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A CHARACTERIZATION OF CERTAIN DOMAINS WITH GOOD BOUNDARY POINTS IN THE SENSE OF GREENE-KRANTZ, II

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Introduction. This is a continuation of our previous paper [12], and we retain the terminology and notation there.

In connection with Rosay-type theorems as in [17], [5] and [10], we shall study in this paper the bounded Reinhardt domain $E(p_1, \cdots, p_n)$ in $\boldsymbol{C}^n$ defined by

$$E(p_1, \cdots, p_n) = \{(z_1, \cdots, z_n) \in \boldsymbol{C}^n \mid |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\} ,$$

where $p_1, \cdots, p_n$ are arbitrarily given positive real numbers. Here we would like to emphasize that the domain $E(p_1, \cdots, p_n)$ is not geometrically convex and moreover its boundary is not smooth in general.

In the special case where all p_i's are positive integers, the domain $E(p_1, \cdots, p_n)$ has C^ω-smooth boundary and the correspondence $(z_1, \cdots, z_n) \mapsto ((z_1)^{p_1}, \cdots, (z_n)^{p_n})$ gives rise to a single-valued holomorphic proper mapping from $E(p_1, \cdots, p_n)$ onto the unit ball B^n in $\boldsymbol{C}^n$. In the previous paper [12], by making use of these facts and some extension theorems of holomorphic mappings due to Rudin [18; p. 311] and Forstnerič-Rosay [4], we succeeded in characterizing the domain $E(p_1, \cdots, p_n)$ with $0 < p_i \in \boldsymbol{Z}$ $(1 \leq i \leq n)$ from the viewpoint of biholomorphic automorphism group. However, if some of p_i's are not integers, then $\partial E(p_1, \cdots, p_n)$ is not smooth and the above correspondence is not even a single-valued holomorphic mapping defined on the whole domain $E(p_1, \cdots, p_n)$. These raise new difficulties to characterize the domain. The main purpose of this paper is to overcome these difficulties and show that exactly the same characterization of the domain $E(p_1, \cdots, p_n)$ as in [12] remains valid for arbitrary real numbers $p_1, \cdots, p_n > 0$. In fact, after some preparations in Section 1, we can prove the following theorem in Section 2, which was announced at the 1989 AMS Summer Institute on Several Complex Variables and Complex Geometry in Santa Cruz, California:

THEOREM I. *Let D be a bounded domain in $\boldsymbol{C}^n$ $(n > 1)$ with a point $z^0 = (z_1^0, \cdots, z_n^0) \in \partial D$. After renumbering the coordinates if necessary, we assume that the following three conditions are satisfied:*

(1) *There exist an integer $k \geq 0$, real numbers p_i with $0 < p_i \neq 1$ $(k+1 \leq i \leq n)$ and an open neighborhood U^0 of z^0 in $\boldsymbol{C}^n$ such that*

(i) *$z^0 \in \partial E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ and*

(ii) $D \cap U^0 = E(1, \cdots, 1, p_{k+1}, \cdots, p_n) \cap U^0$,
where it is understood that $E(1, \cdots, 1, p_{k+1}, \cdots, p_n) = B^n$ *if* $k = n$.

(2) $\#\{i \in Z \mid z_i^0 \neq 0,\ 1 \leq i \leq n\} = j$, *where* $\#$ *denotes the cardinality of a set.*

(3) *The point* z^0 *is a good boundary point of* D *in the sense of Greene-Krantz* [6], *that is, there exist a point* $k^0 \in D$ *and a sequence* $\{\varphi_\nu\}$ *in* $\mathrm{Aut}(D)$ *such that* $\lim_{\nu \to \infty} \varphi_\nu(k^0) = z^0$.

Then we have $1 \leq j \leq k$ *and* $D = E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ *as sets; hence* z^0 *is necessarily of the form* $z^0 = (z_1^0, \cdots, z_k^0, 0, \cdots, 0)$.

As an immediate consequence of this, we obtain the following:

COROLLARY. *For arbitrary real numbers* p_i *with* $0 < p_i \neq 1$ $(1 \leq i \leq n)$, *any bounded domain* D *in* $\boldsymbol{C}^n$ *with a point* $z^0 \in \partial D \cap \partial E(p_1, \cdots, p_n)$ *near which* ∂D *coincides with* $\partial E(p_1, \cdots, p_n)$ *cannot have any* $\mathrm{Aut}(D)$*-orbits accumulating at* z^0.

Clearly this gives an affirmative answer to the following conjecture of Greene-Krantz [6; p. 200]: *Let* x_0 *be a boundary point of the domain* $\Omega_0 = \{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1|^4 + |z_1|^4 < 1\}$. *Then any weakly pseudoconvex bounded domain* Ω *in* $\boldsymbol{C}^2$ *with* $x_0 \in \partial\Omega$ *near which* $\partial\Omega$ *coincides with* $\partial\Omega_0$ *cannot have any* $\mathrm{Aut}(\Omega)$*-orbits accumulating at* x_0.

Next we assume that a complex manifold M can be exhausted by biholomorphic images of a complex manifold N, that is, for any compact subset K of M there exists a biholomorphic mapping f_K from N into M such that $K \subset f_K(N)$. Then, how can we describe M using the data of N? There already exist articles related closely to this problem; for instance, Fornaess-Sibony [2], Fornaess-Stout [3], Fridman [1] and Kodama [10], [11]. In particular, we showed in [11] that *if a hyperbolic manifold* M *in the sense of Kobayashi* [9] *can be exhausted by biholomorphic images of the pseudoconvex domain*

$$E(k, \alpha) = \left\{ z \in \boldsymbol{C}^n \,\Big|\, \sum_{i=1}^{k} |z_i|^2 + \left(\sum_{j=k+1}^{n} |z_j|^2 \right)^{\alpha} < 1 \right\},$$

then M *is biholomorphically equivalent either to* $E(k, \alpha)$ *or to* B^n. The following theorem tells us that the analogue is still valid for the domain $E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ with arbitrary positive real numbers $p_{k+1}, \cdots, p_n \neq 1$.

THEOREM II. *Let* M *be a hyperbolic manifold of complex dimension* n *in the sense of Kobayashi* [9]. *Assume that* M *can be exhausted by biholomorphic images of the domain* $E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$. *Then there exists a subset* $\{q_{l+1}, \cdots, q_n\}$ *of* $\{p_{k+1}, \cdots, p_n\}$ *such that* M *is biholomorphically equivalent to the domain* $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$.

In particular, considering the special case $k = n$, i.e., $E(1, \cdots, 1, p_{k+1}, \cdots, p_n) = B^n$ in Theorem II, we see that if a hyperbolic manifold M can be exhausted by biholomorphic images of B^n, then M is biholomorphically equivalent to B^n. This was first proved by Fornaess-Stout [3].

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1. Preliminaries. In this section we shall recall the structure of $E(p_1, \cdots, p_n)$ with arbitrarily given real numbers $p_1, \cdots, p_n > 0$. To simplify notation, we set $E = E(p_1, \cdots, p_n)$ for the time being.

Recall first that a domain D in $\boldsymbol{C}^n$ is called a *Reinhardt domain* if $((\exp\sqrt{-1}\theta_1)z_1, \cdots, (\exp\sqrt{-1}\theta_n)z_n) \in D$ whenever $(z_1, \cdots, z_n) \in D$ and $(\theta_1, \cdots, \theta_n) \in \boldsymbol{R}^n$. Moreover, it is said to be *complete* if $(z_1, \cdots, z_n) \in D$, $w = (w_1, \cdots, w_n) \in \boldsymbol{C}^n$ and $|w_i| \leq |z_i|$ $(1 \leq i \leq n)$ imply $w \in D$. We now assert that:

(1.1) *E is a taut domain in the sense of Wu* [21].

