i, \mu_i' \in \mathrm{Hom}(X, \boldsymbol{Z})$, where x^{μ_i} and $x^{\mu_i'}$ denote $q^{\bar{\mu}_i(\omega)}$ and $q^{\bar{\mu}_i'(\omega)}$, respectively.

DEFINITION 2. A function $b_\chi(\omega)$ is called a b-function while a function $\Phi(\omega)$ of type (1.14) is called a q-multiplicative function.

2. u_j will denote q^{α_j}. For a function of u_j, v_i and v_i' we denote by $\tilde{Q}_j^{\pm1}$, $\tilde{Q}_{v_i}^{\pm1}$ and $\tilde{Q}_{v_i'}^{\pm1}$ the q-difference operators corresponding to the displacements $u_j\mapsto u_j q^{\pm1}$, $v_i\mapsto v_i q^{\pm1}$ and $v_i'\mapsto v_i' q^{\pm1}$, respectively. Then we have

$$\tilde{Q}_j^{\pm\nu}\Phi=x_j^{\pm\nu}\Phi, \quad \tilde{Q}_{v_i}^{\pm\nu}\Phi=(v_i x^{\mu_i})_\nu^{\pm1}\Phi, \quad \tilde{Q}_{v_i'}^{\pm\nu}\Phi=(v_i' x^{\mu_i'})_\nu^{\mp1}, \tag{2.1}$$

respectively. Consider the operator algebra $\mathscr{A}$ over $\boldsymbol{C}$ generated by $\tilde{Q}_j^{\pm1}$, $\tilde{Q}_{v_i}^{\pm1}$ and $\tilde{Q}_{v_i'}^{\pm1}$ for all i, j. $\mathscr{A}$ acts on $R(\bar{X})$. We denote by V the subspace of $R(\bar{X})$ spanned by $(\kappa\cdot\Phi)/\Phi$ for all $\kappa\in\mathscr{A}$. Then $\Phi\cdot V$ is the smallest $\mathscr{A}$-module in $\Phi\cdot R(\bar{X})$ containing Φ.

For an arbitrary point $\xi=(\xi_1, \cdots, \xi_n)$ of $q^{\bar{X}}$ the X-orbit $X\cdot\xi$

$$X\cdot\xi=\{(q^{\nu_1}\xi_1, \cdots, q^{\nu_n}\xi_n)\,|\,\nu_1, \cdots, \nu_n\in\boldsymbol{Z}\} \tag{2.2}$$

will be denoted by $[0, \xi\infty]_q$ and called an n-dimensional "q-cycle". This terminology may be justified by the following.

DEFINITION 3. The Jackson integral of a function on $q^{\bar{X}}$ over the q-cycle $[0, \xi\infty]_q$

$$\tilde{f}=\int_{[0,\xi\infty]_q} f(x_1,\cdots,x_n)\cdot\Omega \tag{2.3}$$

for $\Omega=(d_qx_1/x_1)\wedge\cdots\wedge(d_qx_n/x_n)$ is defined to be the sum

$$(1-q)^n\sum_{-\infty<\nu_1,\cdots,\nu_n<\infty} f(q^{\nu_1}\xi_1,\cdots,q^{\nu_n}\xi_n) \tag{2.4}$$

if it exists.

It is obvious that

$$\int_{[0,\xi\infty]_q} Q_jf\cdot\Omega=\int_{[0,\xi\infty]_q} f\cdot\Omega\,, \tag{2.5}$$

for each j, and hence

$$\int_{[0,\xi\infty]_q} Q^\chi f\cdot\Omega=\int_{[0,\xi\infty]_q} f\cdot\Omega\,, \tag{2.6}$$

for $Q^\chi=Q_1^{\nu_1}\cdots Q_n^{\nu_n}$.

We are particularly interested in the Jackson integral for Φ:

$$\tilde{\Phi}=\int_{[0,\xi\infty]_q}\Phi\cdot\Omega\,, \tag{2.7}$$

which depends analytically on α_j, v_i, v_i' and ξ.

If Φ has a pole at a point of $[0,\xi\infty]_q$ then (2.7) does not make sense. In this case the q-cycle $[0,\xi\infty]_q$ should be regularized as follows.

First we note:

LEMMA 2.1. *For each i, the function*

$$U_i(\omega)=q^{\mu_i(\omega)^2/2}x_1^{\rho_1}\cdots x_n^{\rho_n}\Theta(v_ix^{\mu_i}) \tag{2.8}$$

is invariant under the displacements $Q_1,\cdots,Q_n$, *where* q^{ρ_j} *denotes* $(-1)^{\mu_i(\chi_j)}\cdot v_i^{\mu_i(\chi_j)}\cdot q^{-\mu_i(\chi_j)/2}$.

PROOF. This follows from (1.12) and the formula $q^{\mu_i(\omega+\chi_j)^2/2}=q^{\mu_i(\omega)^2/2+\mu_i(\chi_j)\mu_i(\omega)+\mu_i(\chi_j)^2/2}$.

Suppose a factor $(v_ix^{\mu_i})_\infty$ of the denominator vanishes at a point of $[0,\xi\infty]_q$ so that Φ has a pole at a point of $[0,\xi\infty]_q$. Since $\Theta(v_ix^{\mu_i})=(v_ix^{\mu_i})_\infty(qv_i^{-1}x^{-\mu_i})_\infty(q)_\infty$, $\Phi U_i(x)$ no longer has the factor $(v_ix^{\mu_i})_\infty$ in the denominator. Moreover it satisfies the same system of difference equations (1.13) as Φ. In this way, the integral $\tilde{\Phi}$ may be replaced by $\widetilde{\Phi U_i}$ so that the zeros of $(v_ix^{\mu_i})_\infty$ are avoided.

This regularization is equivalent to taking the residues of Φ at each pole lying in $[0,\xi\infty]_q$. We call this procedure *the regularization of integration* and the corresponding cycle *the regularized cycle* of $[0,\xi\infty]_q$ which will be denoted by $\mathrm{reg}\,[0,\xi\infty]_q$.

By substitution of integration $x_j \mapsto x_j q$ $(1 \leq j \leq n)$ and by (2.5), we have a formal system of q-difference equations:

$$\prod_{i=1}^{k} (v_i' \tilde{Q}_1^{\mu_i'(\chi_1)} \cdots \tilde{Q}_n^{\mu_i'(\chi_n)})_{\mu_i'(\chi_j)} \tilde{\Phi} = \prod_{i=1}^{k} (v_i \tilde{Q}_1^{\mu_i(\chi_1)} \cdots \tilde{Q}_n^{\mu_i(\chi_n)})_{\mu_i(\chi_j)} u_j^{-1} \tilde{\Phi} \tag{2.9}$$

for each j, $1 \leq j \leq n$ and

$$\tilde{Q}_{v_i}^{\pm 1} \tilde{\Phi} = (1 - v_i \tilde{Q}_1^{\mu_i(\chi_1)} \cdots \tilde{Q}_n^{\mu_i(\chi_n)})^{\pm 1} \tilde{\Phi} \tag{2.10}$$

$$\tilde{Q}_{v_i'}^{\pm 1} \tilde{\Phi} = (1 - v_i' \tilde{Q}_1^{\mu_i'(\chi_1)} \cdots \tilde{Q}_n^{\mu_i'(\chi_n)})^{\mp 1} \tilde{\Phi} . \tag{2.11}$$

One may naturally ask the following questions:

QUESTION 1. *Do* (2.9)–(2.11) *really define a holonomic q-difference system in the variables u_j, v_j and v_j' in the sense of* [A4]? *Namely, do there exist a finite number of elements $\kappa_1, \cdots, \kappa_m$ of $\mathscr{A}$ such that $\mathscr{A} \cdot \tilde{\Phi}$ is contained in the linear space spanned by $\kappa_1 \tilde{\Phi}, \cdots, \kappa_m \tilde{\Phi}$ over $R(\bar{X})$? Or equivalently, does there exist $f_1, \cdots, f_m \in R(\bar{X})$ such that*

$$\kappa \tilde{\Phi} = \sum_{j=1}^{m} f_j \kappa_j \tilde{\Phi} \tag{2.12}$$

for every $\kappa \in \mathscr{A}$? If this is the case, then what is the rank of the system (2.9)–(2.11), *which is defined to be the minimal number among such m?*

For $f = \Phi \cdot \varphi$, $\varphi \in V$, we have:

$$\int_{[0,\xi\infty]_q} \Phi(\omega) \varphi(\omega) \cdot \Omega = \int_{[0,\xi\infty]_q} \Phi(\omega) \cdot b_\chi(\omega) \cdot Q^\chi \varphi(\omega) \cdot \Omega \tag{2.13}$$

because Ω is invariant under the operation Q^χ, i.e.,

$$\int_{[0,\xi\infty]_q} \Phi(\omega)(\varphi(\omega) - b_\chi(\omega) \cdot Q^\chi \varphi(\omega)) \cdot \Omega = 0 . \tag{2.14}$$

This suggests us to consider the residual space

$$V / \left\{ \sum_{\chi \in X} (1 - b_\chi(\omega) Q^\chi) V \right\} \simeq V / \left\{ \sum_{j=1}^{n} (1 - b_{\chi_j}(\omega) Q_j) V \right\} . \tag{2.15}$$

This can be regarded as a q-analogue of the twisted de Rham cohomology group (see [A3]). We shall denote it by $H_\Phi(V, d_q)$ and call it "the q-twisted cohomology group" associated with Φ.

QUESTION 2. *Is $H_\Phi(V, d_q)$ finite dimensional? If so, how can its dimension be determined? How can one find out a basis of $H_\Phi(V, d_q)$?*

QUESTION 3. *What is the dual space of $H_\Phi(V, d_q)$? Is it represented by special kinds of q-cycles? By what kind of q-cycles?*

QUESTION 4. *Find out asymptotic solutions for* $\tilde{\Phi}$ *for* $\alpha_j \to \pm\infty$ *and* $v_i, v'_i \to \pm\infty$. *Classify all different kinds of asymptotics for* $\tilde{\Phi}$.

We do not have any complete answer to these questions. We shall only give a few examples in the next four sections.

