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THE EXPONENT OF CONVERGENCE OF POINCARÉ SERIES OF COMBINATION GROUPS

Dedicated to the memory of the late Professor Tōhru Akaza

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1. Introduction. Let G be a discrete subgroup of the automorphism group $GM(B^{n+1})$ of $(n+1)$-dimensional hyperbolic space B^{n+1}. We shall present in § 3 a certain number $\delta(G)$ which is called the exponent of convergence of Poincaré series associated to G. Let $L(G)$ be the limit set of G and $d(L(G))$ its Hausdorff dimension. It is already known [2], [7] that $\delta(G)=d(L(G))$ for geometrically finite discrete groups. Our motivation is based on the following results. The authors in [3] showed the inequality $d(L(G_1*G_2))>\mathrm{Max}(d(L(G_1)), d(L(G_2)))$ for Shottky groups G_1 and G_2 where G_1*G_2 is the free product of G_1 and G_2. And also Patterson in [6] proved inequality $\delta(G_1*G_2)>\mathrm{Max}(\delta(G_1), \delta(G_2))$ for Fuchsian groups G_1 and G_2 where G_1*G_2 is the free product of G_1 and G_2. In this paper, we extend the above statement generally, that is, the exponent of convergence of Poincaré series of a discrete group G is smaller than that of the discrete group which is obtained by applying the combination theorem with an amalgamated subgroup to G. This is discussed in §§ 4 and 5.

2. Preliminaries. Let $\overline{R^{n+1}}$ be the one point compactification of R^{n+1}. Mobius transformation g in $\overline{R^{n+1}}$ is defined as compositions of even number of reflections in n-spheres or n-planes in $\overline{R^{n+1}}$. Let $GM(n+1)$ be the group of all Mobius transformations in $\overline{R^{n+1}}$. A subgroup of $GM(n+1)$ is called a Mobius group. The identity in $GM(n+1)$ is denoted by I. For a set $E\subset\overline{R^{n+1}}$, we denote by $GM(E)$ the subgroup of $GM(n+1)$ which fixes E, and by $GM|_{\partial E}$ the group $\{f|_{\partial E} \mid f\in GM(E)\}$ where $f|_{\partial E}$ is the restriction of f to ∂E. The two models for E we consider are $H^{n+1}=\{x=(x_1, x_2, \cdots, x_{n+1})\in R^{n+1} \mid x_{n+1}>0\}$, and $B^{n+1}=\{x\in R^{n+1} \mid |x|<1\}$ with respective boundaries $\overline{R^n}=\partial H^{n+1}$ and $S^n=\partial B^{n+1}$. For each $f\in GM(n)$, there exists a unique $\hat{f}\in GM(H^{n+1})$ such that $\hat{f}|_{\partial H^{n+1}}=f$ with the identification $\overline{R^n}=\partial H^{n+1}$. In this way, we have an isomorphism $GM|_{\partial H^{n+1}}\cong GM(n)\cong GM(H^{n+1})$. Hence we identify the elements in $GM(n)$ with the elements in $GM(H^{n+1})$ and use the same letters. Let s be the usual stereographic projection of S^n onto $\overline{R^n}$, then s can be extended to an element of $GM(n+1)$ so that $s(B^{n+1})=H^{n+1}$ ([4]). The conjugation $f\to sfs^{-1}$ is an isomorphism $GM(H^{n+1})$ onto $GM(B^{n+1})$. By this isomorphism, we have isomorphisms $GM(B^{n+1})\cong GM(n)\cong GM|_{\partial B^{n+1}}$.

The elements of $GM(H^{n+1})-\{I\}$ are classified as following three types:

(i) T is elliptic if it has a fixed point in H^{n+1}.

(ii) T is parabolic if it has exactly one fixed point in $\overline{R^n}$.

(iii) T is loxodromic if it has exactly two fixed points, both in $\overline{R^n}$.

For a Mobius transformation $A\in GM(n+1)$, we write $A'(x)$ the Jacobian matrix at $x\in\overline{R^{n+1}}$. Then $A'(x)=kB$ for some $k>0$ and $B\in O(n+1)$. We put $k=|A'(x)|$.