By results of Kiernan [8] and Pflug [15], it suffices to show that E is a bounded complete Reinhardt domain which is pseudoconvex. Since E is obviously a bounded complete Reinhardt domain, we have only to check that the domain

$$B = \{(x_1, \cdots, x_n) \in \boldsymbol{R}^n \mid (\exp x_1, \cdots, \exp x_n) \in E\}$$

is geometrically convex in $\boldsymbol{R}^n$ (cf. [13; p. 120]). To do so, let us take arbitrary points $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ of B and arbitrary real numbers $\lambda, \mu > 0$ such that $\lambda + \mu = 1$. Then, by using Hölder's inequality, we obtain the following:

$$\sum_{i=1}^{n} \exp[2p_i(\lambda x_i + \mu y_i)] \leq \left[\sum_{i=1}^{n} \exp(2p_i x_i)\right]^{\lambda} \cdot \left[\sum_{i=1}^{n} \exp(2p_i y_i)\right]^{\mu} < 1 ,$$

which shows $\lambda x + \mu y \in B$. Thus B is convex, as desired.

(1.2) *For an arbitrarily given point $x = (x_1, \cdots, x_n) \in \partial E$, there exists a local holomorphic peaking function h_x for x of E.*

Since $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ by the definition of E, we have the following cases:

(I) $x_1 \cdots x_n \neq 0$;

(II) $x_1 \cdots x_n = 0$, $x_l \neq 0$ for some index l.

Let us consider the first case (I). Expressing each x_i $(1 \leq i \leq n)$ as

$$x_i = r_i^0 \cdot \exp(\sqrt{-1}\theta_i^0) \qquad \text{with} \quad r_i^0 > 0 , \qquad 0 \leq \theta_i^0 < 2\pi ,$$

we set $\Theta_i = \max\{1, 2p_i\}$ and

$$V_i = \{z_i = r_i \cdot \exp(\sqrt{-1}\theta_i) \mid r_i > 0, |\theta_i - \theta_i^0| < \pi/\Theta_i\} ;$$

$$V(x) = V_1 \times \cdots \times V_n ;$$

$$\varphi_i(z_i) = (r_i^0 r_i)^{p_i} \cdot \exp[\sqrt{-1}p_i(\theta_i - \theta_i^0)], \qquad z_i = r_i \cdot \exp(\sqrt{-1}\theta_i) \in V_i ;$$

$$h_x(z) = \frac{1}{2}\left[1 + \sum_{i=1}^{n} \varphi_i(z_i)\right], \qquad z = (z_1, \cdots, z_n) \in V(x) .$$

Then $V(x)$ is an open neighborhood of x in $\boldsymbol{C}^n$ and each φ_i is a holomorphic function on V_i; hence h_x is a holomorphic function on $V(x)$. (Note that $\varphi_i(z_i)$ is nothing but a branch of $(\bar{x}_i z_i)^{p_i}$ on the simply connected domain $V_i \subset \boldsymbol{C}$.) We now assert that h_x provides a desired peaking function for x. Indeed, it is clear that $h_x(x)=1$. Moreover, since

$$\sum_{i=1}^{n} |\varphi_i(z_i)| = \sum_{i=1}^{n} (r_i^0 r_i)^{p_i} \leq \left[\sum_{i=1}^{n} (r_i)^{2p_i}\right]^{1/2} \leq 1 \tag{1.3}$$

for all $z=(z_1, \cdots, z_n) \in \bar{E} \cap V(x)$, it is obvious that

$$|h_x(z)| \leq 1 \qquad \text{for all} \quad z \in \bar{E} \cap V(x)\,.$$

Therefore, assuming

$$|h_x(y)| = 1 \qquad \text{for some point} \quad y \in \bar{E} \cap V(x)\,,$$

we have only to verify that $y=x$. To this end, write $y=(y_1, \cdots, y_n)$ and $y_i = r_i \cdot \exp(\sqrt{-1}\theta_i)$ for $i=1, \cdots, n$. Then it follows from (1.3) that $\sum_{i=1}^{n} \varphi_i(y_i) = \sum_{i=1}^{n} |\varphi_i(y_i)| = 1$. Clearly this implies that $\theta_i = \theta_i^0$ for $i=1, \cdots, n$ and moreover

$$\sum_{i=1}^{n} [(r_i^0)^{p_i}]^2 = \sum_{i=1}^{n} [(r_i)^{p_i}]^2 = \sum_{i=1}^{n} (r_i^0)^{p_i} \cdot (r_i)^{p_i} = 1\,,$$

which gives $r_i = r_i^0$ for all $i=1, \cdots, n$. Thus $y=x$, as desired.

Next consider the case (II). Let $I := \{i_1, \cdots, i_m\}$ be the proper subset of $J := \{1, \cdots, n\}$ such that

$$x_{i_1} \cdots x_{i_m} \neq 0, \quad \text{while} \quad x_i = 0 \qquad \text{for all} \quad i \in J \setminus I\,.$$

Then the same computation as above tells us that the function

$$h_x(z) = \frac{1}{2}\left[1 + \sum_{j=1}^{m} \varphi_{i_j}(z_{i_j})\right]$$

gives a local holomorphic peaking function for x of E.

Finally, for an integer m with $0 \leq m \leq n$, we consider the function

$$\rho(z) = -1 + \sum_{i=1}^{m} |z_i|^2 + \sum_{i=m+1}^{n} |z_i|^{2p_i}\,, \qquad z=(z_1, \cdots, z_n) \in \boldsymbol{C}^n\,.$$

Let $x=(x_1, \cdots, x_n) \in \partial E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$ be an arbitrary point with $x_{m+1} \cdots x_n \neq 0$. Then, a routine calculation gives the following:

(1.4) *$\rho(z)$ is real analytic near x and $d\rho(x) \neq 0$; and*

$$\sum_{i,j=1}^{n} [\partial^2 \rho(x)/\partial z_i \partial \bar{z}_j] \xi_i \bar{\xi}_j = \sum_{i=1}^{m} |\xi_i|^2 + \sum_{i=m+1}^{n} (p_i)^2 |x_i|^{2(p_i-1)} |\xi_i|^2 \tag{1.5}$$

for all $(\xi_1, \cdots, \xi_n) \in \boldsymbol{C}^n$. Therefore $\partial E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$ is C^ω-smooth and strictly pseudoconvex at the point x.

Combining the above with [12; Theorem A], we have obtained the following:

THEOREM A. *For arbitrarily given real numbers* $p_1, \cdots, p_n > 0$, *the domain* $E(p_1, \cdots, p_n)$ *has the following properties*:

(1) $E(p_1, \cdots, p_n)$ *is taut in the sense of Wu* [21].

(2) *For an arbitrary point* $x \in \partial E(p_1, \cdots, p_n)$, *there exists a local holomorphic peaking function* h_x *for* x *of* $E(p_1, \cdots, p_n)$.

(3) *Assume that* $0 < p_1 \leq \cdots \leq p_n$ *and* $0 < q_1 \leq \cdots \leq q_n$. *Then* $E(p_1, \cdots, p_n)$ *is biholomorphically equivalent to* $E(q_1, \cdots, q_n)$ *if and only if* $p_i = q_i$ *for all* $i = 1, \cdots, n$ (*cf.* [14]).