3. $n=1$, **q-analogue of Jordan-Pochhammer case.** A multiplicative function Φ can be written as

$$\Phi = t^{\alpha} \prod_{j=1}^{m} \frac{(t/x_j)_\infty}{(tq^{\beta_j}/x_j)_\infty} \tag{3.1}$$

for $u=q^{\alpha}$, q^{β_j} and $x_j \in \boldsymbol{C}^*$. The integral over a suitable q-cycle

$$\tilde{\Phi} = \int \Phi \frac{d_q t}{t} \tag{3.2}$$

is a q-analogue of Jordan-Pochhammer integral. We put $\tilde{Q}_u = \tilde{Q}$ and $\tilde{Q}_{x_j} = \tilde{Q}_j$. Then the system (2.9)–(2.11) becomes

$$\prod_{j=1}^{m} \left(1 - \frac{q^{\beta_j}}{x_j} \tilde{Q}\right) \tilde{\Phi} = \prod_{j=1}^{m} \left(1 - \frac{1}{x_j} \tilde{Q}\right) u^{-1} \tilde{\Phi}, \tag{3.3}$$

$$\tilde{Q}_j \tilde{\Phi} = \frac{1 - \dfrac{1}{qx_j} \tilde{Q}}{1 - \dfrac{q^{\beta_j - 1}}{x_j} \tilde{Q}} \tilde{\Phi}, \qquad \tilde{Q}_j^{-1} \tilde{\Phi} = \frac{1 - \dfrac{q^{\beta_j}}{x_j} \tilde{Q}}{1 - \dfrac{1}{x_j} \tilde{Q}} \tilde{\Phi}, \tag{3.4}$$

$$\tilde{Q}_{\beta_j} \tilde{\Phi} = \left(1 - \frac{q^{\beta_j}}{x_j} \tilde{Q}\right) \tilde{\Phi}, \qquad \tilde{Q}_{\beta_j}^{-1} \Phi = \left(1 - \frac{q^{\beta_j - 1}}{x_j} \tilde{Q}\right)^{-1} \tilde{\Phi}. \tag{3.5}$$

$H_\Phi(V, d_q)$ is spanned by a basis consisting of $\varphi_j = (1 - t/x_j)^{-1}$ for $1 \le j \le m$. Hence $\dim H_\Phi(V, d_q) = m$. We denote by $\langle \varphi \rangle$ the integral of $\Phi\varphi$ and put $\langle \Phi \rangle = \tilde{\Phi}$. Then we have

$$\tilde{Q}^{\pm 1}(\langle \varphi_1 \rangle, \cdots, \langle \varphi_m \rangle) = (\langle \varphi_1 \rangle, \cdots, \langle \varphi_m \rangle) A_{\pm}, \tag{3.6}$$

$$\tilde{Q}_j^{\pm 1}(\langle \varphi_1 \rangle, \cdots, \langle \varphi_m \rangle) = (\langle \varphi_1 \rangle, \cdots, \langle \varphi_m \rangle) A_{\pm j}, \tag{3.7}$$

$$\tilde{Q}_{\beta_j}^{\pm 1}(\langle \varphi_1 \rangle, \cdots, \langle \varphi_m \rangle) = (\langle \varphi_1 \rangle, \cdots, \langle \varphi_m \rangle) A_{\pm \beta_j} \tag{3.8}$$

respectively, where $A_{\pm} = ((a_{\pm;k,l}))$, $A_{\pm j} = ((a_{\pm j;k,l}))$, $A_{\pm\beta_j} = ((a_{\pm\beta_j;k,l}))$ denote matrices whose entries are rational functions in u_j, x_j and q^{β_j}. More explicitly:

PROPOSITION 3. *Suppose* x_i/x_j *and* $x_i q^{\beta_j}/x_j$ *are different from* 1, $q^{\pm 1}, q^{\pm 2}, \cdots$ *for each pair* i, j *such that* $i \neq j$. *Then*

(i) $$a_{\beta_r;i,j}=\frac{x_j}{x_r}q^{\beta_r}f_i(x)+\delta_{i,j}\left(1-\frac{x_j}{x_r}q^{\beta_r}\right),$$

(ii) $$a_{+;i,j}=-x_jf_i(x)+x_j\delta_{i,j},$$

(iii) $$a_{r;i,j}=q^{\alpha}\frac{(1-q^{\beta_r})\prod_{\substack{1\le l\le m\\ l\ne r}}\left(1-\frac{x_i}{x_l}q^{\beta_l}\right)}{\left(q\frac{x_r}{x_j}-q^{\beta_r}\right)\prod_{\substack{1\le l\le m\\ l\ne i}}\left(1-\frac{x_i}{x_l}\right)}+\delta_{i,j}\frac{1-\frac{x_i}{qx_r}}{1-\frac{x_i}{x_r}q^{\beta_r-1}},\qquad (r\ne j),$$

$$=q^{\alpha}\frac{\prod_{\substack{1\le l\le m\\ l\ne r}}\left(1-\frac{x_i}{x_l}q^{\beta_l}\right)}{\prod_{\substack{1\le l\le m\\ l\ne i}}\left(1-\frac{x_i}{x_l}\right)},\qquad (j=r),$$

where $f_i(x)$ denotes the rational function

(3.9) $$f_i(x)=\frac{q^{\alpha}(1-q^{\beta_i})}{1-q^{\alpha+\beta_1+\cdots+\beta_m}}\prod_{\substack{1\le l\le m\\ l\ne i}}\frac{\left(1-q^{\beta_l}\frac{x_i}{x_l}\right)}{\left(1-\frac{x_i}{x_l}\right)}.$$

Hence for any $\varphi\in V$ the integral $\langle\varphi\rangle$ is a linear combination of $\langle\varphi_1\rangle,\cdots,\langle\varphi_m\rangle$ over the rational function fields in u, q^{β_j}, x_j. In particular

(3.10) $$\tilde{\Phi}=\sum_{i=1}^{m}f_i(x)\langle\varphi_i\rangle.$$

By substitution $t=x_jq$ in (3.2), the integral of Φ over $[0,x_j\infty]_q$ gives the asymptotic of $\tilde{\Phi}$ for $u\to 0$ ($\alpha\to+\infty$):

(3.11) $$\tilde{\Phi}\sim(1-q)(qx_j)^{\alpha}\prod_{k=1}^{m}\frac{(qx_j/x_k)_\infty}{(q^{\beta_k+1}x_j/x_k)_\infty}$$

since in this case the sum (2.3) runs over only the set $[0,x_j]_q=\{x_jq^{\nu};\ \nu=1,2,3,\cdots\}$. There exist exactly n such asymptotics which correspond to m linearly independent solutions of (3.3). Mimachi [M2] has solved the connection problem attached to these asymptotics.

4. Basic Lemmas and Main Theorem. From now on, *we take as Φ the following function which is attached to the arrangement of A-type root system* (see [A6] for polynomial versions):

$$\Phi = t_1^{\alpha_1} \cdots t_n^{\alpha_n} \prod_{0 \le i \le j \le n} \frac{\left(q^{\beta'_{i,j}} \dfrac{t_j}{t_i}\right)_\infty}{\left(q^{\beta_{i,j}} \dfrac{t_j}{t_i}\right)_\infty}, \tag{4.1}$$

where we let $t_0 = 1$. We consider the integral

$$\tilde{\Phi} = \int \Phi \frac{d_q t_1}{t_1} \wedge \cdots \wedge \frac{d_q t_n}{t_n} \tag{4.2}$$

over a suitable q-cycle. It is a function depending on $u_j = q^{\alpha_j}$, $\beta_{i,j}$, $\beta'_{i,j}$.

Because of symmetry it is convenient to put $\beta'_{j,i} = 1 - \beta_{i,j}$ and $\beta_{j,i} = 1 - \beta'_{i,j}$. We may put $\beta'_{0,j} = 0$.

Many authors have investigated basic hypergeometric functions as generalizations of Heine's hypergeometric function. Except in one variable case, these seem to be included in the set of functions $\tilde{\Phi}$ of type (4.2) *provided that they are not confluent*. In fact, $\tilde{\Phi}$ is an extension of classical Barnes type integrals found, for example, in [S3] and [G1]. The Milne's hypergeometric functions (see [M1]) are similar to our $\tilde{\Phi}$, although they have additional parameters. For the case $q = 1$, see also [G2] and [G3], which study Barnes integrals from the view point of Grassmannian geometry. It is not certain whether our approach is connected with Grassmannian geometry or not.

Assume the following conditions:

($\mathscr{H}$-1) *For arbitrary arguments* $i_0, i_1, \cdots, i_r$, $0 \le i_\nu \le n$, *which are different from each other,*

$$\beta_{i_0, i_1} + \beta_{i_1, i_2} + \cdots + \beta_{i_r, i_0} \notin \boldsymbol{Z}, \tag{4.3}$$

$$\alpha_{i_0} + \alpha_{i_1} + \cdots + \alpha_{i_r} \notin \boldsymbol{Z}. \tag{4.4}$$

($\mathscr{H}$-2) $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are all sufficiently large numbers.*

($\mathscr{H}$-3) *For an arbitrary partition* $\{0, 1, \cdots, n\} = S_1 + S_2$ *such that* $0 \in V(S_1)$,

$$\sum_{j \in V(S_2)} \alpha_j + \sum_{i \in V(S_1), j \in V(S_2)} (\beta_{i,j} - \beta'_{i,j}) \notin \boldsymbol{Z}. \tag{4.5}$$

We denote by $\tilde{Q}_j^{\pm 1}$ the operations $u_j \mapsto u_j q^{\pm 1}$ for functions of $u = (u_1, \cdots, u_n) = q^{\alpha_1}, \cdots, q^{\alpha_n})$ by the displacements of the j-th coordinate u_j. Then the q-difference equations for $\tilde{\Phi}$ in the variables u are given by

$$\prod_{\substack{j=1 \\ j \ne r}}^{n} (\tilde{Q}_j - q^{\beta'_{i,j}} \tilde{Q}_r) u_r^{-1} \tilde{\Phi} = \prod_{\substack{j=1 \\ j \ne r}}^{n} (\tilde{Q}_j - q^{\beta_{j,r}} \tilde{Q}_r) \tilde{\Phi}. \tag{4.6}$$

$$\tilde{Q}_{\beta'_{i,j}} \tilde{\Phi} = (\tilde{Q}_i - q^{\beta'_{i,j}} \tilde{Q}_j)^{-1} \tilde{Q}_i \tilde{\Phi}, \tag{4.7}$$

$$\tilde{Q}^{-1}_{\beta'_{i,j}}\tilde{\Phi}=(\tilde{Q}_i-q^{\beta'_{i,j}-1}\tilde{Q}_j)\tilde{Q}_i^{-1}\tilde{\Phi}\,, \tag{4.8}$$

$$\tilde{Q}_{\beta_{i,j}}\tilde{\Phi}=(\tilde{Q}_i-q^{\beta_{i,j}}\tilde{Q}_j)\tilde{Q}_i^{-1}\tilde{\Phi}\,, \tag{4.9}$$

$$\tilde{Q}^{-1}_{\beta_{i,j}}\tilde{\Phi}=(\tilde{Q}_i-q^{\beta_{i,j}-1}\tilde{Q}_j)^{-1}\tilde{Q}_i\tilde{\Phi}\,, \tag{4.10}$$

where $Q^{\pm 1}_{\beta_{i,j}}$, $Q^{\pm}_{\beta'_{i,j}}$ and $\tilde{Q}^{\pm 1}_{\beta_{i,j}}$, $\tilde{Q}^{\pm 1}_{\beta'_{i,j}}$ are the operations on V and $\tilde{\Phi}\cdot V$ respectively induced by the displacements $\beta_{i,j}\to\beta_{i,j}\pm 1$ and $\beta'_{i,j}\to\beta'_{i,j}\pm 1$. Note that

$$\tilde{Q}^{\pm 1}_{\beta_{i,j}}\langle\varphi\rangle=\langle W^{(\pm)}_{i,j}Q^{\pm 1}_{\beta_{i,j}}\varphi\rangle \tag{4.11}$$

$$\tilde{Q}^{\pm 1}_{\beta'_{i,j}}\langle\varphi\rangle=\langle W'^{(\pm)}_{i,j}Q^{\pm 1}_{\beta_{i,j}}\varphi\rangle \tag{4.12}$$

for $W^{(\pm)}_{i,j}=(Q^{\pm 1}_{\beta_{i,j}}\Phi)/\Phi$ and $W'^{(\pm)}_{i,j}=(Q^{\pm 1}_{\beta'_{i,j}}\Phi)/\Phi$, respectively.

$W^{(\pm)}_{i,j}Q^{\pm 1}_{\beta_{i,j}}$ and $W'^{(\pm)}_{i,j}\tilde{Q}^{\pm 1}_{\beta'_{i,j}}$ are nothing but a *q-analogue of the covariant differentiations.*

Our main result states that this system of q-difference equations is actually *holonomic* and has rank $(n+1)^{n-1}$. This can be shown by the aid of some results in elementary graph theory. Before stating our Theorem, we need a few preliminary lemmas.

