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REMOVABLE SINGULARITIES FOR HOLOMORPHIC VECTOR BUNDLES

Dedicated to Professor Hans Grauert on his sixtieth birthday

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In the pioneering work [U1], Uhlenbeck proved the following removable singularity theorem for Yang-Mills connections:

THEOREM. *Let A be a Yang-Mills connection on a bundle P over the punctured ball $\boldsymbol{B}^* = \boldsymbol{B} \setminus \{0\}$ in $\boldsymbol{R}^4$. If the square integral of the curvature tensor R_A of A is finite, that is,*

$$\int_{\boldsymbol{B}^*} |R_A|^2 < \infty ,$$

then the bundle P and the connection A extend smoothly to the whole ball $\boldsymbol{B}$.

As pointed out by Itoh [I], any Einstein-Hermitian connection of a holomorphic vector bundle on a Kähler surface is a Yang-Mills connection. Hence we get:

COROLLARY. *Let $(E, h) \to \boldsymbol{B}^*$ be an Einstein-Hermitian holomorphic vector bundle over the punctured ball $\boldsymbol{B}^* \subset \boldsymbol{C}^2$. If its curvature is square integrable, then (E, h) extends to the whole ball $\boldsymbol{B}$ as an Einstein-Hermitian holomorphic vector bundle.*

In a sense the assumption of the corollary is too strong. It assumes not only the Yang-Mills equation but also the equation comming from the holomorphy. So it would be natural to try to get rid of the Einstein condition. In this direction there are works by Cornalba-Griffiths [CG], Siu [S2] and Uhlenbeck [U2]. They assumed pointwise estimates of the curvature; boundedness or positivity. We here only assume that the curvature belongs to L^2 and get:

THEOREM 10. *Let $(E, h) \to \boldsymbol{B}^*$ be a Hermitian holomorphic vector bundle over the punctured ball $\boldsymbol{B}^* \subset \boldsymbol{C}^2$. If it satisfies*

$$\int_{\boldsymbol{B}^*} |R_h|^2 < \infty ,$$

the E extends to a holomorphic vector bundle $\bar{E}$ defined on the whole ball $\boldsymbol{B}$. Every holomorphic section of $\bar{E}$ is locally square integrable.

The idea of the proof is rather standard. First we show that E and its dual vector bundle E^* have sufficiently many holomorphic sections on $\boldsymbol{B}^*$ so that we can embed E into a trival vector bundle of sufficiently high rank. Then we extend E as a torsion free sheaf $\mathscr{E}$ over $\boldsymbol{B}$. Since the dimension of the base space is 2, the double dual $\mathscr{E}^{**}$ of $\mathscr{E}$ defines the desired vector bundle $\bar{E}$. The last statement of the theorem is an easy consequence of an analytical lemma.

We remark that the natural inclusions give the following isomorphisms: for an open set U,

$$\Gamma(\bar{E};\, U) \cong \{s \in \Gamma(E;\, U \cap \boldsymbol{B}^*) \mid s \text{ is locally square integrable}\} \cong \Gamma(E;\, U \cap \boldsymbol{B}^*)\,.$$

After this work was completed, Professor Y.-T. Siu pointed out that combined with his slicing theorem [S1, 3], Theorem 10 yields the following result:

THEOREM. *Let S be a closed subset of a ball $\boldsymbol{B}$ in $\boldsymbol{C}^n$ with a finite Hausdorff measure of real codimension 4 and (E, h) be a Hermitian holomorphic vector bundle defined on $\boldsymbol{B}\setminus S$. If its curvature is square integrable, then E uniquely extends to the whole ball $\boldsymbol{B}$ as a reflexive sheaf.*

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1. Extension of line bundles.