LEMMA 1 ([1, p. 19]). *Let g be a Mobius transformation. Then we have*

$$|g(x)-g(y)|^2=|g'(x)||g'(y)||x-y|^2\,. \tag{1}$$

Let $x^*=x\cdot|x|^{-2}$, $x\in R^{n+1}$ $(x\neq 0)$. If $g(\infty)\neq\infty$, then $g(x)=r^2A(x-a)^*+b$ where $a=g^{-1}(\infty)$, $b=g(\infty)$, $r>0$ and A is an orthogonal matrix ([1, p. 21]). The set $I(g)=\{x\in R^{n+1}\,||g'(x)|=1\}$ is an n-sphere centered at $g^{-1}(\infty)$ with radius r. This sphere is called the isometric sphere of g. The chain rule applied to $g^{-1}(g(x))=g(g^{-1}(x))=x$ yields $|(g^{-1})'(g(x))||g'(x)|=|g'(g^{-1}(x))||(g^{-1})'(x)|=1$. From these equalities we have the following facts: $g(\text{ext}\,I(g))=\text{int}\,I(g^{-1})$ and $g^{-1}(\text{ext}\,I(g^{-1}))=\text{int}\,I(g)$, where ext and int denote the exterior and interior, respectively.

3. Discrete groups. Let G be a discrete subgroup of $GM(B^{n+1})$. The points $g(0)$, $g\in G$, are isolated and more generally, if $K\subset B^{n+1}$ is compact there are only finitely many $g\in G$ such that $g(K)\cap K\neq\varnothing$. A point $\zeta\in\overline{B^{n+1}}$ is called a limit point of G if there exists an infinite distinct sequences $g_n\in G$ and a point $a\in B^{n+1}$ such that $g_n(a)\to\zeta$. The set of all limit points of G is the limit set $L=L(G)$. The set of accumulation points of $G(a)=\{g(a)\,|\,g\in G\}$ is denoted by $L(a)$. Clearly, $L=\bigcup L(a)$. Then we have the following fact (see [1]) that $L=L(a)$ for all $a\in B^{n+1}$. The limit set L has the following properties: (i) L is a closed set contained in ∂B^{n+1}. (ii) L is invariant under G and is a perfect set if L contains more than two elements.

An open set F of B^{n+1} is called a fundamental region for a discrete group G acting on B^{n+1} if F satisfies the following conditions:

(i) $F\cap g(F)=\varnothing$ for all $g\in G-\{I\}$,

(ii) $\bigcup_{g\in G}g(\bar{F})\supset B^{n+1}$ where $\bar{F}$ is relative closure of F in B^{n+1}.

The existence of a fundamental region for discrete group acting on B^{n+1} is well known. For instance, the Dirichlet polyhedron is a fundamental region (cf. [5, p. 71]).

Now the exponent of convergence of a discrete group $G\subset GM(B^{n+1})$ is defined as

$$\delta(G)=\inf\{s>0\,|\,\sum_{g\in G}|g'(x)|^s<+\infty\}\,.$$

This does not depend on the choise of $x\in B^{n+1}$ and it satisfies $\delta(G)\leq n$ (see, for instance, [1]).