We denote linear functions of $t_0=1, t_1, \cdots, t_n$, $t_i-q^{\beta'_{i,j}}t_j$, and $t_i-q^{\beta_{i,j}}t_j$ by $(i,j)_+$ and $(i,j)_-$ respectively. A rational function $\varphi=(i_1,j_1)^{-1}_{\varepsilon_1}\cdots(i_r,j_r)^{-1}_{\varepsilon_r}$ for each $\varepsilon_\nu=\pm 1$ defines a graph $G=G_\varphi$ with directed edges $\overline{i_\nu, j_\nu}$ and the set of vertices $\{i_1, j_1, \cdots, i_r, j_r\}$. The edge $\overline{i_\nu, j_\nu}$ is directed from i_ν to j_ν, i.e., $i_\nu\to j_\nu$ or from j_ν to i_ν, i.e., $j_\nu\to i_\nu$ according as $\varepsilon_\nu=+1$ or -1. We denote by $\Delta_G=\prod_{\nu=1}^r(i_\nu, j_\nu)_{\varepsilon_\nu}$, the product of all factors $(i_1,j_1)_{\varepsilon_1}, \cdots, (i_r,j_r)_{\varepsilon_r}$. For an oriented graph Γ we denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges of Γ, respectively. To each edge e of $E(\Gamma)$ there corresponds a unique linear function $(e)=(i,j)_\varepsilon$ for $\varepsilon=-1$ or 1.

DEFINITION 4. Γ is said to be a spanning graph if $V(\Gamma)$ contains all the vertices $\{0, 1, \cdots, n\}$. A forest is a graph without any circuit. A spanning forest F is admissible if and only if the number of edges $|E(F)|$ equals n, i.e., F is a tree. A spanning forest F is said to be subadmissible if $|E(F)|=n-1$. In this case F is a semi-tree, i.e., a disjoint union $F=F_1+F_2$ of only two trees F_1 and F_2 such that $V(F_1)$ contains the root 0 and $V(F_2)$ is disjoint from $\{0\}$ (see [T]).

We denote by $\mathscr{F}_1$ and $\mathscr{F}_2$ the set of all admissible trees and that of all admissible semi-trees, respectively. The evaluation of (e) for $e\in E(\Gamma)$ at some point $t\in q^{\bar{X}}$ will be denoted by $\langle(e), t\rangle$. When Γ is a tree such that $0\in V(\Gamma)$, we denote by $p(j)$ the predecessor of a vertex j of Γ, i.e., the vertex of Γ lying in the path connecting 0 and j such that $\mathrm{dis}(\{p(j)\}, \{0\})=\mathrm{dis}(\{j\}, \{0\})-1$, where dis means the distance between two vertices in the graph Γ.

LEMMA 4.1. *For an arbitrary admissible tree T the equations*

$$\langle(e), t\rangle=0\,, \qquad e\in E(T)\,, \tag{4.13}$$

have a unique solution.

PROOF. Indeed t_j can be uniquely solved by induction on dis($\{0\}$, $\{j\}$). If $j=0$, then $t_j=t_0=1$. Suppose that dis($\{0\}$, $\{j\}$)$=N$ and that all t_k for dis$(0, k)<N$ are already solved. Then t_j is uniquely solved by one of the above equations $(p(j), j)_+=0$ or $(p(j), j)_-=0$.

LEMMA 4.2. *For an arbitrary connected spanning graph Γ containing a circuit, we have a unique partial fraction expansion*

$$\frac{1}{\Delta_\Gamma}=\sum_{e\in E(\Gamma)}\frac{1}{\Delta_{\Gamma_e}}\frac{1}{\langle e, \bar{t}\rangle} \tag{4.14}$$

where $\bar{t}$ is uniquely determined by the equations $\langle (e), \bar{t}\rangle=0$ for all $e\in E(\Gamma_e)$. Moreover each Γ_e is an admissible tree.

PROOF. Indeed, since Γ contains a circuit, the constant 1 is a linear combination of linear functions (e) for $e\in E(\Gamma)$:

$$1=\sum_{e\in E(\Gamma)} a_e(e)\,, \qquad \text{for} \quad a_e\in \boldsymbol{C}\,, \tag{4.15}$$

which is equivalent to (4.14) by division of both sides by Δ_Γ.

Let $\hat{\Gamma}$ be an oriented graph containing Γ, i.e., such that $E(\hat{\Gamma})\supset E(\Gamma)$. $\hat{\Gamma}-\Gamma$ denotes the subgraph complementary to Γ in $\hat{\Gamma}$, i.e., such that $E(\hat{\Gamma}-\Gamma)=E(\hat{\Gamma})-E(\Gamma)$. We put $\tilde{\Delta}_{\hat{\Gamma}-\Gamma}=\prod_{e\in E(\hat{\Gamma}-\Gamma)}(\tilde{e})$, where $(\tilde{e})$ denotes the linear function $(i, j)_{-\varepsilon}$ oppsite to $(e)=(i, j)_\varepsilon$, $\varepsilon=\pm 1$.

Then the following first basic lemma holds.

LEMMA 4.3. *Suppose that Γ is an admissible tree. Then*

$$\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\sum_{T\subset\hat{\Gamma}}\frac{c_T}{\Delta_T} \tag{4.16}$$

where T runs through all admissible spanning trees in $\hat{\Gamma}$. Each c_T is given by

$$c_T=\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_T)}{\Delta_{\hat{\Gamma}-T}(t_T)} \tag{4.17}$$

where $t_T=(t_{T,j})_{1\le j\le n}$ denotes the unique solution of the equations (4.13).

PROOF. We prove the lemma by induction on the number $N=|E(\hat{\Gamma}-\Gamma)|=|E(\hat{\Gamma})|-|E(\Gamma)|$. When $N=0$, then $\hat{\Gamma}$ coincides with Γ so there is nothing to prove. Suppose the lemma has been proved for $N\le M-1$. We must prove it for $N=M$. There exists at least one edge $e_0\in E(\hat{\Gamma}-\Gamma)$. Then there exists a circuit $\mathscr{C}$ in $\hat{\Gamma}$ such that $e_0\in E(\mathscr{C})$ and $E(\mathscr{C}_{e_0})\subset E(\Gamma)$. Then

(4.18)
$$\frac{(\tilde{e}_0)}{\Delta_{\mathscr{C}}}=\sum_{e\subset E(\mathscr{C})} a_e\frac{1}{\Delta_{\mathscr{C}_e}}\,.$$

A fortiori

(4.19)
$$\frac{(\tilde{e}_0)}{\Delta_{\Gamma}}=\sum_{e\in E(\mathscr{C})} a_e\frac{1}{\Delta_{\Gamma_e}}$$

since $(\tilde{e}_0)$ is a linear combination of $e\in E(\mathscr{C})$:

(4.20)
$$(\tilde{e}_0)=\sum_{e\in E(\mathscr{C})} a_e\cdot(e)\,.$$

Hence

(4.21)
$$\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\frac{\tilde{\Delta}_{\hat{\Gamma}_{e_0}-\Gamma}\cdot(\tilde{e}_0)}{\Delta_{\hat{\Gamma}}}=\sum_{e\in E(\mathscr{C})} a_e\frac{\tilde{\Delta}_{\hat{\Gamma}_{e_0}-\Gamma}}{\Delta_{\hat{\Gamma}_e}}\,.$$

First assume $e_0\neq e$, i.e., $e\in E(\Gamma)$. Since $\hat{\Gamma}_{e_0}-\Gamma=\hat{\Gamma}_e-(\Gamma_e\cup\{e_0\})$ and $|E(\hat{\Gamma}_e)-E(\Gamma_e\cup\{e_0\})|=|E(\hat{\Gamma}-\Gamma)|-1$, by the induction hypothesis we get a partial fraction

(4.22)
$$\frac{\tilde{\Delta}_{\hat{\Gamma}_{e_0}-\Gamma}}{\Delta_{\hat{\Gamma}_e}}=\sum_{T\subset\hat{\Gamma}_e} a_T^*\frac{1}{\Delta_T}$$

where T runs through all admissible spanning trees of $\hat{\Gamma}_e$. On the other hand if $e=e_0$, then $\hat{\Gamma}_{e_0}\supset\Gamma$ and we have again $|E(\hat{\Gamma}_{e_0}-\Gamma)|=|E(\hat{\Gamma}-\Gamma)|-1$. Hence by the induction hypothesis

(4.23)
$$\frac{\tilde{\Delta}_{\hat{\Gamma}_{e_0}-\Gamma}}{\Delta_{\hat{\Gamma}_{e_0}}}=\sum_{T\subset\hat{\Gamma}_{e_0}} a_T^*\frac{1}{\Delta_T}\,.$$

Summing up (4.22) and (4.23), we get

(4.24)
$$\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\Gamma}}=\sum_{e\in E(\mathscr{C})} a_e\frac{\tilde{\Delta}_{\hat{\Gamma}_{e_0}-\Gamma}}{\Delta_{\hat{\Gamma}_e}}=\sum_{e\in E(\mathscr{C})} a_e\sum_{T\subset\hat{\Gamma}_e} a_T^*\frac{1}{\Delta_T}\,.$$

Any admissible spanning tree of $\hat{\Gamma}_e$ being also an admissible tree, we have finally the formula (4.16). The expression of (4.16) is unique. Indeed by residue calculus on both sides of (4.16), c_T is equal to (4.17).

The second basic lemma is as follows:

LEMMA 4.4. *Let $\Gamma=\Gamma_1+\Gamma_2$ be a semi-tree such that $0\in V(\Gamma_1)$ and 0 is disjoint from $V(\Gamma_2)$. Let $\hat{\Gamma}$ be an admissible graph containing Γ. Then*

(4.25)
$$\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\sum_{T\in\mathscr{F}_1}\frac{a_T}{\Delta_T}+\sum_{F\in\mathscr{F}_2,F_1\subset\Gamma_1}\frac{b_F}{\Delta_F}$$

for

$$a_T=\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_T)}{\Delta_{\hat{\Gamma}-T}(t_T)}\quad and \quad b_F=\lim_{\lambda\to\infty}\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_F(\lambda))}{\Delta_{\hat{\Gamma}-F}(t_F(\lambda))}, \tag{4.26}$$

where $F=F_1+F_2$ *such that* $0\in V(F_1)$ *and where* $t_F(\lambda)$ *denotes a non-zero solution of the equations*

$$\langle (e), t\rangle=0 \qquad for\ any \quad e\in E(F). \tag{4.27}$$

This solution is not unique and can be written as $t=t_F(\lambda)=t_F^{(0)}+\lambda t_F^{(1)}$ *for an arbitrary parameter* $\lambda\in \boldsymbol{R}$. $t_F^{(0)}$ *and* $t_F^{(1)}$ *denote real constants.* $t_{F,j}=t_{F,j}^{(0)}$ *is unique for* $j\in F_1$ *and* $t_{F,j}^{(0)}=0$ *for* $j\in V(F_2)$. $t_{F,j}^{(1)}=0$ *for* $j\in V(F_1)$ *and* $t_{F,j}^{(1)}$, $j\in V(F_2)$, *differ from zero and are determined uniquely except for a scalar factor.*