THEOREM 1. *Let S be an analytic subset of at least codimension 2 of the ball $\boldsymbol{B}$ in $\boldsymbol{C}^n$, and (L, h) be a holomorphic Hermitian line bundle defined on $\boldsymbol{B}\setminus S$. If the curvature ω of (L, h) is integrable, then L extends to the whole ball $\boldsymbol{B}$ as a holomorphic line bundle.*

PROOF. We define a (1,1)-current $\bar{\omega}$ on $\boldsymbol{B}$ as follows: For a smooth $2(n-1)$-form θ with compact support in $\boldsymbol{B}$,

$$\bar{\omega}(\theta) = \int_{\boldsymbol{B}\setminus S} \omega \wedge \theta\,.$$

If $S=\{0\}$, we take a sequence of radial cut-off functions η_ε such that $\eta_\varepsilon=0$ for $|z|<\varepsilon$, $=1$ for $|z|>2\varepsilon$, and $|d\eta_\varepsilon|<2\varepsilon^{-1}$. For θ with compact support we define a form θ_0 with constant coefficients which coincides with θ at the origin. Since $H^2(\boldsymbol{B}\setminus\{0\})=0$ and $d\theta_0=0$, we get

$$\bar{\omega}(d\eta_\varepsilon \wedge \theta_0) = 0\,.$$

Hence

$$\bar{\omega}(d\theta)=\lim_{\varepsilon\to 0}\bar{\omega}(\eta_\varepsilon d\theta)=\lim\bar{\omega}(d(\eta_\varepsilon\theta)-d\eta_\varepsilon\wedge\theta)=\lim\bar{\omega}(d\eta_\varepsilon\wedge(\theta_0-\theta))=0\,.$$

This meanes that $\bar{\omega}$ is d-closed. For general S we use the stratification of S by smooth subvarieties. Since along the strata of maximal dimension the situation for the transversal direction is the same as the case of $S=0$, we can use the above argument. By induction on the dimension of the strata, one can show the d-closedness of $\bar{\omega}$ in general. Thus there exists a (0, 0)-current u such that

$$\sqrt{-1}\partial\bar{\partial}u=\bar{\omega}\,.$$

The regularity theorem says that u is smooth on $\boldsymbol{B}\diagdown S$. If we replace the Hermitian metric h on L by he^u, then its curvature vanishes. This means that L comes from a representation of $\pi_1(\boldsymbol{B}\diagdown S)=\{1\}$. Thus L is a trivial line bundle on $\boldsymbol{B}\diagdown S$, which clearly extends to the whole ball $\boldsymbol{B}$ as a line bundle.

2. Solving $\bar{\partial}$-equations. From now on we work under the assumption of the theorem. We may assume that $\boldsymbol{B}=\{z\in\boldsymbol{C}^2\mid |z|^2<1\}$ and (E, h) is defined on a larger punctured ball.

Let ρ be a smooth $\bar{\partial}$-closed E-valued (0,1)-form which has compact support in $\boldsymbol{B}^*$. We are to solve the equation

$$\bar{\partial}u=\rho \qquad \text{on} \quad \boldsymbol{B}^*\,,$$

with $u\in H^1$, namely, u and its covariant derivative ∇u are square integrable. First we solve the $\bar{\partial}$-Neumann problem: with the formal adjoint ϑ of $\bar{\partial}$

$$\square\phi=(\bar{\partial}\vartheta+\vartheta\bar{\partial})\phi=\rho \qquad \text{on} \quad \boldsymbol{B}^*\,,$$

with $\phi\in H^1$ which satisfies the $\bar{\partial}$-Neumann condition at the boundary $\partial\boldsymbol{B}$. We need to specify a base metric and a fiber metric. We fix the base metric to be the standard Euclidean metric and the fiber metric to be $h_K=he^{-K|z|^2}$ with a sufficiently large constant K to be chosen later.

For a small number $\varepsilon>0$, we solve the Dirichlet-$\bar{\partial}$-Neumann problem on $\boldsymbol{B}_\varepsilon^*=\{z\in\boldsymbol{C}^2\mid \varepsilon<|z|^2<1\}$, i.e., we put the $\bar{\partial}$-Neumann condition on $\{|z|^2=1\}$ and the Dirichlet condition on $\{|z|^2=\varepsilon\}$.

LEMMA 2. *If we take K large enough, then for a section ϕ which satisfies the Dirichlet-$\bar{\partial}$-Neumann condition, we get*

$$(\square\phi,\phi)=\|\bar{\partial}\phi\|^2+\|\vartheta\phi\|^2\geq\|\phi\|^2\,.$$

In particular, we can solve the equation $\square\phi=\rho$, with $\|\phi\|$, $\|\vartheta\phi\|$, $\|\bar{\partial}\phi\|\leq\|\rho\|$.