4. Free product with amalgamated subgroup. Following the statement in [5, Chap. VII] we give some definitions. Let G_1 and G_2 be subgroups of $GM(B^{n+1})$ with a common subgroup H. We also assume throughout §4 that $G_m - H \neq \varnothing$ ($m = 1, 2$). A normal form is a word of the form $g_1 g_2 \cdots g_i g_{i+1} \cdots g_n$ where $g_i \in G_1 - H$ for even i and $g_j \in G_2 - H$ for odd j, or vice versa, that is, the element of $G_1 - H$ or that of $G_2 - H$ appear in a normal form alternatively. A normal form $g_1 g_2 \cdots g_n$ is said to be in a (p, q) form if $g_1 \in G_p - H$ and $g_n \in G_q - H$ for $p, q = 1, 2$. There is a natural identification of normal forms as follows. If $h \in H$, then we regard the forms $g_1 g_2 \cdots g_n$ and $g_1 g_2 \cdots (g_k h)(h^{-1} g_{k+1}) \cdots g_n$ as being equivalent. Using the above equivalence, the product of two normal forms is equivalent to either a normal form, or an element of H. The set of equivalence classes of normal forms together with the elements of H, is called the free product of G_1 and G_2, with amalgamated subgroup H, and written as $G_1 *_H G_2$. Let $\langle G_1, G_2 \rangle$ be the group generated by G_1 and G_2. Then there exists a natural homomorphism $\Phi: G_1 *_H G_2 \to \langle G_1, G_2 \rangle$ given by regarding juxtaposition of words as composition of mapping, that is, $\Phi(g_1 g_2 \cdots g_n) = g_1 \circ g_2 \circ \cdots \circ g_n$. It is clear that equivalent normal forms are mapped onto the same transformation. If Φ is an isomorphism, then we say that $\langle G_1, G_2 \rangle = G_1 *_H G_2$, and we do not distinguish between $\langle G_1, G_2 \rangle$ and $G_1 *_H G_2$. If $\langle G_1, G_2 \rangle = G_1 *_H G_2$, and H is trivial, then every non-trivial element of $\langle G_1, G_2 \rangle$ has a unique normal form, while if H is non-trivial, the normal form of an element of $\langle G_1, G_2 \rangle$ is clearly not unique.

PROPOSITION. *Let G_i ($i = 1, 2$) be a discrete subgroup of $GM(B^{n+1})$ acting on B^{n+1} with a fundamental region F_i satisfying the geometric condition*

$$(*) \qquad F_1^c \cap F_2^c = \varnothing ,$$

where F_i^c is the complement of the set of F_i with respect to B^{n+1}. Then the group $G = \langle G_1, G_2 \rangle$ is the free product $G_1 * G_2$ with the amalgamated subgroup $\{I\}$ and $F_1 \cap F_2$ is precisely invariant under the identity in G.

PROOF. The geometric conditions $(*)$ implies $F_1 \cup F_2 = B^{n+1}$. Furthermore, we see that $F_1 \cap F_2 \neq \varnothing$. Hence we are done by Theorem A. 13 in [5, p. 139].

5. The case *H* trivial. Let G_1 and G_2 be discrete subgroups of $GM(B^{n+1})$ with fundamental regions F_1 and F_2, respectively, satisfying the geometric conditions $(*)$ and let $G = \langle G_1, G_2 \rangle$. Then by Proposition, $G = G_1 * G_2$ and $F_1 \cap F_2$ ($\neq \varnothing$) is precisely invariant under $\{I\}$ in G.

Now under the conditions stated above, we have the following lemma.

LEMMA 2. *For $g \in G_k$ ($k = 1, 2$), we define the number $\beta_{k,3-k}(g)$ by*

$$(2)\qquad \beta_{k,3-k}(g)=\operatorname*{Sup}_{x\in F_1\cap F_2}\left\{\operatorname*{Inf}_{w\in F_k^c}|x-w|^2 j(g^{-1},x)\right\}\left\{\operatorname*{Sup}_{w\in F_{3-k}^c}|g^{-1}(x)-w|\right\}^{-2}$$

where $j(g,x)=|g'(x)|$. *Assume that*

$$\sum_{g_1\in G_1-\{I\}}\beta_{12}(g_1)^s\sum_{g_2\in G_2-\{I\}}\beta_{21}(g_2)^s>1\,,\qquad \textit{then }\delta(G_1*G_2)\geqq s\,.$$