PROOF. Choose an edge $(e_0)\in E(\hat{\Gamma})$ outside $E(\Gamma)$, such that $\Gamma\cup\{e_0\}$ is a spanning tree. Since $\hat{\Gamma}\supset\Gamma\cup\{e_0\}$, by the preceding lemma we have

$$\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma\cup\{e_0\}}\cdot(\tilde{e}_0)}{\Delta_{\hat{\Gamma}}}=\sum_{T\in\mathscr{F}_1,T\subset\hat{\Gamma}}a_T\frac{(\tilde{e}_0)}{\Delta_T}, \tag{4.28}$$

for $a_T\in \boldsymbol{C}$. Since each $(\tilde{e}_0)$ is a linear combination of (e) for $e\in E(\mathrm{T})$ modulo constants: $(\tilde{e}_0)=c_0+\sum_{e\in E(T)}c_e\cdot(e)$ for $c_e\in \boldsymbol{C}$, and since $(e)/\Delta_T=1/\Delta_{T_e}$, each $(\tilde{e}_0)/\Delta_T$ can be written as

$$\frac{(\tilde{e}_0)}{\Delta_T}=\sum_{e\in E(T)}a_e\frac{1}{\Delta_{T_e}}+\frac{\text{const}}{\Delta_T}. \tag{4.29}$$

T_e is a semi-tree: $T_e\in\mathscr{F}_2$. Hence we have from (4.28) an expression

$$\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\sum_{T\in\mathscr{F}_1}\frac{c_T}{\Delta_T}+\sum_{F\in\mathscr{F}_2}\frac{c_F}{\Delta_F}. \tag{4.30}$$

Through residue calculus, c_T and c_F are given by $\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_T)/\Delta_{\hat{\Gamma}-T}(t_T)$ and $\lim_{\lambda\to\infty}\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_F(\lambda))/\Delta_{\hat{\Gamma}-F}(t_F(\lambda))$, respectively. We must show that $F_1\subset\Gamma_1$ for $F=F_1+F_2$. Suppose the contrary is true: $F_1\not\subset\Gamma_1$, i.e., there exists an edge $e\in E(F_1)-E(\Gamma_1)$. Since for any $e\in E(F_1)$,

$$\begin{aligned}\lim_{\lambda\to\infty}\langle(\tilde{e}), t_F(\lambda)\rangle/\lambda&=0 \qquad \text{for} \quad e\in E(F_1),\\ &=\text{non-zero constant} \qquad \text{for} \quad e\in E(F_2),\end{aligned} \tag{4.31}$$

we have

$$\lim_{\lambda\to\infty}\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_F(\lambda))}{\Delta_{\hat{\Gamma}-F}(t_F(\lambda))}=0. \tag{4.32}$$

Hence c_F must vanish unless $E(F_1)\subset E(\Gamma_1)$. The proof of the lemma is now complete.

One can formulate the third main lemma as follows:

LEMMA 4.5. *Γ be a spanning forest with two components Γ_1 and Γ_2 such that $0 \in V(\Gamma_1)$ and $j \in V(\Gamma_2)$. Let $\hat{\Gamma}$ be an admissible graph containing Γ. Then*

$$t_j^{-1}\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\sum_{T\in\mathscr{F}_1} a_T\frac{1}{\Delta_T}+\sum_{S\in\mathscr{F}_2} b_S t_j^{-1}\frac{1}{\Delta_S} \tag{4.33}$$

where $S\in\mathscr{F}_2$ denotes a forest with two components: $S=S_1+S_2$ such that $E(S_2)\subset E(\Gamma_2)$, $0\in V(S_1)$ and $j\in V(S_2)$.

PROOF. According to (4.25),

$$t_j^{-1}\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\sum_{T\in\mathscr{F}_1} a_T\frac{1}{t_j\Delta_T}+\sum_{F\in\mathscr{F}_2, F_1\subset\Gamma_1} b_F\frac{1}{t_j\Delta_F} \tag{4.34}$$

$a_T, b_F\in \boldsymbol{C}$, where $j\in V(F_2)$ since $V(S_2)\subset V(F_2)$. For each T on the right hand side we have

$$1=c_0 t_j+\sum_{e\in E(T)} c_e(e)\,,\qquad \text{for some } c_0 \text{ and } c_e\in\boldsymbol{C}. \tag{4.35}$$

Hence

$$\frac{1}{t_j\Delta_T}=c_0\frac{1}{\Delta_T}+\sum_{e\in E(T)} c_e\frac{1}{t_j\Delta_{T_e}}. \tag{4.36}$$

Since $T_e\in\mathscr{F}_2$, from (4.34) and (4.36) $t_j^{-1}\Delta_{\hat{\Gamma}-\Gamma}/\Delta_{\hat{\Gamma}}$ can be reexpressed as

$$\frac{\hat{\Delta}_{\hat{\Gamma}-\Gamma}}{t_j\Delta_\Gamma}=\sum_{T\in\mathscr{F}_1} a_T^*\frac{1}{\Delta_T}+\sum_{F\in\mathscr{F}_2} b_F^*\frac{1}{t_j\Delta_F}, \tag{4.37}$$

for some $a_T^*, b_F^*\in\boldsymbol{C}$. a_T^* and b_F^* are uniquely determined by the residue formulae:

$$a_T^*=\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_T)}{t_{T,j}\Delta_{\hat{\Gamma}-T}(t_T)}\quad\text{and}\quad b_F^*=\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_F)}{\Delta_{\hat{\Gamma}-F}(t_F)} \tag{4.38}$$

where $t_T=(t_{T,j})_{1\le j\le n}$ denotes the solution of the equations $\langle(e),t\rangle=0$ for all $e\in E(T)$, while $t_F=(t_{F,j})_{1\le j\le n}$ denotes that of the equations $\langle(e),t\rangle=0$, for all $e\in E(F)$ together with $t_j=0$. Clearly, $t_{F,k}$ vanish for $k\in V(F_2)$. Hence $\tilde{\Delta}_{\hat{\Gamma}-\Gamma}(t_F)$ vanishes if it contains a factor $(e)\in E(F_2)$, i.e., b_F^* vanishes if $E(\hat{\Gamma}-\Gamma)\cap E(F_2)\neq\varnothing$. In other words, if b_F^* differs from zero, then $E(F_2)\subset E(\Gamma_1)\cup E(\Gamma_2)$. Being a tree such that $j\in V(F_2)$, F_2 must be contained in Γ_2. In this way (4.33) has been proved.

DEFINITION 5. *An admissible labelled tree Γ is called terminal if every edge $e\in E(\Gamma)$ is directed towards the vertex 0.*

We denote by $\mathscr{B}$ the linear space spanned by admissible forms φ_Γ associated with admissible labelled trees Γ with directed edges. We also denote by $\mathscr{B}_0$ the linear space spanned by terminal admissible forms φ_Γ for labelled trees with terminal directed edges.

The inclusion $\iota: \mathscr{B}_0 \mapsto V$ gives rise to a homomorphism

$$\iota_*: \mathscr{B}_0 \mapsto H_\Phi(V, d_q)\,. \tag{4.39}$$

Then our Main Theorem can be stated as follows:

THEOREM. *Under the assumptions* $(\mathscr{H}\text{-}1) \sim (\mathscr{H}\text{-}3)$, ι_* *is an isomorphism. Hence* $\dim H_\Phi(V, d_q) = (n+1)^{n-1}$.

5. Proof of Theorem.

LEMMA 5.1. *Suppose Γ is an admissible tree.*

$$b_\chi \cdot Q^\chi \varphi_\Gamma \not\equiv 0 \quad \mathrm{mod}\, \mathscr{B} \tag{5.1}$$

for any $\chi \in X^+$ if and only if Γ is terminal, i.e., φ_Γ does not admit any transformation $\varphi_\Gamma \mapsto b_\chi \cdot Q^\chi \varphi_\Gamma$ for $\chi \in X^+$, where X^+ denotes the abelian semigroup generated by $\chi_1, \cdots, \chi_n$ in X.

PROOF. Suppose Γ is terminal. We take an arbitrary $\chi = \sum_{j=1}^n \nu_j \chi_j \in X^+$. Let k be the vertex nearest to 0 in $V(\Gamma)$ such that $\nu_k > 0$. Then $b_\chi Q^\chi \varphi_\Gamma$ contains $(t_{p(k)} - q^{\beta'_{p(k),k}} t_k)^{-1} \cdots (t_{p(k)} - q^{\beta'_{p(k),k} + \nu_k} t_k)^{-1}$ as an irreducible factor. Hence (5.1) holds. The converse is proved below.

The first main result which we want to prove is the following.

PROPOSITION 4. *An arbitrary admissible form φ_Γ which is not terminal is cohomologous to a linear combination of terminal admissible forms. More precisely,*

$$\mathscr{B} = \mathscr{B}_0 + \mathscr{B} \cap \{ \sum_{\chi \in X^+} (1 - b_\chi Q^\chi) \mathscr{B} \}\,. \tag{5.2}$$

PROOF. Assume that φ_Γ is not terminal. Then Γ being a spanning tree, there exists an edge $e = (i, j)_-$ directed from i to j such that $p(j) = i$. The deleted graph Γ_e is divided into two components Γ_1 and Γ_2 such that $0 \in V(\Gamma_1)$ and that $V(\Gamma_2)$ is disjoint from $\{0\}$ (see Figure 1). We apply the transformation $t_k \mapsto t_k q$ for all $k \in V(\Gamma_2)$. Then

$$\frac{1}{\Delta_\Gamma} \Omega - q^{\alpha_{\Gamma_2} - |E(\Gamma_2)|} \frac{\tilde{\Delta}_{\hat{\Gamma} - \Gamma}}{\Delta_{\hat{\Gamma}}} \Omega \equiv 0 \quad \mathrm{mod}\, \mathscr{B} \cap \sum_{\chi \in X^+} (1 - b_\chi Q^\chi) \mathscr{B} \tag{5.3}$$

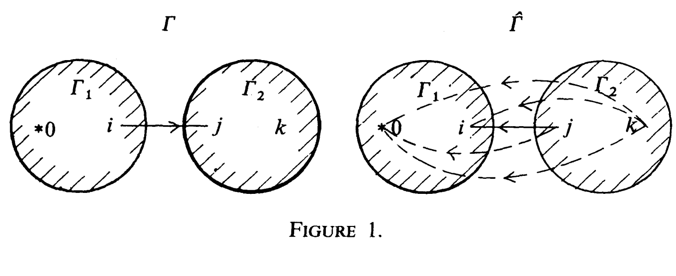

FIGURE 1.

where $\hat{\Gamma}$ denotes a graph such that (i) $V(\hat{\Gamma})=V(\Gamma)$ and (ii) $E(\hat{\Gamma})=E(\Gamma_1)\cup E(\Gamma_2)\cup\bigcup_{h\in V(\Gamma_1),k\in V(\Gamma_2)}(h,k)_+$. From Proposition 1 we have

$$\frac{1}{\Delta_\Gamma}\Omega-q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|}\sum_{T\in\mathscr{F}_1}a_T\frac{\Omega}{\Delta_T}\equiv 0 \mod \mathscr{B}\cap\sum_{\chi\in X^+}(1-b_\chi Q^\chi)\mathscr{B}, \tag{5.4}$$

where in particular $a_\Gamma=1$. Hence the relation (5.3) is rewritten as

$$(1-q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|})\frac{\Omega}{\Delta_\Gamma}\equiv q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|}\sum_{T\in\mathscr{F}_1,T\neq\Gamma}a_T\frac{\Omega}{\Delta_T} \mod \mathscr{B}\cap\sum_{\chi\in X^+}(1-b_\chi Q^\chi)\mathscr{B}. \tag{5.5}$$

In this way we have $(2^n-1)(n+1)^{n-1}$ relations corresponding to non-terminal admissible forms. ($\mathscr{H}$-1)$\sim$($\mathscr{H}$-3) enable us to solve these equations with regard to non-terminal admissible forms, i.e., each non-terminal admissible form is cohomologous to a linear combination of terminal admissible forms. This is exactly what we wanted to prove.