(4) $\partial E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$ *is* C^ω-*smooth and strictly pseudoconvex at every point* $x = (x_1, \cdots, x_n) \in \partial E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$ *with* $x_{m+1} \cdots x_n \neq 0$ $(0 \leq m \leq n)$.

(5) *Assume that* $m \geq 1$. *Let* $\{x^\nu\}$ *be a sequence in* $E := E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$ *converging to a boundary point*

$$x^0 = (x_1^0, \cdots, x_n^0) \qquad \textit{with} \quad (x_{m+1}^0, \cdots, x_n^0) = (0, \cdots, 0)\,.$$

Then there exists a sequence $\{\psi_\nu\}$ *in* $\operatorname{Aut}(E)$ *such that*

$$\psi_\nu(x^\nu) = (0, \cdots, 0, y_{m+1}^\nu, \cdots, y_n^\nu)\,, \qquad 0 \leq y_{m+1}^\nu, \cdots, y_n^\nu < 1$$

for all ν. *Moreover, the inverse mapping* ψ_ν^{-1} *of each* ψ_ν *can be expressed as*

$$\psi_\nu^{-1}(z', z'') = (A^\nu(z'), B^\nu(z', z''))\,, \qquad (z', z'') \in (\boldsymbol{C}^m \times \boldsymbol{C}^{n-m}) \cap E\,,$$

where $A^\nu \in \operatorname{Aut}(B^m)$ *and* $B^\nu \colon E \to \boldsymbol{C}^{n-m}$ *is a suitable holomorphic mapping* (*cf.* [7], [19], [12; (1.2)]).

2. Proofs of Theorems I and II. Our proofs are based on the normal family arguments developed in our previous papers [11], [12]. Although there are some overlaps with those papers, we carry out the proofs in detail for the sake of completeness and self-containedness.

Before undertaking the proofs of the theorems, we need a preparation. Let $p_{k+1}, \cdots, p_n > 0$ be the real numbers appearing in the theorems. For an integer m with $k+1 \leq m \leq n$, consider the correspondence $h_{(1,\cdots,1,p_{k+1},\cdots,p_m,1,\cdots,1)}$ defined by

$$(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_k, (z_{k+1})^{p_{k+1}}, \cdots, (z_m)^{p_m}, z_{m+1}, \cdots, z_n)\,.$$

If all the p_i's are integers, this is a single-valued holomorphic mapping from $\boldsymbol{C}^n$ onto itself. However, if some of the p_i's are irrationals, then it provides an infinitely-many-valued holomorphic mapping from $(\boldsymbol{C} \setminus \{0\})^n$ onto itself. We thus need to introduce the concept of principal branch of $h_{(1,\cdots,1,p_{k+1},\cdots,p_m,1,\cdots,1)}$. For this purpose, let us fix an arbitrary point

$$x^0=(x_1^0, \cdots, x_n^0)\in \boldsymbol{C}^n \qquad \text{with} \quad x_{k+1}^0\cdots x_m^0\neq 0 .$$

Write each x_i^0 $(k+1\leq i\leq m)$ in the form

$$x_i^0=r_i^0\cdot\exp(\sqrt{-1}\theta_i^0) \qquad \text{with} \quad r_i^0>0 , \quad 0\leq\theta_i^0<2\pi$$

and set

$$\text{(2.1)} \qquad W_i(x_i^0)=\{z_i=r_i\cdot\exp(\sqrt{-1}\theta_i)\,|\,r_i>0, |\,\theta_i-\theta_i^0\,|<\pi\}=\boldsymbol{C}\diagdown\{sx_i^0\,|\,s\leq 0\} ;$$

$$\text{(2.2)} \qquad W(x^0)=\boldsymbol{C}^k\times W_{k+1}(x_{k+1}^0)\times\cdots\times W_m(x_m^0)\times \boldsymbol{C}^{n-m} ;$$

$$\text{(2.3)} \qquad H_i(z_i)=(r_i)^{p_i}\cdot\exp(\sqrt{-1}p_i\theta_i) , \qquad z_i=r_i\cdot\exp(\sqrt{-1}\theta_i)\in W_i(x_i^0) ;$$

$$\text{(2.4)} \quad H_{(1,\cdots,1,p_{k+1},\cdots,p_m,1,\cdots,1)}(z)=(z_1, \cdots, z_k, H_{k+1}(z_{k+1}), \cdots, H_m(z_m), z_{m+1}, \cdots, z_n)$$

for all $z=(z_1, \cdots, z_n)\in W(x^0)$.

Then $W(x^0)$ is an open neighborhood of x^0 and each H_i is a single-valued holomorphic function on $W_i(x_i^0)$; hence $H_{(1,\cdots,1,p_{k+1},\cdots,p_m,1,\cdots,1)}$ is a single-valued holomorphic mapping from $W(x^0)$ into $\boldsymbol{C}^n$. Moreover, it is injective on a small open neighborhood of x^0, since its Jacobian determinant does not vanish at x^0.

DEFINITION. We call $H_{(1,\cdots,1,p_{k+1},\cdots,p_m,1,\cdots,1)}$: $W(x^0)\to\boldsymbol{C}^n$ the *principal branch of* $h_{(1,\cdots,1,p_{k+1},\cdots,p_m,1,\cdots,1)}$ *on* $W(x^0)$.

2.1. Some lemmas. The following lemma can be proved with exactly the same argument as in Rudin [18; Lemma 15.2.2., p. 306]:

LEMMA 1. *Assume that*

(1) *X is a domain in $\boldsymbol{C}^m$ and $\{x^i\}_{i=1}^{\infty}$ is a sequence in X converging to some point $x_0\in X$;*

(2) *Y is a bounded domain in $\boldsymbol{C}^n$ and y_0 is a boundary point of Y, for which there exists a local holomorphic peaking function;*

(3) *$\{F_i\}_{i=1}^{\infty}$ is a sequence of holomorphic mappings from X into Y, and $F_i(x^i)\to y_0$ as $i\to\infty$.*

Then $F_i(x)\to y_0$ uniformly on every compact subset of X.

From now on D, $z^0\in U^0$, k, j, p_i, $\cdots$ are as in Theorem I.

LEMMA 2. *The domain D is biholomorphically equivalent either to B^n or to $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ with $\{q_{l+1}, \cdots, q_n\}\subset\{p_{k+1}, \cdots, p_n\}$, where $1\leq\max\{k, j\}\leq l\leq n-1$.*

PROOF. We set $E=E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ for the sake of simplicity. Thanks to Lemma 1 and (2) of Theorem A, $\{\varphi_\nu\}$ converges uniformly on compact subsets of D to the constant mapping C_{z^0}: $D\to\boldsymbol{C}^n$ defined by $C_{z^0}(z)=z^0$ for all $z\in D$. Now we fix a family of relatively compact subdomains D_μ of D such that

(2.5) $$D=\bigcup_{\mu=1}^{\infty} D_\mu \supset \cdots \supset D_{\mu+1} \supset D_\mu \supset \cdots \supset D_1 \ni k^0$$

and choose an arbitrary integer $\mu \geq 1$. Then, since $\varphi_\nu(z) \to z^0$ uniformly on D_μ, there is an integer $\nu(\mu)$ such that

(2.6) $$\varphi_\nu(D_\mu) \subset D \cap U^0 = E \cap U^0 \qquad \text{for all} \quad \nu \geq \nu(\mu) .$$

We first assume $k=n$, so that $E=B^n$. Then z^0 is a C^ω-smooth and strictly pseudoconvex boundary point of D; and consequently, by a well-known result of Rosay [17] D is biholomorphically equivalent to B^n.