PROOF. The chain rule applied to $g\circ h(x)=g(h(x))$ and $g^{-1}(g\circ h(x))=h(x)$ yield $j(gh,x)=j(g,h(x))j(g^{-1},gh(x))^{-1}j(h,x)$. Using the equality (1) stated in §2, we have $|g^{-1}(x')-h(x)|^2=|g^{-1}(x')-g^{-1}(gh(x))|^2=j(g^{-1},x')j(g^{-1},gh(x))|x'-gh(x)|^2$. Thus we have

$$(3)\qquad j(gh,x)=j(g^{-1},x')j(h,x)|x'-gh(x)|^2|g^{-1}(x')-h(x)|^{-2}\,.$$

Suppose $g\in G_1-\{I\}$, $x, x'\in F_1\cap F_2$ and $h(x)\in F_2^c$, then $h(x)\in F_1$ and $gh(x)\in F_1^c$. Therefore we have

$$(4)\qquad j(gh,x)\geqq j(h,x)\beta_{12}(g)\,.$$

Similarly, we have

$$(5)\qquad j(gh,x)\geqq j(h,x)\beta_{21}(g)\,,$$

for $g\in G_2-\{I\}$, $x\in F_1\cap F_2$, $h\in G$ such that $h(x)\in F_1^c$. If $g=g_1^{(1)}g_1^{(2)}\cdots g_k^{(1)}g_k^{(2)}$ is (1, 2) form stated in §4 and if $x\in F_1\cap F_2$ then $g_1^{(2)}\cdots g_k^{(1)}g_k^{(2)}(x)\in F_2^c$. Hence $j(g,x)\geqq j(g_1^{(2)}\cdots g_k^{(1)}g_k^{(2)},x)\beta_{12}(g_1^{(1)})$ by (4). Furthermore, since $g_2^{(1)}g_2^{(2)}\cdots g_k^{(1)}g_k^{(2)}(x)\in F_1^c$ for $x\in F_1\cap F_2$, we see that

$$j(g_1^{(2)}\cdots g_k^{(1)}g_k^{(2)},x)\geqq j(g_2^{(1)}g_2^{(2)}\cdots g_k^{(1)}g_k^{(2)},x)\beta_{21}(g_1^{(2)})$$

by (5). Continuing this argument, we have $j(g,x)\geqq\beta_{12}(g_1^{(1)})\beta_{21}(g_1^{(2)})\cdots\beta_{12}(g_k^{(1)})j(g_k^{(2)},x)$ for $x\in F_1\cap F_2$. Hence the sum of s-th power of $j(g,x)$ for the elements g of (1, 2) form in G_1*G_2 is not smaller than

$$\sum_{k\geqq 0}\left\{\sum_{g_1\in G_1-\{I\}}\beta_{12}(g_1)^s\right\}^{k+1}\left\{\sum_{g_2\in G_2-\{I\}}\beta_{21}(g_2)^s\right\}^{k}\sum_{g\in G_2-\{I\}}j^s(g,x)\,.$$

Therefore we have the following inequality considering all (p, q) forms,

$$\begin{aligned}\sum_{f\in G_1*G_2}j^s(f,x)\geqq 1+&\left[\sum_{k\geqq 0}\left\{\left(\sum_{g_1\in G_1-\{I\}}\beta_{12}(g_1)^s\right)^k\left(\sum_{g_2\in G_2-\{I\}}\beta_{21}(g_2)^s\right)^k\right\}\right]\\ &\times\left\{\left(\sum_{g_2\in G_2-\{I\}}j^s(g_2,x)\right)\left(1+\sum_{g_1\in G_1-\{I\}}\beta_{12}(g_1)^s\right)\right.\\ &\left.+\left(\sum_{g_1\in G_1-\{I\}}j^s(g_1,x)\right)\left(1+\sum_{g_2\in G_2-\{I\}}\beta_{21}(g_2)^s\right)\right\}.\end{aligned}$$

Thus we have our assertion by this inequality.

Now we have the following theorem from Lemma 2.