LEMMA 5.2. *Let Γ be an arbitrary spanning forest with two components, $\Gamma\in\mathscr{F}_2$. Then $\varphi_\Gamma=\Omega/\Delta_\Gamma$ is cohomologous to a linear combination of admissible forms, i.e.,*

$$\varphi_\Gamma\equiv 0 \mod \mathscr{B}+\sum_{\chi\in X}(1-b_\chi Q^\chi)V. \tag{5.6}$$

PROOF. Γ consists of two disjoint trees Γ_1 and Γ_2 such that $0\in V(\Gamma_1)$ and 0 is disjoint from $V(\Gamma_2)$. The lemma can be proved by induction on $|E(\Gamma_1)|$. Indeed, we can apply to Ω/Δ_Γ the substitution $t_j\to t_jq$ for all $j\in V(\Gamma_2)$. Then as in (5.3),

$$\frac{\Omega}{\Delta_\Gamma}-q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|}\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}\Omega\equiv 0 \mod \sum_{\chi\in X}(1-b_\chi Q^\chi)V. \tag{5.7}$$

By Proposition 2, $\tilde{\Delta}_{\hat{\Gamma}-\Gamma}/\Delta_\Gamma$ can be written as

$$\sum_{T\in\mathscr{F}_1}a_T\frac{1}{\Delta_T}+\sum_{S\in\mathscr{F}_2}b_S\frac{1}{\Delta_S} \tag{5.8}$$

where $S=S_1+S_2$ runs through the set of all the semi-trees such that $E(S_1)\subset E(\Gamma_1)$. a_T and b_S are given by the formula (4.26). Hence we have

$$\frac{\Omega}{\Delta_T}-q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|}\left\{\sum_{T\in\mathscr{F}_1}a_T\frac{\Omega}{\Delta_T}+\sum_{S\in\mathscr{F}_2}b_S\frac{\Omega}{\Delta_S}\right\}\equiv 0 \mod \sum_{\chi\in X}(1-b_\chi Q^\chi)V, \tag{5.9}$$

where b_Γ is given by $\sum_{h\in V(\Gamma_1),k\in V(\Gamma_2)}\beta_{h,k}-\beta'_{h,k}$. Then (5.9) can be rewritten as

$$\begin{aligned}(1-q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|+\Sigma_{h\in V(\Gamma_1),k\in V(\Gamma_2)}\beta_{h,k}-\beta'_{h,k}})\frac{\Omega}{\Delta_\Gamma}\\ \equiv q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|}\left\{\sum_{T\in\mathscr{F}_1}a_T\frac{\Omega}{\Delta_T}+\sum_{S\in\mathscr{F}_2,S_1\subsetneqq\Gamma_1}b_S\frac{\Omega}{\Delta_S}\right\}\end{aligned} \tag{5.10}$$

$$\equiv q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|}\sum_{S\in\mathscr{F}_2, S_1\subsetneqq\Gamma_1} b_S\frac{\Omega}{\Delta_S} \quad \bmod \mathscr{B}+\sum_{\chi\in X}(1-b_\chi Q^\chi)V.$$

Since each Ω/Δ_S in the last part is cohomologous to an element of $\mathscr{B}$ by the induction hypothesis, so is Ω/Δ_Γ. The proof is now complete.

LEMMA 5.3. *For an arbitrary admissible form φ_Γ and an arbitrary j, $1\le j\le n$, $t_j\varphi_\Gamma$ is cohomologous to a linear combination of admissible forms, i.e.,*

$$t_j\varphi_\Gamma\sim 0 \quad \bmod \mathscr{B}. \tag{5.11}$$

PROOF. Indeed, there exists a unique path $[j_0, j_1, \cdots, j_{m-1}, j], j_0=0$ and $j_m=j$, in a tree Γ so that t_j can be written as

$$t_j=c_0+\sum_{\nu=1}^{m} c_\nu(e_\nu), \tag{5.12}$$

for $c_0, c_\nu\in \boldsymbol{C}$ and $(e_\nu)=(j_{\nu-1}, j_\nu)_+$ so that

$$\frac{t_j}{\Delta_\Gamma}=\frac{c_0}{\Delta_\Gamma}+\sum_{\nu=1}^{m} c_\nu\frac{1}{\Delta_{\Gamma_{e_\nu}}}. \tag{5.13}$$

Since Γ_{e_ν} is a spanning semi-tree, we can apply Lemma 4.4 to $\Omega/\Delta_{\Gamma_{e_\nu}}$ so that $\Omega/\Delta_{\Gamma_{e_\nu}}\sim 0 \bmod \mathscr{B}$. This shows $(t_j/\Delta_\Gamma)\Omega\sim 0 \bmod \mathscr{B}$, since $\Omega/\Delta_\Gamma\in\mathscr{B}$.

Similarly, we have:

LEMMA 5.4. *Under the same circumstance as in Lemma 4.5, we have $t_j^{-1}\Omega/\Delta_\Gamma\sim 0 \bmod \mathscr{B}$.*

PROOF. We can apply the substitution $t_k\mapsto t_k q$ for all $k\in V(\Gamma_2)$. Then as in (5.3)

$$t_j^{-1}\frac{\Omega}{\Delta_\Gamma}\sim q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|-1}\, t_j^{-1}\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}\Omega. \tag{5.14}$$

By Lemma 4.4,

$$t_j^{-1}\frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}}=\sum_{T\in\mathscr{F}_1} a_T\frac{\Omega}{\Delta_T}+\sum_{S\in\mathscr{F}_2} b_S t_j^{-1}\frac{\Omega}{\Delta_S}, \tag{5.15}$$

since S is a semi-tree with two components S_1, S_2 such that $j\in V(S_2)$, $E(S_2)\subset E(\Gamma_2)$ and $0\in V(S_1)$. a_T and b_S are given by (4.25) for the solutions t_T and t_S of the equations: $\langle(e), t_T\rangle=0$ for $e\in E(T)$ and $\langle(e), t_S\rangle=0$ for $e\in E(S)$ together with $t_j=0$, respectively. b_S vanishes unless $E(S_2)\subset E(\Gamma_2)$. Hence

$$t_j^{-1}\frac{\Omega}{\Delta_\Gamma}\sim q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|-1}\left\{\sum_{T\in\mathscr{F}_1} a_T\frac{\Omega}{\Delta_T}+\sum_{S\in\mathscr{F}_2, S_2\subset\Gamma_2} b_S t_j^{-1}\frac{\Omega}{\Delta_S}\right\} \tag{5.16}$$

or equivalently,

$$(5.17)\qquad (1-q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|-1})\, t_j^{-1}\frac{\Omega}{\Delta_\Gamma}$$

$$\sim q^{\alpha_{\Gamma_2}-|E(\Gamma_2)|-1}\left\{\sum_{T\in\mathscr{F}_1} a_T\frac{\Omega}{\Delta_T}+\sum_{S\in\mathscr{F}_2, S_2\subsetneq\Gamma_2} b_S t_j^{-1}\frac{\Omega}{\Delta_S}\right\},$$

since $b_\Gamma=1$. By induction, the system of equations (5.17) for all the forms $t_j^{-1}\varphi_\Gamma$, with φ_Γ admissible, can be solved concerning $t_j^{-1}\varphi_\Gamma$ in such a way that $t_j^{-1}\varphi_\Gamma$ is cohomologous to a linear combination of admissible ones. This implies the lemma.

PROPOSITION 5. *For an arbitrary admissible $\varphi_\Gamma=\Omega/\Delta_\Gamma$ and any j, $1\leq j\leq n$,* we have $t_j^{-1}\varphi_\Gamma\sim 0 \bmod \mathscr{B}$.

PROOF. As in the proof of Lemma 5.3 there exists a unique path $[j_0, j_1, \cdots, j_{m-1}, j]$ in Γ such that (5.12) holds. (5.12) implies

$$(5.18)\qquad \frac{1}{t_j\prod_{\nu=1}^{m}(e_\nu)}=\frac{1}{c_0}\frac{1}{\Delta_\Gamma}-\sum_{\nu=1}^{m}\frac{c_\nu}{c_0}\frac{1}{\prod_{k\neq\nu}^{m}(e_k)}$$

(remark that $c_0\neq 0$ by hypothesis), i.e.,

$$(5.19)\qquad \frac{1}{t_j\Delta_\Gamma}=\frac{1}{c_0\Delta_\Gamma}-\sum_{\nu=1}^{m}\frac{c_\nu}{c_0}\frac{1}{\Delta_{\Gamma_{e_\nu}}}.$$

From Lemma 4.4 $\Omega/\Delta_{\Gamma_{e_\nu}}\sim 0 \bmod \mathscr{B}$, whence Proposition 5 follows.

COROLLARY. $W_{0,j}^{(+)}Q_{\beta_{0,j}}\varphi\sim 0 \bmod \mathscr{B}$, $W_{i,j}^{(+)}Q_{\beta_{i,j}}\varphi\sim 0 \bmod \mathscr{B}$, $W_{i,j}^{\prime(-)}Q_{\beta'_{i,j}}^{-1}\varphi\sim 0 \bmod \mathscr{B}$ *for an admissible* φ.

PROOF. Indeed, $W_{\beta_{0,j}}^{(+)}Q_{\beta_{0,j}}\varphi_\Gamma=(1-q^{\beta_{0,j}-1}Q_j)\varphi_\Gamma$ or $(1-q^{\beta_{0,j}}Q_j)\varphi_\Gamma$ according as $(0,j)_-\in E(\Gamma)$ or not. Similarly, $W_{i,j}^{(-)}Q_{\beta_{i,j}}\varphi_\Gamma=Q_i^{-1}(Q_i-q^{\beta_{i,j}-1}Q_j)\varphi_\Gamma$ or $Q_i^{-1}(Q_i-q^{\beta_{i,j}}Q_j)\varphi_\Gamma$ according as $(i,j)_-\in E(\Gamma)$ or not, while $W_{i,j}^{\prime(-)}Q_{\beta'_{i,j}}^{-1}\varphi_\Gamma=Q_i^{-1}(Q_i-q^{\beta'_{i,j}}Q_j)\varphi_\Gamma$ or $Q_i^{-1}(Q_i-q^{\beta'_{i,j}-1}Q_j)\varphi_\Gamma$ according as $(i,j)_+\in E(\Gamma)$ or not.

PROPOSITION 6. (i) $W_{i,j}^{\prime(+)}Q_{\beta'_{i,j}}\varphi_\Gamma\sim 0 \bmod \mathscr{B}$.
(ii) $W_{i,j}^{(-)}Q_{\beta_{i,j}}^{-1}\varphi_\Gamma\sim 0 \bmod \mathscr{B}$, *for* $0\leq i\leq j\leq n$.