PROOF. Let $R_K=R_h+K$ be the curvature tensor of the metric h_K, and $0\leq\eta\leq 1$ be a cut-off function which is equal to 1 near the origin. Then,

$$(\Box\phi, \phi) = \|\bar{\partial}\phi\|^2 + \|\vartheta\phi\|^2 \geq \int |\nabla^{0,1}\phi|^2 + (R_K\phi, \phi)$$

$$\geq \int \left\{ \frac{1}{2} |\nabla^{0,1}(\eta\phi)|^2 - |\nabla^{0,1}\eta|^2 |\phi|^2 \right\} + (R_K\phi, \phi)$$

$$= \int \left\{ \frac{1}{4} (|\nabla^{0,1}(\eta\phi)|^2 + |\nabla^{1,0}(\eta\phi)|^2) - |\nabla^{0,1}\eta|^2 |\phi|^2 \right\}$$

$$- \frac{1}{4} (\eta^2 \operatorname{tr} R_K\phi, \phi) + (R_K\phi, \phi)$$

$$\geq \int \left\{ \frac{1}{4} |\nabla(\eta\phi)|^2 + \frac{K}{2} |\phi|^2 - |\nabla^{0,1}\eta|^2 |\phi|^2 \right\}$$

$$- \frac{1}{4} (\eta^2 \operatorname{tr} R_h\phi, \phi) + (R_h\phi, \phi),$$

where tr is taken in the form part. Since $\eta\phi$ has compact support, we can apply the Sobolev inequality and get, with a positive constant S,

$$\left(\int |\eta\phi|^4 \right)^{1/2} \leq S \int |\nabla(\eta\phi)|^2 .$$

Choose η in such a way that its support is so small that

$$\int_{\operatorname{supp}\eta} |R_h|^2 \leq \frac{1}{64S^2},$$

and take K large enough. Then we get

$$(\Box\phi, \phi) \geq \int \left(\frac{K}{2} - |\nabla^{0,1}\eta|^2 \right) |\phi|^2 + ((1-\eta^2)R_h\phi, \phi) \geq \|\phi\|^2 .$$

Thus the Dirichlet-$\bar{\partial}$-Neumann problem has solutions with the desired properties. (cf. [FK].)

Letting $\varepsilon \to 0$ we get:

LEMMA 3. *We have a solution $\phi \in H^1$ of the $\bar{\partial}$-Neumann problem on $\boldsymbol{B}^*$.*

By Moser's iteration argument one can get the following lemma. (cf. [BKN, Lemma (5.8), Lemma (5.9)].)

LEMMA 4. *Let f be a square integrable non-negative function on $\boldsymbol{B}^*$, and u be a locally H^1 non-negative function on $\boldsymbol{B}^*$ such that with a positive constant c*

$$\Delta u \geq -fu - c, \qquad on \quad \boldsymbol{B}^*,$$

where $\Delta=\sum\partial^2/(\partial x^i)^2$. *If* $u\in H^1$ *or* $\int_{B(r)}u^p=o(r^2)$ *as* $r\to 0$ *with* $p>1$, *where* $B(r)=\{|z|<r\}$, *then we have* $u\in L^q$ *for all* $q>1$ *on* $B(1/2)$.

The equation $\square\phi=\rho$ implies $\Delta\phi=2R_h\phi-\operatorname{tr}R_h\phi-2\rho$, hence $\Delta|\phi|\geq -4|R_h||\phi|-2|\rho|$. The lemma yields that $\phi\in L^q$ for all $q>1$. By integration by parts with a cut-off function η we get

$$\int|\nabla(\eta\phi)|^2\leq 4\left(\int|R_h|^2\right)^{1/2}\left(\int|\eta\phi|^4\right)^{1/2}+2\left(\int|\eta\rho|^2\right)^{1/2}\left(\int|\eta\phi|^2\right)^{1/2}$$
$$+\left(\int|\nabla\eta|^4\right)^{1/2}\left(\int_{\operatorname{supp}\nabla\eta}|\phi|^4\right)^{1/2}.$$