Thus it suffices to prove Lemma 2 when $0 \leq k \leq n-1$. We have two cases to consider:

Case I. $(z_{k+1}^0, \cdots, z_n^0)=(0, \cdots, 0)$.

First notice that $z^0=(z_1^0, \cdots, z_n^0) \neq (0, \cdots, 0)$; hence $1 \leq j \leq k$ in this case. Now, by virtue of (5) of Theorem A, there exists a sequence $\{\psi_\nu\}$ in Aut(E) such that each point $\psi_\nu(\varphi_\nu(k^0))$, $\nu \geq \nu(\mu)$, can be expressed as

$$\psi_\nu(\varphi_\nu(k^0))=(0, \cdots, 0, u_{k+1}^\nu, \cdots, u_n^\nu) , \qquad 0 \leq u_{k+1}^\nu, \cdots, u_n^\nu < 1 .$$

Define the biholomorphic mappings $f^\nu: D_\mu \to E$ from D_μ into E by

$$f^\nu = \psi_\nu \circ (\varphi_\nu \,|\, D_\mu) \qquad \text{for all} \quad \nu \geq \nu(\mu) .$$

Then, after taking a subsequence and changing the notation if necessary, we are in one of the following two cases:

(A) $f^\nu(k^0) \to u^0 \in E$;

(B) $f^\nu(k^0) \to u^0=(0, \cdots, 0, u_{k+1}^0, \cdots, u_n^0) \in \partial E$.

In Case (A) we would like to show that D is biholomorphically equivalent to E. Recall that E is a taut domain by (1) of Theorem A. Then the normality of $\{f^\nu\}$, combined with the fact that $\{f^\nu(k^0)\}_{\nu \geq \nu(\mu)}$ lies in a compact subset of E, guarantees that some subsequence of $\{f^\nu\}$ converges uniformly on compact subsets of D_μ to a holomorphic mapping $f(\mu): D_\mu \to E$. Since μ was arbitrary and $\{D_\mu\}$ increases to D monotonously, by the usual diagonal argument we may assume that $\{f^\nu\}$ itself converges uniformly on D_μ to the holomorphic mapping $f(\mu)$ for all $\mu=1, 2, \cdots$. Hence, we can define a holomorphic mapping f: $D \to E$ by $f(z)=f(\mu)(z)$, $z \in D_\mu$ for $\mu=1, 2, \cdots$.

Setting $E_\nu=\psi_\nu(E \cap U^0)=\psi_\nu(D \cap U^0)$ for all ν, we here consider the holomorphic mappings g^ν: $E_\nu \to D$ defined by

$$g^\nu=\varphi_\nu^{-1} \circ (\psi_\nu^{-1} \,|\, E_\nu) \qquad \text{for all} \quad \nu .$$

Then it is clear that

(2.7) $$g^\nu \circ f^\nu = \mathrm{id}_{D_\mu} \quad \text{and} \quad f^\nu \circ (g^\nu \,|\, f^\nu(D_\mu)) = \mathrm{id}_{f^\nu(D_\mu)}$$

for all $\nu \geq \nu(\mu)$, $\mu=1, 2, \cdots$. Let E' be an arbitrary subdomain of E with compact closure in E. Then, since

$$f^{\nu}(k^0) \to u^0 \in E \quad \text{and} \quad \psi_{\nu}^{-1}(f^{\nu}(k^0)) \to z^0 \in \partial E$$

as $\nu \to \infty$, we have by Lemma 1 and (2) of Theorem A that

$$\psi_{\nu}^{-1}(E') \subset E \cap U^0 , \qquad \text{or equivalently,} \quad E' \subset E_{\nu}$$

for all sufficiently large ν. We may therefore assume that $\{g^{\nu}\}$ converges uniformly on every compact set in E to a holomorphic mapping $g: E \to \bar{D} \subset \boldsymbol{C}^n$. Once $g(E) \subset D$ is shown, the assertion (2.7) implies that $g \circ f = \mathrm{id}_D$ and $f \circ g = \mathrm{id}_E$; consequently, f gives a biholomorphic mapping from D onto E. Thus we have only to show that $g(E) \subset D$. To this end, take a subdomain E' of E with compact closure in E such that $f(\bar{D}_1) \cup f^{\nu}(\bar{D}_1) \subset E'$ for all $\nu \geq \nu_0$, where D_1 is the domain appearing in (2.5) and ν_0 is a large integer. Then, for any point $z \in D_1$, there exist a sequence $\{x^i\} \subset E'$ and a subsequence $\{g^{\nu_i}\} \subset \{g^{\nu}\}$ such that $g^{\nu_i}(x^i) = z$ for all i and $x^i \to x^0$ for some point $x^0 \in \bar{E}' \subset E$. So

$$z = \lim_{i \to \infty} g^{\nu_i}(x^i) = g(x^0) \in g(E) .$$

Hence, $D_1 \subset g(E)$. On the other hand, being the local uniform limit of regular holomorphic mappings g^{ν}, the mapping g is either regular on E or the Jacobian determinant of g vanishes identically on E (cf. [13; p. 80]). However, $g(E)$ now contains the non-empty open set D_1; hence $g: E \to \boldsymbol{C}^n$ is regular on E. Therefore we conclude by [3; Lemma 0] or [13; p. 79] that $g(E) \subset D$, completing the proof in Case (A).

In Case (B), i.e., $\lim_{\nu \to \infty} f^{\nu}(k^0) = u^0 = (0, \cdots, 0, u^0_{k+1}, \cdots, u^0_n) \in \partial E$, we wish to show that D is biholomorphically equivalent either to B^n or to some $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ with $\{q_{l+1}, \cdots, q_n\} \subset \{p_{k+1}, \cdots, p_n\}$, where $1 \leq \max\{k, j\} < l \leq n-1$. Since $(u^0_{k+1}, \cdots, u^0_n) \neq (0, \cdots, 0)$, the proof will be divided into the following two cases:

(B-1) $u^0_{k+1} \cdots u^0_n \neq 0$; and

(B-2) $u^0_{k+1} \cdots u^0_n = 0$, $u^0_s \neq 0$ for some s with $k+1 \leq s \leq n$.

In the first case (B-1), we claim that D is biholomorphically equivalent to B^n. To prove our claim, choose an open neighborhood W of u^0 in $\boldsymbol{C}^n$ so small that the restriction

$$H := H_{(1, \cdots, 1, p_{k+1}, \cdots, p_n)} | W : W \to H(W)$$

is a biholomorphic mapping, where $H_{(1, \cdots, 1, p_{k+1}, \cdots, p_n)}$ is the principal branch of $h_{(1, \cdots, 1, p_{k+1}, \cdots, p_n)}$ defined by (2.4) with $x^0 = u^0$ and $m = n$. Since

$$(2.8) \qquad |H_i(z_i)|^2 = |z_i|^{2p_i} , \qquad z_i \in W_i(u^0_i) \qquad \text{for} \quad i = k+1, \cdots, n$$

by (2.3), we then have

$$(2.9) \qquad H(W \cap E) = H(W) \cap B^n ;$$

$$(2.10) \qquad H(f^{\nu}(k^0)) \to H(u^0) \in \partial B^n \qquad \text{as} \quad \nu \to \infty .$$