PROOF. Suppose first that $E(\Gamma)$ does not contain the form $(i,j)_+$. We denote by $\hat\Gamma$ the graph obtained from Γ by adding the edge $(i,j)_+$ to Γ such that $E(\hat\Gamma)=E(\Gamma)\cup\{(i,j)_+\}$ and $V(\hat\Gamma)=V(\Gamma)$. $\hat\Gamma$ contains a circuit $\mathscr{C}$ which itself contains $(i,j)_+$. Then from Lemma 4.2,

$$(5.20)\qquad \frac{1}{\Delta_{\hat\Gamma}}=\sum_{e\in E(\mathscr{C})} a_e\frac{1}{\Delta_{\Gamma_e}}.$$

Since each Γ_e is a tree such that $0 \in V(\Gamma_e)$, Ω/Δ_{Γ_e} is admissible, i.e., $W'^{(+)}_{i,j} Q_{\beta'_{i,j}} \Omega/\Delta_\Gamma \sim 0 \mod \mathscr{B}$. Suppose on the contrary $E(\Gamma)$ contains the form $(i, j)_+$. Then

$$W'^{(+)}_{i,j} Q_{\beta'_{i,j}} \frac{\Omega}{\Delta_\Gamma} = \frac{\Omega}{(t_i - q^{\beta'_{i,j}} t_j)(t_i - q^{\beta'_{i,j}+1} t_j) \prod_{e \in E(\Gamma_e)} (e)} . \tag{5.21}$$

$\Gamma_{(i,j)_+}$ consists of two components of disjoint trees Γ_1 and Γ_2 such that $\{0, i\} \subset V(\Gamma_1)$ and $\{j\} \subset V(\Gamma_2)$. We apply to $W'^{(+)}_{i,j} Q_{\beta'_{i,j}} \Omega/\Delta_\Gamma$ the substitution $t_k \mapsto q^{-1} t_k$ for all $k \in V(\Gamma_2)$. Then

$$W'^{(+)}_{i,j} Q_{\beta'_{i,j}} \frac{\Omega}{\Delta_\Gamma} \sim q^{-\alpha_{\Gamma_2} + |E(\Gamma_2)|} \frac{\tilde{\Delta}_{\hat{\Gamma}-\Gamma}}{\Delta_{\hat{\Gamma}}} \Omega , \tag{5.22}$$

where $\hat{\Gamma}$ is a graph containing Γ such that

$$V(\hat{\Gamma}) = V(\Gamma) , \tag{5.23}$$

$$E(\hat{\Gamma}) = E(\Gamma_1) \cup E(\Gamma_2) \cup \bigcup_{h \in V(\Gamma_1), k \in V(\Gamma_2)} (h, k)_- \cup \{(i, j)_+\} , \tag{5.24}$$

where $(h, k) \neq (i, j)$. From Lemma 4.3 we have the partial fraction on the right hand side of (5.21). Hence the proposition follows.

From Propositions 3 and 4 applied to an arbitrary admissible form φ_Γ

$$Q_i^{\pm 1} \varphi_\Gamma \sim 0 \mod \mathscr{B}_0 \tag{5.25}$$

$$W'^{(\pm)}_{i,j} Q^{\pm 1}_{\beta'_{i,j}} \varphi_\Gamma \sim 0 \mod \mathscr{B}_0 \tag{5.26}$$

$$W^{(\pm)}_{i,j} Q^{\pm 1}_{\beta_{i,j}} \varphi_\Gamma \sim 0 \mod \mathscr{B}_0 . \tag{5.27}$$

Since $\Phi V = \mathscr{A}\Phi = \mathscr{A}(\Phi \mathscr{B}_0)$, an arbitrary element $\varphi \in V$ is cohomologous to an element of $\mathscr{B}_0$: $\varphi \sim 0 \mod \mathscr{B}_0$. This implies the following:

Proposition 7. *The map ι_* defined in* (4.39) *is a surjection.*

We can now prove the Theorem in Section 4.

Proof of Theorem. For each unoriented admissible labelled tree $\hat{T}$, the point $\bar{t} = (\bar{t}_j)_{1 \le j \le n} \in q^{\bar{X}}$ is defined by the equations: $\bar{t}_{p(j)} = q^{\beta'_{p(j),j}} \bar{t}_j$, and $\bar{t}_0 = 1$. We can construct a cycle $\mathfrak{c}(T) = \mathfrak{c}(\bar{t})$ consisting of countable points given by

$$q^{\beta'_{p(j),j}} t_j / t_{p(j)} \in q^{\mathbf{Z}^+} . \tag{5.28}$$

To each $\hat{T}$ corresponds a unique terminal admissible tree and vice versa. Thus the set of unoriented admissible labelled trees is in one-to one correspondence with that of terminal admissible forms. The number of such trees is equal to $\mu = (n+1)^{n-1}$. Let $T_1, \cdots, T_\mu$ be the totality of them. We must prove that these are linearly independent in $H_\Phi(V, d_q)$. It is sufficient to prove that the determinant of the period matrix

$M=((\varphi_{T_i}, c(T_j)))_{1\le i,j\le\mu}$ does not vanish. This can be shown by asymptotic argument as follows.

We consider the integration of the functions $\Phi\varphi$, $\varphi\in\mathscr{B}_0$, over the cycle $c(T)$. The function Φ has no pole on $c(T)$ if and only if T is standard, i.e., $p(j)<j$ for each $j\in V(T)$. If T is not standard, we replace $c(T)$ by its regularization reg $c(T)$ by taking the residues of $\Phi\varphi$ at the poles of $\Phi\varphi$. The crucial fact is the following:

LEMMA 5.5. *For* $\alpha_j=\eta_j N+\alpha'_j$ $(\eta_j\in\boldsymbol{Z}^+, \alpha'_j\in\boldsymbol{C})$, $N\to+\infty$, *the integral of an terminal admissible form* φ_{T^*}

$$\int_{c(T)}\Phi\varphi_{T^*}\Omega\sim(1-q)^n(q)_\infty^n\,\bar{t}_1^{\alpha_1-\delta_1}\cdots\bar{t}_n^{\alpha_n-\delta_n}\left(1+O\left(\frac{1}{N}\right)\right)\tag{5.29}$$

or

$$\sim(1-q)^n(q)_\infty^n\,\bar{t}_1^{\alpha_1-\delta_1}\cdots\bar{t}_n^{\alpha_n-\delta_n}\,O\left(\frac{1}{N}\right),\tag{5.30}$$

according as $T^*=T$ *or* $T^*\ne T$, *where* δ_j+1 *denotes the degree of the vertex j in* T^*. *The same holds for the integration over* reg $c(T)$.

PROOF. The function Φ has an expression

$$\Phi=(t_1^{\eta_1}\cdots t_n^{\eta_n})^N t_1^{\alpha'_1}\cdots t_n^{\alpha'_n}\prod_{0\le i<j\le n}\frac{(q^{\beta'_{i,j}}t_j/t_i)_\infty}{(q^{\beta_{i,j}}t_j/t_i)_\infty}.\tag{5.31}$$

By assumption the function $|t_1^{\eta_1}\cdots t_n^{\eta_n}|$ has maximal value at $t=\bar{t}$ on $c(T)$ or reg $c(T)$. It is unique, i.e., $|t_1^{\eta_1}\cdots t_n^{\eta_n}|<|\bar{t}_1^{\eta_1}\cdots\bar{t}_n^{\eta_n}|$ on $c(T)-\{\bar{t}\}$. If $T^*\ne T$, then the factors $1-q^{\beta'_{i,j}}t_j/t_{p(j)}$ appear in the numerator of Φ/Δ_T, while if $T^*=T$, all the factors $1-q^{\beta'_{p(j),j}}t_j/t_{p(j)}$ disappear. Since all these factors vanish on $c(T)$ or reg $c(T)$, Φ vanishes at $t=\bar{t}(T^*)$ for $T^*\ne T$, while Φ is equal to

$$\bar{t}_1^{\alpha_1}\cdots\bar{t}_n^{\alpha_n}\frac{(q)_\infty^n}{\prod_{j=1}^n(q^{\beta_{i,j}}\bar{t}_j/\bar{t}_{p(j)})}\qquad\text{for}\quad T^*=T.\tag{5.32}$$

This shows that the period matrix M is asymptotically equal to a diagonal matrix whose entries are represented by the principal terms in (5.29) for each unoriented admissible labelled tree T. In other words, the matrix M is non-singular for sufficiently large $\alpha_1,\cdots,\alpha_n$. Hence $\varphi_{T_1},\cdots,\varphi_{T_n}$ are linearly independent in $H_\Phi(V,d_q)$. The theorem has been proved.

COROLLARY. $\langle\varphi_{T_1}\rangle,\cdots,\langle\varphi_{T_\mu}\rangle$ *satisfy the normal holonomic q-difference equations*

$$\tilde{Q}_j^{\pm1}(\langle\varphi_{T_1}\rangle,\cdots,\langle\varphi_{T_\mu}\rangle)=(\langle\varphi_{T_1}\rangle,\cdots,\langle\varphi_{T_\mu}\rangle)A_j^{\pm},\quad 1\le j\le n,\tag{5.33}$$

(5.34) $\tilde{Q}^{\pm 1}_{\beta_{i,j}}(\langle\varphi_{T_1}\rangle, \cdots, \langle\varphi_{T_\mu}\rangle)=(\langle\varphi_{T_1}\rangle, \cdots, \langle\varphi_{T_\mu}\rangle)A_{\pm\beta_{i,j}}, \quad 0\leq i<j\leq n,$

(5.35) $\tilde{Q}^{\pm 1}_{\beta'_{i,j}}(\langle\varphi_{T_1}\rangle, \cdots, \langle\varphi_{T_\mu}\rangle)=(\langle\varphi_{T_1}\rangle, \cdots, \langle\varphi_{T_\mu}\rangle)A_{\pm\beta'_{i,j}}, \quad 1\leq i<j\leq n,$

respectively. Here $A^{\pm}_j$, $A_{\pm\beta_{i,j}}$ *and* $A_{\pm\beta'_{i,j}}$ *denote matrices of degree* μ *over the rational function field* $\boldsymbol{C}((u_l, q^{\beta_{k,l}}, q^{\beta'_{k,l}})_{0\leq k<l\leq n})$. *These are equivalent to* (4.6)~(4.10).

REMARK. The set of all directions $\eta=(\eta_1, \cdots, \eta_n)\in\boldsymbol{Z}^n-\{0\}$ giving inequivalent asymptotic behaviours of $\tilde{\Phi}$ are divided into a finite set of rational polyhedral cones in $\boldsymbol{Q}^n$. This defines an n-dimensional toric variety which may be singular in general (see [O1] for the definition). The connection coefficients among asymptotic solutions along different directions η can be described in terms of transition matrices on this variety. The combinatorial structure of them will be presented elsewhere (see [A5]).