THEOREM 1. *Let G_1 and G_2 be discrete subgroups of $GM(B^{n+1})$ with the fundamental regions F_1 and F_2 respectively, satisfying the geometric condition* (*). *Assume that $\delta(G_1)\geqq\delta(G_2)$ and $\sum_{g\in G_1}j^{\delta(G_1)}(g, x)=+\infty$. Then $\delta(G_1 * G_2)>\delta(G_1)$.*

PROOF. Let r be the radius of a ball B_r which is contained in $F_1\cap F_2$. Then by (2), we have $\beta_{k,3-k}(g)\geqq r^2 j(g^{-1}, x)/4$ for $k=1, 2, x\in B_r$ and $g\in G_1 * G_2$. Therefore we have

$$(6)\quad \sum_{g_1\in G_1-\{I\}}\beta_{12}(g_1)^s\sum_{g_2\in G_2-\{I\}}\beta_{21}(g_2)^s\geqq\left(\frac{r^2}{4}\right)^{2s}\sum_{g_1\in G_1-\{I\}}j^s(g_1, x)\sum_{g_2\in G_2-\{I\}}j^s(g_2, x)\quad(x\in B_r)\,.$$

By the assumption we see $\lim_{s\to\delta(G_1)}\sum_{g\in G_1}j^s(g, x)=+\infty$, so that the right hand side of (6) is greater than 1 for some $s_0>\delta(G)$. Hence by Lemma 2, we have $\delta(G_1 * G_2)\geqq s_0>\delta(G_1)$. This completes the proof.

REMARK. The assumption $\sum_{g\in G}j^{\delta(G)}(g, x)=+\infty$ in Theorem 1 is satisfied by convex cocompact groups and geometrical finite groups.

6. The case H non-trivial. Throughout this section, all groups we consider are subgroups of $GM(H^3)$. From § 2, we have isomorphisms $GM(B^3)\cong GM(H^3)\cong GM|_{\partial H^3}$. As $\bar{C}=C\cup\{\infty\}$ is identified with ∂H^3, $GM|_{\partial H^3}$ is the class of orientation preserving Mobius transformations $\bar{C}$ onto itself and denote it $M(\bar{C})$. A discrete subgroup of $M(\bar{C})$ is called a Kleinian group.

Let G_1 and G_2 be Kleinian groups acting on $\bar{C}$ with a common subgroup H and let $G_m-H\neq\varnothing$ for $m=1, 2$. An interactive pair of sets (X_1, X_2), consists of two non-empty disjoint sets X_1 and X_2 in $\bar{C}$, where X_k ($k=1, 2$) is invariant under H, every element of G_1-H maps X_1 into X_2, and every element of G_2-H maps X_2 into X_1. Note that if (X_1, X_2) is an interactive pair, then X_k is precisely invariant under H in G_k ($k=1, 2$).

From § 4, any element $g\in G_1 *_H G_2-H$ is represented by a normal form $g=g_1g_2\cdots g_n$. Every normal form has a length, $n=|g_1\cdots g_n|$. If $h\in H$, then $g_1\cdots g_kg_{k+1}\cdots g_n$ and $g_1\cdots(g_kh)(h^{-1}g_{k+1})\cdots g_n$ are equivalent. Therefore equivalent normal forms have the same length, so if $G=\langle G_1, G_2\rangle=G_1 *_H G_2$, then $|g|$ is well defined for all elements of G (if $h\in H$, we put $|h|=0$). Thus we have the following lemma due to Maskit.

LEMMA 3. *Let $G=\langle G_1, G_2\rangle$ be a Kleinian group with $G=G_1 *_H G_2$. Let X_1 and X_2 be mutually disjoint topological closed discs in $\bar{C}$ bounded by a simple closed curve W and let $(\mathring{X}_1, \mathring{X}_2)$ be an interactive pair, where $\mathring{X}_i$ is the interior of X_i. Furthermore, assume that $W=\partial X_1=\partial X_2$ is precisely invariant under H in either G_1 or G_2. Then there is a loxodromic element of G with one fixed point in $\mathring{X}_1$ and the other in $\mathring{X}_2$.*

PROOF. Let g be an element of G such that $|g|>1$ and $|g|$ is minimal among all conjugates of g in G. Then g is a $(3-k, k)$ form and $g(X_k) \subset g_1 g_2(X_k) \subset \mathring{X}_k$ $(k=1, 2)$, as in [5, p. 150]. Hence we see that g is a loxodromic element with one fixed point in $\mathring{X}_1$ and the other in $\mathring{X}_2$ (see [5, p. 150]).