In view of the homogeneity of B^n, one can extract a sequence $\{\Psi_\nu\}$ in $\mathrm{Aut}(B^n)$ such that

$$\Psi_\nu(H(f^\nu(k^0)))=o\in B^n \tag{2.11}$$

for all sufficiently large ν, where o stands for the origin of $\boldsymbol{C}^n$. On the other hand, since there is a local holomorphic peaking function for $u^0\in\partial E$, Lemma 1 guarantees that

$$f^\nu(D_\mu)\subset W\cap E \qquad \text{for all} \quad \nu\geq\tilde{\nu}(\mu)\,,$$

where $\tilde{\nu}(\mu)$ is a large integer depending on $\mu=1, 2, \cdots$. Thus, one can define holomorphic mappings $F^\nu\colon D_\mu\to B^n$ by setting

$$F^\nu=\Psi^\nu\circ H\circ(f^\nu\,|\,D_\mu) \qquad \text{for all} \quad \nu\geq\tilde{\nu}(\mu)\,.$$

Since B^n is taut and $F^\nu(k^0)=o\in B^n$ for all $\nu\geq\tilde{\nu}(\mu)$, $\mu=1, 2, \cdots$, we may assume, by taking a subsequence if necessary, that $\{F^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $F\colon D\to B^n$. We now show that F is a biholomorphic mapping from D onto B^n. Choose a relatively compact subdomain B' of B^n arbitrarily. It follows from Lemma 1 that $\psi_\nu^{-1}(B')\subset H(W)\cap B^n$ for all sufficiently large ν. Moreover,

$$\psi_\nu^{-1}(W\cap E)\subset E\cap U^0=D\cap U^0 \qquad \text{for all large} \quad \nu\,.$$

Indeed, each ψ_ν^{-1} can be written in the form

$$\psi_\nu^{-1}(z', z'')=(A^\nu(z'), B^\nu(z', z''))\,, \qquad (z', z'')\in(\boldsymbol{C}^k\times\boldsymbol{C}^{n-k})\cap E\,,$$

where $A^\nu\in\mathrm{Aut}(B^k)$ and $B^\nu\colon E\to\boldsymbol{C}^{n-k}$ is a suitable holomorphic mapping. Since

$$\psi_\nu^{-1}(0, \cdots, 0, u_{k+1}^\nu, \cdots, u_n^\nu)\to z^0=(z_1^0, \cdots, z_k^0, 0, \cdots, 0)\in\partial E\,,$$

we have

$$A^\nu(0, \cdots, 0)\to(z_1^0, \cdots, z_k^0)\in\partial B^k\,.$$

Consequently, for an arbitrary compact subset K of B^k we conclude by Lemma 1 that

$A^\nu(z')\to(z_1^0, \cdots, z_k^0)$ uniformly on K; and hence

$B^\nu(z)\to(0, \cdots, 0)$ uniformly on $(K\times\boldsymbol{C}^{n-k})\cap E$.

Since we may assume the set $\{w'\in\boldsymbol{C}^k\,|\,(w', w'')\in W\}$ to have compact closure in B^k, these facts imply that $\psi_\nu^{-1}(z)\to z^0$ uniformly on $W\cap E$, as desired. Therefore one can define holomorphic mappings $G^\nu\colon B'\to D$ by

$$G^\nu=\varphi_\nu^{-1}\circ\psi_\nu^{-1}\circ H^{-1}\circ(\Psi_\nu^{-1}\,|\,B') \qquad \text{for all large} \quad \nu\,.$$

Since B' is arbitrary, we may now assume that $\{G^\nu\}$ converges uniformly on compact subsets to a holomorphic mapping $G\colon B^n\to\bar{D}\subset\boldsymbol{C}^n$. We can show $G(B^n)\subset D$, $F\circ G=\mathrm{id}_{B^n}$ and $G\circ F=\mathrm{id}_D$ with exactly the same argument as in Case (A).

In the case (B-2) we may rename the indices so that for some integer m, $k+1\leq m\leq n-1$, one has

$$u_{k+1}^0\cdots u_m^0\neq 0\,, \qquad \text{while} \quad u_{m+1}^0=\cdots=u_n^0=0\,.$$

The restriction

$$H := H_{(1,\cdots,1,p_{k+1},\cdots,p_m,1,\cdots,1)} \,|\, W\colon W \to H(W)$$

of the principal branch to some open neighborhood W of u^0 provides a biholomorphic mapping from W onto $H(W)$ satisfying

$$\text{(2.12)} \qquad H(W \cap E) = H(W) \cap E(1, \cdots, 1, p_{m+1}, \cdots, p_n);$$

$$\text{(2.13)} \qquad H(f^\nu(k^0)) \to H(u^0) =: (v_1^0, \cdots, v_n^0) \in \partial E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$$

with $(v_{m+1}^0, \cdots, v_n^0) = (0, \cdots, 0)$. Taking the assertion (5) of Theorem A into account and passing to a subsequence if necessary, we may assume that

$$w^\nu := \Psi_\nu(H(f^\nu(k^0))) = (0, \cdots, 0, w_{m+1}^\nu, \cdots, w_n^\nu), \qquad \nu = 1, 2, \cdots$$

for some sequence $\{\Psi_\nu\}$ in $\mathrm{Aut}(E(1, \cdots 1, p_{m+1}, \cdots, p_n))$. If $\{w^\nu\}$ accumulates at some point $w^0 \in E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$, then we conclude by the same reasoning as in Case (A) that D is biholomorphically equivalent to $E(1, \cdots, 1, p_{m+1}, \cdots, p_n)$. On the other hand, if a subsequence $\{w^{\nu_i}\} \subset \{w^\nu\}$ can be so chosen that

$$w^{\nu_i} \to w^0 = (0, \cdots, 0, w_{m+1}^0, \cdots, w_n^0) \in \partial E(1, \cdots, 1, p_{m+1}, \cdots, p_n),$$

then we are in the same situation as in Case (B), but with $m > k$ and see that D is biholomorphically equivalent either to B^n or to some $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ with $\{q_{l+1}, \cdots, q_n\} \subsetneqq \{p_{m+1}, \cdots, p_n\}$, as desired.

Case II. $(z_{k+1}^0, \cdots, z_n^0) \neq (0, \cdots, 0)$.

First assume $z_{k+1}^0 \cdots z_n^0 \neq 0$. Then z^0 is a C^ω-smooth and strictly psuedoconvex boundary point of D; hence D is biholomorphically equivalent to B^n by a result of Rosay [17]. Moreover, if $z_{k+1}^0 \cdots z_n^0 = 0$, then it can be shown in exactly the same way as in the case (B-2) that D is biholomorphically equivalent either to B^n or to some $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ with $\{q_{l+1}, \cdots, q_n\} \subset \{p_{k+1}, \cdots, p_n\}$, where $1 \leq \max\{k, j\} \leq l \leq n-1$.