6. The basic hypergeometric function of third order. The case $n=2$ is given by the basic hypergeometric function

$$(6.1)\qquad {}_3\varphi_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}\middle| x\right)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n(c;q)_n}{(d;q)_n(e;q)_n(q;q)_n}x^n,$$

for $a, b, c, d, e\in\boldsymbol{C}$ and $(a;q)_n=(a)_\infty/(aq^n)_\infty$ etc., such that $d, e\neq 1, q^{-1}, q^{-2}, \cdots$. It has an integral representation

$$(6.2)\qquad {}_3\varphi_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}\middle| x\right)=\frac{(a_1)_\infty(a_2)_\infty(b_1/a_1)_\infty(b_2/a_2)_\infty}{(b_1)_\infty(b_2)_\infty(q)^2_\infty(1-q)^2}\cdot$$

$$\int_{1\geq\tau_1\geq\tau_2>0}\tau_1^{\alpha_1-\alpha_2}\tau_2^{\alpha_2}\frac{(\tau_1 q)_\infty(q\tau_2/\tau_1)_\infty(a_0\tau_2 x)_\infty}{(b_1\tau_1/a_1)_\infty(b_2\tau_2/(a_2/\tau_1))_\infty(\tau_2 x)_\infty}\frac{d_q\tau_1\wedge d_q\tau_2}{\tau_1\tau_2}$$

for $b=q^{\alpha_1}$ and $c=q^{\alpha_2}$. This integral coincides with (4.2) by putting $\alpha_1\mapsto\alpha_1-\alpha_2$, $\alpha_2\mapsto\alpha_2$, $q^{\beta'_{0,1}}=q$, $q^{\beta'_{0,2}}=a_0x$, $q^{\beta_{0,1}}=b_1/a_1$, $q^{\beta_{0,2}}=x$, $q^{\beta'_{1,2}}=q$ and $q^{\beta_{1,2}}=b_2/a_2$ in (4.2). For brevity we put $\beta_{0,1}=\beta_1$, $\beta_{0,2}=\beta_2$, $\beta_{1,2}=\beta'$ and $\beta_{1,2}=\beta$. We have $\dim\mathscr{B}_0=3$ due to the Theorem. The basis is given by

$$(6.3)\quad \varphi_{T_1}=\frac{\Omega}{(1-t_1)(1-t_2)},\quad \varphi_{T_2}=\frac{\Omega}{(1-t_1)(t_1-q^{\beta'}t_2)}\quad\text{and}\quad \varphi_{T_3}=\frac{\Omega}{(1-t_2)(t_1-q^{\beta-1}t_2)}$$

corresponding to the terminal admissible trees T_1, T_2 and T_3, respectively as in Figure 2. In addition to these it is also convenient to consider the forms

$$(6.4)\qquad \varphi_{T_4}=\frac{\Omega}{(1-t_1)(t_1-q^{\beta'}t_2)}\quad\text{and}\quad \varphi_{T_5}=\frac{\Omega}{(1-t_1)(t_1-q^{\beta-1}t_2)}$$

corresponding to the admissible trees T_4 and T_5 which are not terminal (see Figure 2). There are two linear relations among them as follows:

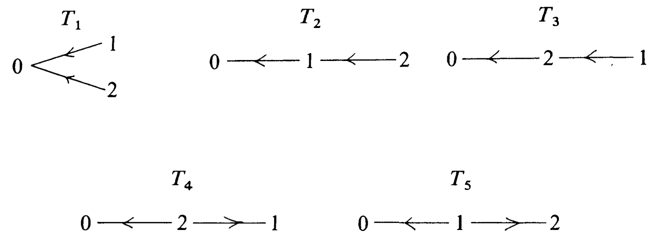

FIGURE 2.

(6.5) $$\varphi_{T_4} \sim q^{\alpha_1 - 1}\left\{\frac{1-q^{\beta_1}}{1-q^{\beta-1}}\varphi_{T_1} + \frac{1-q^{\beta_1+\beta-1}}{1-q^{\beta-1}}\varphi_{T_3} + \frac{1-q^{\beta_1}}{1-q^{1-\beta}}\varphi_{T_5}\right\},$$

(6.6) $$\varphi_{T_5} \sim q^{\alpha_2}\left\{\frac{1-q^{\beta_2}}{1-q^{\beta'}}\varphi_{T_1} + \frac{q^{\beta_2}-q^{\beta'}}{1-q^{\beta'}}\varphi_{T_2} + \frac{1-q^{\beta_2}}{1-q^{\beta'}}\varphi_{T_4}\right\}.$$

From these relations one can solve φ_{T_4} aand φ_{T_5} as linear combinations of φ_{T_1}, φ_{T_2} and φ_{T_3}, provided $(1-q^{1-\beta})(1-q^{\beta'}) - q^{\alpha_1+\alpha_2-1}(1-q^{\beta_1})(1-q^{\beta_2}) \neq 0$, i.e.,

(6.7) $$\varphi_{T_4} \sim 0 \mod \mathscr{B}_0 \quad \text{and} \quad \varphi_{T_5} \sim 0 \mod \mathscr{B}_0 .$$

To find the formulae for $\tilde{Q}_1$ and $\tilde{Q}_2$ one needs the following:

LEMMA 6.1. *We have the relations*

(6.8) $$\begin{aligned}(1-q^{\alpha_1+\beta_1})\left\langle\frac{\Omega}{1-t_2}\right\rangle &+ q^{\alpha_1+\beta_1}(q^{\beta-1}-q^{\beta'-1})\left\langle\frac{\Omega}{t_1-q^{\beta-1}t_2}\right\rangle \\ &= q^{\alpha_1}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta'-1})}{1-q^{\beta-1}}\langle\varphi_{T_1}\rangle + \frac{(1-q^{\beta_1})(1-q^{\beta'-\beta})}{1-q^{1-\beta}}\langle\varphi_{T_5}\rangle\right. \\ &\quad \left. + \frac{(q^{\beta-1}-q^{\beta'-1})(1-q^{\beta_1+\beta-1})}{1-q^{\beta-1}}\langle\varphi_{T_3}\rangle\right\}.\end{aligned}$$

(6.9) $$\begin{aligned}(1-q^{\alpha_1+\alpha_2+\beta_1+\beta_2-1})&\left\langle\frac{\Omega}{t_1-q^{\beta'}t_2}\right\rangle \\ &= q^{\alpha_1+\alpha_2-1}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle + \frac{(1-q^{\beta_1})(q^{\beta_2}-q^{\beta'})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle\right. \\ &\quad \left. + \frac{(1-q^{\beta_2})(1-q^{\beta_1+\beta'})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\},\end{aligned}$$

$$(1-q^{\alpha_2+\beta_2+\beta-\beta'})\left\langle\frac{\Omega}{1-t_1}\right\rangle+q^{\alpha_2+\beta_2}(1-q^{\beta-\beta'})\left\langle\frac{\Omega}{t_1-q^{\beta'}t_2}\right\rangle \tag{6.10}$$

$$=q^{\alpha_2}\left\{\frac{(1-q^{\beta_2})(1-q^{\beta})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle+\frac{(1-q^{\beta_2-\beta'})(q^{\beta}-q^{\beta'})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle+\frac{(1-q^{\beta_2})(q^{\beta'}-q^{\beta})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\},$$

$$\left\langle\frac{\Omega}{t_1-q^{\beta-1}t_2}\right\rangle=q^{\alpha_2}\left\{(1-q^{\beta_2})\langle\varphi_{T_4}\rangle+q^{\beta_2}\left\langle\frac{\Omega}{t_1-q^{\beta'}t_2}\right\rangle\right\}. \tag{6.11}$$

(6.8)–(6.11) can be derived as in the proof of Lemma 5.2. They enable us to express $\langle\Omega/(1-t_1)\rangle$, $\langle\Omega/(1-t_2)\rangle$, $\langle\Omega/(t_1-q^{\beta'}t_2)\rangle$ and $\langle\Omega/(t_1-q^{\beta-1}t_2)\rangle$ in terms of $\langle\varphi_{T_j}\rangle$, $1\le j\le 5$. Since

$$\tilde{Q}_1\langle\varphi_{T_1}\rangle=\langle\varphi_{T_1}\rangle-\left\langle\frac{\Omega}{1-t_2}\right\rangle,\quad \tilde{Q}_1\langle\varphi_{T_2}\rangle=\langle\varphi_{T_2}\rangle-\left\langle\frac{\Omega}{t_1-q^{\beta'}t_2}\right\rangle, \tag{6.12}$$

$$\tilde{Q}_1\langle\varphi_{T_4}\rangle=\left\langle\frac{\Omega}{1-t_2}\right\rangle-q^{\beta'}\left\langle\frac{\Omega}{t_1-q^{\beta'}t_2}\right\rangle+q^{\beta'}\langle\varphi_{T_4}\rangle, \tag{6.13}$$

$$\tilde{Q}_2\langle\varphi_{T_1}\rangle=\langle\varphi_{T_1}\rangle-\left\langle\frac{\Omega}{1-t_1}\right\rangle, \tag{6.14}$$

$$\tilde{Q}_2\langle\varphi_{T_2}\rangle=q^{-\beta'}\left\{\langle\varphi_{T_2}\rangle-\left\langle\frac{\Omega}{1-t_1}\right\rangle-\left\langle\frac{\Omega}{t_1-q^{\beta'}t_2}\right\rangle\right\}, \tag{6.15}$$

$$\tilde{Q}_2\langle\varphi_{T_4}\rangle=\langle\varphi_{T_4}\rangle-\left\langle\frac{\Omega}{t_1-q^{\beta'}t_2}\right\rangle, \tag{6.16}$$

we get from the formulae (6.8)–(6.11) the following:

LEMMA 6.2.

$$\tilde{Q}_1\langle\varphi_{T_2}\rangle=\langle\varphi_{T_2}\rangle-\frac{q^{\alpha_1+\alpha_2-1}}{1-q^{\alpha_1+\alpha_2+\beta_1+\beta_2-1}}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle+\frac{(1-q^{\beta_1})(q^{\beta_2}-q^{\beta'})}{(1-q^{\beta'})}\langle\varphi_{T_2}\rangle+\frac{(1-q^{\beta_1+\beta'})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\}, \tag{6.17}$$

$$\tilde{Q}_1\langle\varphi_{T_1}\rangle+\frac{q^{\beta}-q^{\beta'}}{1-q^{\alpha_1+\beta_1}}\tilde{Q}_1\langle\varphi_{T_2}\rangle=\frac{1-q^{\alpha_1}}{1-q^{\alpha_1+\beta_1}}\langle\varphi_{T_1}\rangle+\frac{q^{\beta}-q^{\beta'}}{1-q^{\alpha_1+\beta_1}}\langle\varphi_{T_2}\rangle+\frac{q^{\beta'}-q^{\beta}}{1-q^{\alpha_1+\beta_1}}\langle\varphi_{T_4}\rangle, \tag{6.18}$$

$$(6.19)\qquad \tilde{Q}_1\langle\varphi_{T_1}\rangle - q^{\beta'}\tilde{Q}_1\langle\varphi_{T_2}\rangle + \tilde{Q}_1\langle\varphi_{T_4}\rangle = \langle\varphi_{T_1}\rangle - q^{\beta'}\langle\varphi_{T_2}\rangle + q^{\beta'}\langle\varphi_{T_4}\rangle .$$

$$(6.20)\qquad \tilde{Q}_2\langle\varphi_{T_1}\rangle + \frac{q^{\alpha_2+\beta_2}(1-q^{\beta-\beta'})}{1-q^{\alpha_2+\beta_2+\beta-\beta'}}\tilde{Q}_2\langle\varphi_{T_4}\rangle$$
$$= \langle\varphi_{T_1}\rangle + \frac{q^{\alpha_2+\beta_2}(1-q^{\beta-\beta'})}{1-q^{\alpha_2+\beta_2+\beta-\beta'}}\langle\varphi_{T_4}\rangle - \frac{q^{\alpha_2}}{1-q^{\alpha_2+\beta_2+\beta-\beta'}}\left\{\frac{(1-q^{\beta_2})(1-q^{\beta})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle\right.$$
$$\left. - \frac{(1-q^{\beta-\beta'})(q^{\beta'}-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle + \frac{(1-q^{\beta_2})(q^{\beta'}-q^{\beta})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\}$$

$$(6.21)\qquad \tilde{Q}_2\langle\varphi_{T_2}\rangle - q^{-\beta'}\tilde{Q}_2\langle\varphi_{T_1}\rangle = -q^{-\beta'}\langle\varphi_{T_1}\rangle + q^{-\beta'}\langle\varphi_{T_2}\rangle$$
$$-\frac{q^{\alpha_1+\alpha_2-\beta'-1}}{1-q^{\alpha_1+\alpha_2+\beta_1+\beta_2-1}}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{(1-q^{\beta'})}\langle\varphi_{T_1}\rangle + \frac{(1-q^{\beta_1})(q^{\beta_2}-q^{\beta'})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle\right.$$
$$\left.+\frac{(1-q^{\beta_1+\beta'})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\},$$

$$(6.22)\qquad \tilde{Q}_2\langle\varphi_{T_4}\rangle = \langle\varphi_{T_4}\rangle - \frac{q^{\alpha_1+\alpha_2-1}}{(1-q^{\alpha_1+\alpha_2+\beta_1+\beta_2-1})}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle\right.$$
$$\left.+\frac{(1-q^{\beta_1})(q^{\beta_2}-q^{\beta'})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle + \frac{(1-q^{\beta_2})(1-q^{\beta_1+\beta'})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\},$$

so that

$$(6.23)\qquad \tilde{Q}_2\langle\varphi_{T_2}\rangle - q^{-\beta'}\tilde{Q}_2\langle\varphi_{T_1}\rangle - q^{-\beta'}\tilde{Q}_2\langle\varphi_{T_4}\rangle = q^{-\beta'}\{\langle\varphi_{T_2}\rangle - \langle\varphi_{T_1}\rangle - \langle\varphi_{T_4}\rangle\} .$$

To compute the formulae for $\tilde{Q}_1^{-1}$ and $\tilde{Q}_2^{-1}$, one needs the following two lemmas, which can be obtained as in the proof of lemma 5.4.