As $\phi\in L^q$ for any $q>1$, the Hölder inequality gives $\int_{B(r)}|\phi|^4=O(r^{4-2\delta})$ for any positive δ. Thus $\int_{B(r)}|\nabla\phi|^2=O(r^{2-\delta})$. Taking $\bar\partial$ of the equation $\square\phi=\rho$, we get $0=\bar\partial\square\phi=\bar\partial\vartheta\bar\partial\phi=\square\bar\partial\phi$, and $\Delta|\bar\partial\phi|\geq -2|R_K||\bar\partial\phi|$. Applying Lemma 4 with $u=|\bar\partial\phi|$ and $1<p<2$, we get $\bar\partial\phi\in L^q$ for any $q>1$. Taking a cut-off function $\eta=\eta_r$ such that $\eta(z)=1$ for $|z|>2r$, $=0$ for $|z|<r$ and $|\nabla\eta|<2/r$, we get

$$0=(\bar\partial\vartheta\bar\partial\phi,\eta^2\bar\partial\phi)=(\vartheta\bar\partial\phi,\vartheta(\eta^2\bar\partial\phi))=\|\eta\vartheta\bar\partial\phi\|^2+2(\eta\vartheta\bar\partial\phi,\nabla^{1,0}\eta*\bar\partial\phi),$$

hence

$$\|\eta\vartheta\bar\partial\phi\|^2\leq 4\left(\int|\nabla\eta|^3\right)^{2/3}\left(\int|\bar\partial\phi|^6\right)^{1/3}\to 0\qquad\text{as}\quad r\to 0\,.$$

Thus putting $u=\vartheta\phi$ we get:

LEMMA 5. *For a given smooth $\bar\partial$-closed* (0, 1)*-form* ρ *with compact support in* $\boldsymbol{B}^*$, *we can solve the $\bar\partial$-equation*

$$\bar\partial u=\rho\qquad\text{on}\quad \boldsymbol{B}^*\,,$$

with $\|u\|\leq\|\rho\|$.

Let s be a holomorphic section of E defined in a neighborhood of $z_0\in\boldsymbol{B}^*$, and η be a cut-off function with compact support in $\boldsymbol{B}^*$ which is equal to 1 in a neighborhood of z_0 and that ηs makes a smooth section on $\boldsymbol{B}^*$ by putting $\eta s=0$ where s is not defined. Then $\rho=\bar\partial(\eta s)$ is a $\bar\partial$-closed (0, 1)-form which vanishes in a neighborhood of z_0. We take a pluri-subharmonic function $w(z)=\log|z-z_0|^2$ as a weight function, and use a fiber metric $he^{-K|z|^2-2w}$ instead of $he^{-K|z|^2}$ in the proof of Lemma 5. Since $\rho\in L^2$ in the new metric, the solution u also belongs to the L^2-space. This means that u vanishes at z_0. Thus $\eta s-u$ is a holomorphic L^2-section (with respect to the original metric h) of E on $\boldsymbol{B}^*$ which is equal to s at z_0.

LEMMA 6. *For any point* $z_0\in\boldsymbol{B}^*$ *we can find a family* $\{s_1,s_2,\cdots,s_r\}$, $r=\operatorname{rank}E$, *of holomorphic L^2-sections on* $\boldsymbol{B}^*$ (L^2 *with respect to the given metric* h) *which gives a* *base of E_{z_0} at z_0.*

LEMMA 7. *There exists a family $\{s_1, s_2, \cdots, s_m\}$ of a finite number of holomorphic L^2-sections on $\boldsymbol{B}^*$ such that $\{s_1(z), s_2(z), \cdots, s_m(z)\}$ spans E_z for each point $z \in B(1/2)^*$.*