Lemma 3. *Assume that D is biholomorphically equivalent to $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ in Lemma 2, and let $F\colon D \to E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ be a biholomorphic mapping. Then there exist at least two points*

$z'' = (z_1'', \cdots, z_n'') \in U^0 \cap \partial D$, $z_1'' \cdots z_n'' \neq 0$;

$w'' = (w_1'', \cdots, w_n'') \in \partial E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$, $w_1'' \cdots w_n'' \neq 0$

and an open neighborhood U'' of z'' in $\mathbf{C}^n$ satisfying the following conditions:

(1) *$U'' \subset U^0$ and $U'' \cap D$ is a connected open subset of D*;

(2) *F has a continuous extension $\tilde{F}\colon U'' \cap \bar{D} \to \bar{E}(1, \cdots, 1, q_{l+1}, \cdots, q_n)$, the closure of $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$, with $\tilde{F}(z'') = w''$*;

(3) *$\tilde{F}(U'' \cap \bar{D})$ is an open neighborhood of w'' in $\bar{E}(1, \cdots, 1, q_{l+1}, \cdots, q_n)$*;

(4) *$\tilde{F}\colon U'' \cap \bar{D} \to \tilde{F}(U'' \cap \bar{D})$ is a homeomorphism.*

Proof. We set $E = E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ throughout the proof. Now, let us choose a point $z'=(z'_1, \cdots, z'_n)\in U^0\cap\partial D$ and an open neighborhood U' of z' in $\boldsymbol{C}^n$ in such a way that

$$z'_1\cdots z'_n\neq 0 \quad \text{and} \quad U'\subset U^0\cap(\boldsymbol{C}\setminus\{0\})^n .$$

Then every point $x\in U'\cap\partial D$ is a C^ω-smooth and strictly pseudoconvex boundary point of D by (4) of Theorem A. Hence, by a theorem on the boundary continuity of proper holomorphic mappings due to Forstnerič-Rosay [4], the proof of Lemma 3 is now reduced to showing the following assertion:

(*) *There exist a point $z''\in U'\cap\partial D$ and a sequence $\{z^i\}$ in D such that $z^i\to z''$, $F(z^i)\to w''=(w''_1, \cdots, w''_n)\in\partial E$ for some boundary point w'' with $w''_1\cdots w''_n\neq 0$.*

Suppose not. First of all, representing F by coordinates $F=(f_1, \cdots, f_n)$, we claim that there exist an open neighborhood $U(z')\subset U'$ of z' in $\boldsymbol{C}^n$ and a closed complex submanifold S of $U(z')$, of complex dimension one, satisfying the following conditions:

(2.14) $S\cap D$ is a relatively compact subdomain of S with C^ω-smooth boundary;

(2.15) $S\cap D$ is biholomorphically equivalent to the unit disc $\Delta=\{\xi\in\boldsymbol{C}\,|\,|\xi|<1\}$ in $\boldsymbol{C}$;

(2.16) $f_1(p)\cdots f_n(p)\neq 0$ for some point $p\in S\cap D$.

To prove our claim, recall that z' is a C^ω-smooth and strictly pseudoconvex boundary point of D. Then there exist an open set U in $\boldsymbol{C}^n$, $z'\in U\subset U'$, and a biholomorphic mapping $\varphi: U\to\boldsymbol{C}^n$ from U into $\boldsymbol{C}^n$ satisfying the following (cf. [16; p. 61]):

(2.17) $\varphi(U)=B^n$ and $\varphi(z')=o$, the origin of $\boldsymbol{C}^n$;

(2.18) $\varphi(U\cap\partial D)$ is a strictly convex C^ω-smooth real hypersurface of B^n.

Moreover, it is clear that the set $\{z\in D\,|\,f_1(z)\cdots f_n(z)=0\}$ is a nowhere dense analytic subset of D. One can then easily choose a complex affine line L in $\boldsymbol{C}^n$ such that

(2.19) $L\cap\varphi(U\cap D)$ is a relatively compact, simply connected subdomain of $L\cap B^n$ with C^ω-smooth boundary;

(2.20) L is transversal to $\varphi(U\cap\partial D)$;

(2.21) $\varphi^{-1}(L\cap\varphi(U\cap D))=\varphi^{-1}(L\cap B^n)\cap D$ contains a point p with $f_1(p)\cdots f_n(p)\neq 0$.

Therefore, the open neighborhood U of z' and the closed submanifold $\varphi^{-1}(L\cap B^n)$ of U satisfy all the conditions in (2.14), (2.15) and (2.16), as desired.

According to (2.15), we fix a biholomorphic mapping $\psi: \Delta\to S\cap D$, and define holomorphic functions $\Gamma: D\to\boldsymbol{C}$, $\tilde{\Gamma}: \Delta\to\boldsymbol{C}$ by setting

$$\Gamma(z)=f_1(z)\cdots f_n(z)\,, \qquad z\in D \quad \text{and} \quad \tilde{\Gamma}(\xi)=\Gamma(\psi(\xi))\,, \quad \xi\in\Delta\,.$$

It then follows that

(2.22) $$\tilde{\Gamma}(\xi^*)\neq 0 \qquad \text{for} \quad \xi^*:=\psi^{-1}(p)\in\Delta$$

by (2.16), and $\tilde{\Gamma}$ is a bounded holomorphic function on Δ. Hence, by Fatou's theorem there is a set Ω of full measure in the unit circle $\partial\Delta$ on which the function $\tilde{\Gamma}$ has finite non-tangential limits. For points $b\in\Omega$, we denote these limits by $\tilde{\Gamma}(b)$. Now, let $\xi^0\in\Omega$ and let $\{\xi^i\}$ be a sequence of points of Δ, which converges to ξ^0 non-tangentially. Set

$$z^i=\psi(\xi^i)\in S\cap D, \qquad i=1, 2, \cdots.$$

By passing to a subsequence if necessary, we may assume that

$$z^i\to z''\in S\cap\partial D\subset U'\cap\partial D \quad \text{and} \quad F(z^i)\to w''=(w''_1, \cdots, w''_n)\in\partial E.$$

Here, since (*) is assumed not to hold, it follows that $w''_1\cdots w''_n=0$ and consequently

$$\tilde{\Gamma}(\xi^0)=\lim_{i\to\infty}\tilde{\Gamma}(\xi^i)=\lim_{i\to\infty}[f_1(z^i)\cdots f_n(z^i)]=0.$$

Since $\xi^0\in\Omega$ is arbitrary, this, combined with F. and M. Riesz' Theorem [20; p. 137], guarantees that

$$\tilde{\Gamma}(\xi)=0 \qquad \text{for all} \quad \xi\in\Delta,$$

which contradicts (2.22). q.e.d.

2.2. Proof of Theorem I. In view of Lemma 2, we shall divide the proof into two cases as follows:

Case I. *D is biholomorphically equivalent to B^n.* By the homogeneity of D, we may assume that $z^0_1\cdots z^0_n\neq 0$. Fixing a biholomorphic mapping $F\colon D\to B^n$, we choose a sequence $\{z^i\}$ in D in such a way that

$$z^i\to z^0 \quad \text{and} \quad F(z^i)\to w^0 \qquad \text{for some point} \quad w^0\in\partial B^n.$$

Then both z^0 and w^0 are C^ω-smooth and strictly pseudoconvex boundary points. Thus, applying again the theorem of Forstnerič-Rosay [4] to the biholomorphic mappings $F\colon D\to B^n$ and $F^{-1}\colon B^n\to D$, we can find an open neighborhood V of z^0 such that

(2.23) $V\subset U^0$ and $V\cap D$ is a connected open subset of D;

(2.24) F has a continuous extension $\tilde{F}\colon V\cap\bar{D}\to\bar{B}^n$ with $\tilde{F}(z^0)=w^0$;

(2.25) $\tilde{F}(V\cap\bar{D})$ is an open neighborhood of w^0 in $\bar{B}^n$;

(2.26) $\tilde{F}\colon V\cap\bar{D}\to\tilde{F}(V\cap\bar{D})$ is a homeomorphism.