LEMMA 6.3.

$$(6.24)\qquad (1-q^{\alpha_1+\beta'-\beta-1})\left\langle\frac{\Omega}{t_1(1-t_2)}\right\rangle$$
$$= q^{\alpha_1-1}\left\{\frac{1-q^{\beta'-1}}{1-q^{\beta-1}}\langle\varphi_{T_1}\rangle + \frac{1-q^{\beta'-\beta}}{1-q^{\beta-1}}\langle\varphi_{T_3}\rangle + \frac{1-q^{\beta'-\beta}}{1-q^{1-\beta}}\langle\varphi_{T_5}\rangle\right\},$$

$$(6.25)\qquad (1-q^{\alpha_1+\alpha_2-2})\left\langle\frac{\Omega}{t_1(t_1-q^{\beta'}t_2)}\right\rangle$$
$$= -q^{\alpha_1+\alpha_2-\beta'-2}(1-q^{\beta_2})\left\langle\frac{\Omega}{t_1(1-t_2)}\right\rangle + q^{\alpha_1+\alpha_2-2}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle\right.$$
$$\left.+\frac{(1-q^{\beta_1})(q^{\beta_2}-q^{\beta'})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle + q^{-\beta'}\frac{(1-q^{\beta_1+\beta'})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\},$$

$$(6.26)\qquad (1-q^{\alpha_1+\alpha_2-2})\left\langle\frac{\Omega}{t_1(t_1-q^{\beta-1}t_2)}\right\rangle$$

$$=-q^{\alpha_1+\alpha_2-\beta-1}(1-q^{\beta_2})\left\langle\frac{\Omega}{t_1(1-t_2)}\right\rangle+q^{\alpha_1+\alpha_2-2}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{1-q^{\beta-1}}\langle\varphi_{T_1}\rangle\right.$$

$$\left.+\frac{(1-q^{\beta_1})(1-q^{\beta_2-\beta+1})}{1-q^{1-\beta}}\langle\varphi_{T_5}\rangle+q^{1-\beta}\frac{(1-q^{\beta_1+\beta-1})(1-q^{\beta_2})}{(1-q^{\beta-1})}\langle\varphi_{T_3}\rangle\right\}.$$

LEMMA 6.4.

$$(6.27)\qquad (1-q^{\alpha_2-1})\left\langle\frac{\Omega}{(1-t_1)t_2}\right\rangle=q^{\alpha_2-1}\left\{\frac{(q^{\beta}-q^{\beta'})(q^{\beta'}-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle\right.$$

$$\left.+\frac{(1-q^{\beta})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle+\frac{(q^{\beta'}-q^{\beta})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\}$$

$$(6.28)\qquad (1-q^{\alpha_1+\alpha_2-2})\left\langle\frac{\Omega}{t_2(t_1-q^{\beta'}t_2)}\right\rangle$$

$$=q^{\alpha_1+\alpha_2-2}(1-q^{\beta_1})\left\langle\frac{\Omega}{t_2(1-t_1)}\right\rangle+q^{\alpha_1+\alpha_2-2}\left\{q^{\beta'}\frac{(1-q^{\beta_1})(q^{\beta_2}-q^{\beta'})}{(1-q^{\beta'})}\langle\varphi_{T_2}\rangle\right.$$

$$\left.+\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle+\frac{(1-q^{\beta_1+\beta'})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\},$$

$$(6.29)\qquad (1-q^{\alpha_1+\alpha_2-2})\left\langle\frac{\Omega}{t_2(t_1-q^{\beta-1}t_2)}\right\rangle$$

$$=q^{\alpha_1+\alpha_2-2}(1-q^{\beta_1})\left\langle\frac{\Omega}{t_2(1-t_1)}\right\rangle+q^{\alpha_1+\alpha_2-2}\left\{\frac{(1-q^{\beta_1})(1-q^{\beta_2})}{1-q^{\beta-1}}\langle\varphi_{T_1}\rangle\right.$$

$$\left.+\frac{(1-q^{\beta_1+\beta-1})(1-q^{\beta_2})}{1-q^{\beta-1}}\langle\varphi_{T_3}\rangle+q^{\beta-1}\frac{(1-q^{\beta_1})(1-q^{\beta_2-\beta+1})}{1-q^{1-\beta}}\langle\varphi_{T_5}\rangle\right\}.$$

From these two lemmas one can express

$$(6.30)\qquad \left\langle\frac{\Omega}{t_1(1-t_2)}\right\rangle,\quad \left\langle\frac{\Omega}{t_1(t_1-q^{\beta'}t_2)}\right\rangle,\quad \left\langle\frac{\Omega}{t_1(t_1-q^{\beta-1}t_2)}\right\rangle$$

and

$$(6.31)\qquad \left\langle\frac{\Omega}{t_2(1-t_1)}\right\rangle,\quad \left\langle\frac{\Omega}{t_2(t_1-q^{\beta'}t_2)}\right\rangle,\quad \left\langle\frac{\Omega}{t_2(t_1-q^{\beta-1}t_2)}\right\rangle$$

as linear combinations of $\langle\varphi_{T_1}\rangle$, $\langle\varphi_{T_2}\rangle$, $\langle\varphi_{T_3}\rangle$, $\langle\varphi_{T_4}\rangle$, and $\langle\varphi_{T_5}\rangle$. Since we have

(6.32) $$\tilde{Q}_1^{-1}\langle\varphi_{T_1}\rangle=\langle\varphi_{T_1}\rangle+\left\langle\frac{\Omega}{t_1(1-t_2)}\right\rangle,$$

(6.33) $$\tilde{Q}_1^{-1}\langle\varphi_{T_2}\rangle=\langle\varphi_{T_2}\rangle+\left\langle\frac{\Omega}{t_1(t_1-q^{\beta'}t_2)}\right\rangle,$$

(6.34) $$\tilde{Q}_1^{-1}\langle\varphi_{T_4}\rangle=-q^{-\beta'}\left\langle\frac{\Omega}{t_1(1-t_2)}\right\rangle+\left\langle\frac{\Omega}{t_1(t_1-q^{\beta'}t_2)}\right\rangle+q^{-\beta'}\langle\varphi_{T_4}\rangle,$$

(6.35) $$\tilde{Q}_2^{-1}\langle\varphi_{T_1}\rangle=q^{\alpha_2-1}\left\{\left\langle\frac{\Omega}{t_2(1-t_1)}\right\rangle+\frac{(1-q^{\beta})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_1}\rangle\right.$$
$$\left.+\frac{(q^{\beta'}-q^{\beta})(q^{\beta_2}-q^{\beta'})}{1-q^{\beta'}}\langle\varphi_{T_2}\rangle+\frac{(q^{\beta'}-q^{\beta})(1-q^{\beta_2})}{1-q^{\beta'}}\langle\varphi_{T_4}\rangle\right\},$$

(6.36) $$\tilde{Q}_2^{-1}\langle\varphi_{T_2}\rangle=\left\langle\frac{\Omega}{t_2(1-t_1)}\right\rangle+\left\langle\frac{\Omega}{t_2(t_1-q^{\beta'}t_2)}\right\rangle+q^{-\beta'}\langle\varphi_{T_2}\rangle,$$

(6.37) $$\tilde{Q}_2^{-1}\langle\varphi_{T_4}\rangle=\left\langle\frac{\Omega}{t_2(1-t_1)}\right\rangle+\left\langle\frac{\Omega}{t_2(t_1-q^{\beta-1}t_2)}\right\rangle+q^{1-\beta}\langle\varphi_{T_5}\rangle,$$

we can conclude:

Proposition 8. *$\tilde{Q}_1^{\pm1}\langle\varphi_{T_j}\rangle$ and $\tilde{Q}_2^{\pm1}\langle\varphi_{T_j}\rangle$, $1\le j\le 3$, are written as linear combinations of $\langle\varphi_{T_1}\rangle$, $\langle\varphi_{T_2}\rangle$, $\langle\varphi_{T_3}\rangle$, $\langle\varphi_{T_4}\rangle$, $\langle\varphi_{T_5}\rangle$, respectively.*

Since $\tilde{Q}_{\beta'}^{-1}$ and $\tilde{Q}_{\beta}$ are written by using $\tilde{Q}_1^{\pm1}$ and $\tilde{Q}_2$ as

(6.38) $$\tilde{Q}_{\beta'}^{-1}=\tilde{Q}_1^{-1}(\tilde{Q}_1-q^{\beta'-1}\tilde{Q}_2)\quad\text{for}\quad\langle\varphi_{T_1}\rangle,\ \langle\varphi_{T_3}\rangle,$$

(6.39) $$\tilde{Q}_{\beta'}^{-1}=\tilde{Q}_1^{-1}(\tilde{Q}_1-q^{\beta'}\tilde{Q}_2)\quad\text{for}\quad\langle\varphi_{T_2}\rangle,$$

(6.40) $$\tilde{Q}_{\beta}=\tilde{Q}_1^{-1}(\tilde{Q}_1-q^{\beta}\tilde{Q}_2)\quad\text{for}\quad\langle\varphi_{T_1}\rangle,\ \langle\varphi_{T_2}\rangle,$$

(6.41) $$\tilde{Q}_{\beta}=\tilde{Q}_1^{-1}(\tilde{Q}_1-q^{\beta-1}\tilde{Q}_2)\quad\text{for}\quad\langle\varphi_{T_3}\rangle,$$

we get the following:

Proposition 9. *$\tilde{Q}_{\beta'}^{-1}\langle\varphi_{T_j}\rangle$ and $\tilde{Q}_{\beta}\langle\varphi_{T_j}\rangle$, $1\le j\le 3$, are written explicitly as linear combinations of $\langle\varphi_{T_1}\rangle$, $\langle\varphi_{T_2}\rangle$, $\langle\varphi_{T_3}\rangle$, $\langle\varphi_{T_4}\rangle$ and $\langle\varphi_{T_5}\rangle$ through the formulae* (6.38)–(6.41). *The latter are expressible as linear combinations of $\langle\varphi_{T_1}\rangle$, $\langle\varphi_{T_2}\rangle$ and $\langle\varphi_{T_3}\rangle$ through* (6.5)–(6.6).

The formulae for $\tilde{Q}_i^{\pm1}$, $\tilde{Q}_{\beta}$ and $\tilde{Q}_{\beta'}^{-1}$ give *a complete system of contiguous relations* for the basic hypergeometric series ${}_3\varphi_2$.

Remark. To prove the Theorem we have used asymptotic behaviours of integrals. However it is desirable and is probably possible to give a *purely algebraic proof* of the

Theorem.

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