PROOF. Theorem 1 says that the determinant line bundle $\det E$ extends to the whole ball as a holomorphic line bundle, which we continue to call $\det E$. Fix a point $z_0 \in \boldsymbol{B}^*$ and construct holomorphic L^2-sections $s_1, s_2, \cdots, s_r$ on $\boldsymbol{B}^*$ which give a base at z_0. Then $\sigma = s_1 \wedge s_2 \wedge \cdots \wedge s_r$ gives a section of $\det E$ on $\boldsymbol{B}^*$. Since by Hartogs' theorem σ extends to $\boldsymbol{B}$ as a holomorphic section, which we continue to call σ, the divisor $(\sigma) = \{z \in \boldsymbol{B} \mid \sigma(z) = 0\}$ has finitely many irreducible components D_i $(i = 1, \cdots, l)$ in $B(4/5)$. Take a point $z_i \in B(4/5)^*$ in each component D_i, and construct a family $\{s_{i,1}, s_{i,2}, \cdots, s_{i,r}\}$ of holomorphic L^2-sections on $\boldsymbol{B}^*$, which spans E_{z_i} at z_i. Then $\{s_1, s_2, \cdots, s_r, s_{i,1}, s_{i,2}, \cdots, s_{i,r} (i = 1, \cdots, l)\}$ spans E in $B(3/5)^*$ except at a finite number of points $\{z'_j\}$. Again we construct a finite number of holomorphic L^2-sections $\{s'_1, s'_2, \cdots, s'_{l'}\}$ on $\boldsymbol{B}^*$ to make them span E at $\{z'_j\}$. Then $\{s_1, s_2, \cdots, s_r, s_{i,1}, s_{i,2}, \cdots, s_{i,r}, s'_1, s'_2, \cdots, s'_{l'}\}$ is the desired family.

Since E^* also have the square integrable curvature, we have:

LEMMA 8. *There exists a family $\{t_1, t_2, \cdots t_n\}$ of a finite number of holomorphic L^2-sections on $\boldsymbol{B}^*$ such that $\{t_1(z), t_2(z), \cdots, t_n(z)\}$ spans E^*_z for each point $z \in B(1/2)^*$.*

3. Extension of holomorphic vector bundles. We embed the vector bundle $E|_{\boldsymbol{B}^*}$, $\boldsymbol{B} = \boldsymbol{B}(1/2)$ into the trivial vector bundle $\boldsymbol{C}^n$ by

$$E \ni s \mapsto (\langle s, t_1\rangle, \langle s, t_2\rangle, \cdots, \langle s, t_n\rangle) \in \boldsymbol{C}^n .$$

Then it is generated by the images $\{\tilde{s}_i\}$ of $\{s_i\}_{i=1}^m$. By Hartogs' theorem $\tilde{s}_i$ extends to the whole ball $\boldsymbol{B}$ as a holomorphic section. We define a coherent subsheaf $\mathscr{E}$ of $\boldsymbol{C}^n$ on $\boldsymbol{B}$ to be the one generated by $\{\tilde{s}_i\}$. Since $\dim \boldsymbol{B} = 2$, the double dual $\mathscr{E}^{**}$ of $\mathscr{E}$, which coincides with $\mathscr{E}$ except at the origin, comes from a holomorphic vector bundle $\bar{E}$. Then there exists a non-zero polynomial P such that for any holomorphic section $\tilde{s}$ of $\bar{E}$, $P\tilde{s}$ belongs to $\mathscr{E}$. This implies that the restriction of $P\tilde{s}$ to $\boldsymbol{B}^*$ is square integrable with respect to the metric h. We fix an arbitrary smooth fiber metric $\bar{h}$ on $\bar{E}$. Then $\log^+ \operatorname{tr}_{\bar{h}} h$ belongs to the L^q-space for any $q > 1$. A calculation shows

$$\Delta \log^+ \operatorname{tr}_{\bar{h}} h \geq -2(|\operatorname{tr} R_h| + |\operatorname{tr} R_{\bar{h}}|) .$$

We solve the equation

$$\Delta v = -2(|\operatorname{tr} R_h| + |\operatorname{tr} R_{\bar{h}}|) \in L^2 , \qquad v|_{\partial B} = \log^+ \operatorname{tr}_{\bar{h}} h|_{\partial B} .$$

Then we get $v \in H^2$ and $\log^+ \operatorname{tr}_{\bar{h}} h \leq v$. We apply the following lemma to see that $\operatorname{tr}_{\bar{h}} h$ belongs to the L^q-space for any $q > 1$.

LEMMA 9. *Let v be a function in the H^2-space on a real 4-dimensional ball. Then*

$\exp v$ *belongs to the* L^q*-space for any* $q > 1$.

Thus we complete the proof of Theorem 10 in the introduction.

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