Now we consider the principal branch

$$H:=H_{(1,\cdots,1,p_{k+1},\cdots,p_n)}\colon W(z^0)\to \boldsymbol{C}^n$$

of $h_{(1,\cdots,1,p_{k+1},\cdots,p_n)}$ defined by (2.4) with $x^0=z^0$ and $m=n$. Shrinking V if necessary, one may assume that $V \subset W(z^0)$ and H gives rise to a biholomorphic mapping from V onto $H(V)$. Set $G = (H | V \cap \bar{D})^{-1}$ and consider the homeomorphism

$$\Psi := \tilde{F} \circ G:\ H(V \cap \bar{D}) \to \tilde{F}(V \cap \bar{D}) .$$

(Note that $H(V \cap \bar{D}) \cup \tilde{F}(V \cap \bar{D}) \subset \bar{B}^n$ by (2.8).) An extension theorem due to Rudin [18; p. 311] can now be applied to obtain an element $\Phi \in \mathrm{Aut}(B^n)$ such that $\Psi(z) = \Phi(z)$, $z \in H(V \cap D)$, or equivalently,

$$\Phi^{-1} \circ F(z) = H(z) \qquad \text{for all} \quad z \in V \cap D . \tag{2.27}$$

Consider now the sets A and D' defined by

$$A = \{ z \in D \,|\, z_{k+1} \cdots z_n = 0 \} \quad \text{and} \quad D' = D \setminus A .$$

Then A is a nowhere dense analytic subset of D; hence D' is a connected open dense subset of D (cf. [13; p. 50]). Writing

$$\Phi^{-1} \circ F = (\alpha_1, \cdots, \alpha_n):\ D \to B^n ,$$

we here define the functions $\alpha:\ D \to \boldsymbol{R}$ and $\beta:\ \boldsymbol{C}^n \to \boldsymbol{R}$ by

$$\alpha(z) = |\alpha_1(z)|^2 + \cdots + |\alpha_n(z)|^2, \qquad z \in D ; \tag{2.28}$$

$$\beta(z) = |z_1|^2 + \cdots + |z_k|^2 + |z_{k+1}|^{2p_{k+1}} + \cdots + |z_n|^{2p_n}, \qquad z \in \boldsymbol{C}^n . \tag{2.29}$$

Then $\alpha(z) < 1$ for all $z \in D$ and α, β are obviously real analytic on the domain D'. Moreover, we see by (2.8), (2.27) that

$$\alpha(z) = \beta(z) \qquad \text{for all} \quad z \in V \cap D = V \cap D' ;$$

consequently $\alpha(z) = \beta(z)$ for all $z \in D'$ by analytic continuation. Here D' is dense in D and α, β are continuous on D, so that

$$\beta(z) = \alpha(z) < 1 \qquad \text{for all} \quad z \in D .$$

This means that $D \subset E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$. More precisely, we now assert that $D = E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$. Indeed, pick an arbitrary point $x^0 \in \partial D$ and choose a sequence $\{x^i\}$ in D converging to x^0. Then

$$\beta(x^0) = \lim_{i \to \infty} \beta(x^i) = \lim_{i \to \infty} \alpha(x^i) = 1 ,$$

because $\Phi^{-1} \circ F:\ D \to B^n$ is a biholomorphic mapping from D onto B^n. Therefore there exists no boundary point of D in $E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ and so $D = E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$, as desired. Since $E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ is now biholomorphically equivalent to B^n, we conclude by (3) of Theorem A that $k = n$ and $D = E(1, \cdots, 1) = B^n$, completing the proof in Case I.

Case II. *D is biholomorphically equivalent to $E := E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ with $\{q_{l+1}, \cdots, q_n\} \subset \{p_{k+1}, \cdots, p_n\}$, where $1 \leq \max\{k, j\} \leq l \leq n-1$.* Let us fix a biholomorphic mapping $F: D \to E$ from D onto E. By Lemma 3 we can choose two points

$$z'' = (z''_1, \cdots, z''_n) \in U^0 \cap \partial D\,, \qquad z''_1 \cdots z''_n \neq 0\,;$$
$$w'' = (w''_1, \cdots, w''_n) \in \partial E\,, \qquad w''_1 \cdots w''_n \neq 0$$

and an open neighborhood U'' of z'' in C^n satisfying the conditions (1) through (4) described in Lemma 3. By a change of coordinates if necessary, we may assume that $(q_{l+1}, \cdots, q_n) = (p_{l+1}, \cdots, p_n)$. Consider now the principal branches

$$H := H_{(1,\cdots,1,p_{k+1},\cdots,p_n)}:\ W(z'') = C^k \times W_{k+1}(z''_{k+1}) \times \cdots \times W_n(z''_n) \to C^n$$

and

$$\tilde{H} := H_{(1,\cdots,1,p_{l+1},\cdots,p_n)}:\ W(w'') = C^l \times W_{l+1}(w''_{l+1}) \times \cdots \times W_n(w''_n) \to C^n$$

of $h_{(1,\cdots,1,p_{k+1},\cdots,p_n)}$ and $h_{(1,\cdots,1,p_{l+1},\cdots,p_n)}$, respectively. Shrinking U'' if necessary, one may assume that

(2.30) $U'' \subset W(z'')$ and the restriction $H \mid U'': U'' \to H(U'')$ is a biholomorphic mapping;

(2.31) $\tilde{H}$ is biholomorphic on an open neighborhood of $\tilde{F}(U'' \cap \bar{D})$, where $\tilde{F}: U'' \cap \bar{D} \to \bar{E}$ is the continuous extension of F appearing in Lemma 3.

Set $G = (H \mid U'' \cap \bar{D})^{-1}$. Then, by our construction we can apply Rudin's theorem [18; p. 311] to the homeomorphism

$$\Psi := \tilde{H} \circ \tilde{F} \circ G:\ H(U'' \cap \bar{D}) \to \tilde{H}(\tilde{F}(U'' \cap \bar{D}))$$

from $H(U'' \cap \bar{D}) \subset \bar{B}^n$ onto $\tilde{H}(\tilde{F}(U'' \cap \bar{D})) \subset \bar{B}^n$. Hence there exists an element $\Phi \in \mathrm{Aut}(B^n)$ satisfying

(2.32) $$\Phi^{-1} \circ \tilde{H} \circ F(z) = H(z) \qquad \text{for all} \quad z \in U'' \cap D\,.$$

For later purpose, we here wish to show:

(2.33) $E' := E \cap W(w'')$ is a connected open dense subset of E.

Since $W(w'')$ is open dense in C^n by (2.1) and (2.2), so is E' in E. Thus it suffices to verify the connectedness of E'. To this end, let us set

$$\Delta(a) = \{\xi \in C \mid |\xi| < a\} \qquad (a > 0)\,;$$
$$\Delta_i(a) = \Delta(a) \setminus \{s w''_i \mid s \leq 0\}\,, \qquad i = l+1, \cdots, n\,; \text{ and}$$
$$aX = \{ax \mid x \in X\} \qquad (a > 0,\ X \subset C^n)\,.$$

Then one can choose an $r > 0$ so small that $(\Delta(r))^n \subset E$; hence

$$E' \cap (\Delta(r))^n = (\Delta(r))^l \times \Delta_{l+1}(r) \times \cdots \times \Delta_n(r)$$

by (2.1) and (2.3). Clearly, this says that $E' \cap (\Delta(r))^n$ is a connected and simply connected subset of E'. Moreover, it can be checked that

$$sE' \subset E' \ (0 < s \leq 1) \quad \text{and} \quad tE' \subset E' \cap (\Delta(r))^n$$

for a sufficiently small $t > 0$. Hence E' is a connected subset of E, as desired.