By Lemma 3, we have the following theorem.

THEOREM 2. *Let the Kleinian group* $G=\langle G_1, G_2\rangle$ *be* $G_1 *_H G_2$ *and let the topological closed discs* X_1 *and* X_2 *satisfy the hypothesis in Lemma* 3. *Then there exist fundamental regions* F_1 *and* F *of* G, *and a loxodromic cyclic subgroup of* G, *respectively, satisfying the geometric condition* $(*)$.

PROOF. By Lemma 3, there is a loxodromic element g in G with one fixed point ζ in $\mathring{X}_1$. Suppose that a fundamental region F_H of H contains a given fundamental region F_1 of G_1. As $\zeta \notin L(H)$, and $\mathring{X}_1 = \bigcup_{h \in H} h(\bar{F}_1 \cap \mathring{X}_1)$, there is an element h of H such that one fixed point $h(\zeta)$ of hgh^{-1} in $\Delta = \bar{F}_1 \cap \mathring{X}_1$ and also $h(\zeta)$ is not an isolated point of $L(G)$. Hence we can find two disjoint open balls V_1, V_2 in Δ both of which intersect $L(G)$. Thus, by [5, p. 96], we have a loxodromic element g in G with one fixed point in V_1 and the other in V_2. If we consider sufficiently large k then the isometric spheres g^k and g^{-k} are contained in V_1 and V_2, respectively. Thus, putting $f=g^k$ and $F=(\text{ext } I(f)) \cap (\text{ext } I(f^{-1}))$, we have our theorem.

Finally we have the following theorem from Theorems 1 and 2.

THEOREM 3. *Let the Kleinian group* $G=\langle G_1, G_2\rangle$ *be the free product of* G_1 *and* G_2 *with an amalgamated subgroup* H *and let the topological closed discs* X_1 *and* X_2 *satisfy the hypothesis in Lemma* 3. *Suppose that* $\delta(G_1) \geqq \delta(G_2)$ *and* $\sum_{g_1 \in G_1} j^{\delta(G_1)}(g_1, x) = +\infty$, *then* $\delta(G_1 *_H G_2) > \delta(G_1)$.

PROOF. By Theorem 2, there exist fundamental regions F_1 and F of G_1 and a loxodromic cyclic subgroup $\langle f\rangle$ of G, respectively, satisfying the geometric condition $(*)$. Hence $\delta(G_1 * \langle f\rangle) > \delta(G_1)$ by Theorem 1. Furthermore, since $G_1 * \langle f\rangle$ is a subgroup of G and since $G=G_1 *_H G_2$, we have $\delta(G_1 *_H G_2) \geqq \delta(G_1 * \langle f\rangle) > \delta(G_1)$.

REFERENCES

[1] L. V. AHLFORS, Möbius Transformations in Several Dimensions, Univ. of Minnesota Lecture Notes, Minnesota, 1981.
[2] T. AKAZA, Local property of the singular sets of some Kleinian groups, Tôhoku Math. J. 25 (1973), 1–22.
[3] T. AKAZA AND T. SHIMAZAKI, The Hausdorff dimension of the singular sets of combination groups, Tôhoku, Math. J. 25 (1973), 61–68.
[4] A. F. BEARDON, The Geometry of Discrete Groups, Springer Verlarg, New York-Heidelberg-Berlin, 1983.
[5] B. MASKIT, Kleinian Groups, Springer Verlarg, New York-Heidelberg-Berlin, 1987.
[6] S. J. PATTERSON, The exponent of convergence of Poincaré series, Monatsh. F. Math. 82 (1976), 297–315.

[7] S. J. PATTERSON, Lectures on measures on limit sets of Kleinian groups, in Analytical and Geometric Aspects of Hyperbolic Space (D. B. Epstein, ed.), London Math. Soc. Lecture Notes 111 (1984), 281–323.

[8] N. J. WIELENBERG, Discrete Möbius groups: fundamental polyhedra and convergence, Amer. J. Math. 99 (1977), 861–877.

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