As the final step, we set

$$D' = F^{-1}(E') , \qquad D'' = \{z \in D' \mid z_{k+1} \cdots z_n \neq 0\} ;$$
$$\Phi^{-1} \circ \tilde{H} \circ F(z) = (\tilde{\alpha}_1(z), \cdots, \tilde{\alpha}_n(z)) , \qquad z \in D' ;$$
$$\tilde{\alpha}(z) = |\tilde{\alpha}_1(z)|^2 + \cdots + |\tilde{\alpha}_n(z)|^2 , \qquad z \in D' .$$

Obviously D' (resp. D'') is a connected open dense subset of D (resp. D'); hence so is D'' in D. Thus, taking (2.32) into account and repeating the same argument as in Case I, we see that

$$\beta(z) = \tilde{\alpha}(z) < 1 \qquad \text{for all} \quad z \in D'' ,$$

where β: $\boldsymbol{C}^n \to \boldsymbol{R}$ is the function defined in (2.29). Let us now pick an arbitrary point $(z, x) \in D \times \partial D$ and take a sequence $\{(z^i, x^i)\}$ in $D'' \times D''$ converging to (z, x). Then

$$\beta(z) = \lim_{i \to \infty} \beta(z^i) = \lim_{i \to \infty} \tilde{\alpha}(z^i) \leq 1$$

and

$$\beta(x) = \lim_{i \to \infty} \beta(x^i) = \lim_{i \to \infty} \tilde{\alpha}(x^i) = 1 .$$

Since the interior of the closure of $E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ coincides with $E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ itself, these imply that $D = E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$; and consequently $1 \leq j \leq k = l \leq n-1$ by (3) of Theorem A.

We shall complete the proof by showing that z^0 must have the form $z^0 = (z_1^0, \cdots z_k^0, 0, \cdots, 0)$. Indeed, assume that $z_s^0 \neq 0$ for some integer s with $k+1 \leq s \leq n$. It follows then from Case II in the proof of Lemma 2 that $D = E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ is biholomorphically equivalent to some $E(1, \cdots, 1, q_{m+1}, \cdots, q_n)$ with $\{q_{m+1}, \cdots, q_n\} \subsetneqq \{p_{k+1}, \cdots, p_n\}$, which contradicts the assertion (3) of Theorem A. Therefore $z_s^0 = 0$ for all $s = k+1, \cdots, n$. q.e.d.

REMARK. In Theorem I, assume the following condition (1)′ instead of (1):

(1)′ There exist an integer $k \geq 0$, real numbers p_i with $0 < p_i \neq 1$ $(k+1 \leq i \leq n)$, open neighborhoods U^0, V^0 of z^0 in $\boldsymbol{C}^n$ and a biholomorphic mapping f: $U^0 \to V^0$ such that

(i) $z^0 \in \partial E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ and

(ii)′ $f(D \cap U^0) = E(1, \cdots, 1, p_{k+1}, \cdots, p_n) \cap V^0$.

Then, a glance at the proof of Theorem I tells us that $1 \leq j \leq k$, and D is biholomorphically equivalent to $E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$.

2.3. Proof of Theorem II. We set $E=E(1, \cdots, 1, p_{k+1}, \cdots, p_n)$ and fix a family of relatively compact subdomains M_μ of M such that

$$(2.34) \qquad M=\bigcup_{\mu=1}^{\infty} M_\mu \supset \cdots \supset M_{\mu+1} \supset M_\mu \supset \cdots \supset M_1 \ni k^0 ,$$

where k^0 is an arbitrary point of M. Since M can be exhausted by biholomorphic images of E, there exists a sequence $\{\psi_\nu\}_{\nu=1}^{\infty}$ of biholomorphic mappings from E into M such that

$$M_\nu \subset \psi_\nu(E) , \qquad \nu=1, 2, \cdots .$$

We set

$$\varphi_\nu=\psi_\nu^{-1} : \psi_\nu(E) \to E , \qquad \nu=1, 2, \cdots .$$

Without loss of generality, we may assume that $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges uniformly on every compact set in M to a holomorphic mapping $\varphi: M \to \bar{E} \subset \boldsymbol{C}^n$. Replacing ψ_ν, φ_ν by suitable holomorphic mappings of the form $\psi_\nu \circ \sigma_\nu^{-1}$, $\sigma_\nu \circ \varphi_\nu$ with some $\sigma_\nu \in \mathrm{Aut}(E)$, if necessary, we may further assume that

$$\varphi_\nu(k^0)=(0, \cdots, 0, u_{k+1}^\nu, \cdots, u_n^\nu) , \qquad 0 \leq u_{k+1}^\nu, \cdots, u_n^\nu < 1$$

for all $\nu=1, 2, \cdots$. We now have two cases to consider.

Case I. $\lim_{\nu\to\infty} \varphi_\nu(k^0)=\varphi(k^0) \in E$. We shall prove that M is biholomorphically equivalent to E in this case. Notice first that $\varphi(M) \subset E$, because E is taut and $\varphi_\nu(k^0) \to \varphi(k^0) \in E$. Once it is shown that $\varphi: M \to E$ is injective, we can regard M as a bounded domain in $\boldsymbol{C}^n$. Hence, replacing the system $(\{f^\nu\}, \{g^\nu\}, D, \{D_\mu\})$ by $(\{\varphi_\nu\}, \{\psi_\nu\}, M, \{M_\mu\})$ in Case I, (A), of the proof of Lemma 2, we can prove that M is biholomorphically equivalent to E. Therefore it is enough to show that $\varphi: M \to E$ is injective. Assume that $\varphi(x_1)=\varphi(x_2)=z$ for $x_1, x_2 \in M$. It follows then from the distance decreasing property of holomorphic mappings (cf. [9; p. 45]) that

$$d_E(\varphi_\nu(x_1), \varphi_\nu(x_2))=d_{\psi_\nu(E)}(\psi_\nu(\varphi_\nu(x_1)), \psi_\nu(\varphi_\nu(x_2)))=d_{\psi_\nu(E)}(x_1, x_2) \geq d_M(x_1, x_2)$$

for all ν, where d_E (resp. $d_{\psi_\nu(E)}$, resp. d_M) denotes the Kobayashi pseudodistance of E (resp. $\psi_\nu(E)$, resp. M). Consequently, we have $x_1=x_2$, because M is hyperbolic and $d_E(\varphi_\nu(x_1), \varphi_\nu(x_2)) \to d_E(z, z)=0$ as $\nu \to \infty$. Thus $\varphi: M \to E$ is injective, as desired.

Case II. $\lim_{\nu\to\infty} \varphi_\nu(k^0)=\varphi(k^0)=:(0, \cdots, 0, u_{k+1}^0, \cdots, u_n^0) \in \partial E$. In this case, we can show that M is biholomorphically equivalent to some $E(1, \cdots, 1, q_{l+1}, \cdots, q_n)$ with $\{q_{l+1}, \cdots, q_n\} \subsetneqq \{p_{k+1}, \cdots, p_n\}$ with exactly the same arguments as in Case I, (B), of the proof of Lemma 2, if we consider the mapping $\Psi_\nu \circ H \circ \varphi_\nu$ (resp. $\psi_\nu \circ H^{-1} \circ (\Psi_\nu^{-1} | B'))$ instead of F^ν (resp. G^ν). q.e.d.

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