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ON THE CLASSIFICATION OF SMOOTH PROJECTIVE TORIC VARIETIES

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Abstract. We investigate the problem of the classification of smooth projective toric varieties V of dimension d with a given Picard number ρ over an algebraically closed field. For that purpose we introduce a convenient combinatorial description of such varieties by means of primitive relations among $d+\rho$ integral generators of the associated complete regular fan of convex cones in d-dimensional real space. The main conjecture asserts that the number of the primitive relations is bounded by an absolute constant depending only on ρ. We prove this conjecture for $\rho \leq 3$ and give the classification of d-dimensional smooth complete toric varieties with $\rho = 3$.

1. Introduction. Let k be an arbitrary algebraically closed field. A d-dimensional algebraic torus T is a product of d copies of the multiplicative group k^* of k. A toric variety V is a normal algebraic variety containing T as a Zariski open dense subset with an algebraic action of T on V which extends the group law of T. Any toric variety can be described by a finite system of cones spanned by integer points in the real space $\boldsymbol{R}^d$. The reader is referred to [1] for the precise definitions.

In this paper we restrict ourselves to complete smooth toric varieties V. Moreover, we shall often assume that V is a projective toric variety.

One can notice that any description of smooth toric varieties has two sides: the combinatorial structure of the corresponding fan and unimodularity conditions on its generators. The weighted triangulations of $(d-1)$-dimensional sphere introduced in [7] is an example of such a description. One of our objectives is to give a new description of complete smooth toric varieties.

In §2 we introduce the notion of a *primitive collection* of generators and the notion of an associated *primitive relations* among generators. We use these notions to describe toric varieties. If a toric variety V is projective we define also the *degree* of a primitive relation and the *distance* between a generator and a d-dimensional cone of the corresponding fan $\Sigma(V)$.

All these notions are used in §3 to get some properties of the combinatorial structure of a d-dimensional fan $\Sigma(V)$ associated with a toric variety V. It should be remarked that if the Picard number $\rho(V) \geq 3$ there exist combinatorial types of simplicial polytopes which do not give rise to any complete regular fan defining a smooth toric variety [2]. We prove that an arbitrary d-dimensional projective regular fan of cones has a primitive

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collection $\mathscr{P}=\{x_1,\ldots,x_k\}$ of its generators such that $x_1+\cdots+x_k=0$. The last statement is a generalization of a result of Oda in [7] for $d=2$.

Our next purpose is the classification of several types of smooth complete toric varieties. This problem for $d\leq 3$ was investigated by Oda and Miyake in [7]. They obtained the list of all 3-dimensional smooth complete toric varieties with the Picard number $\rho\leq 5$ which cannot be blown down. It is easy to see that the projective space is the unique smooth complete d-dimensional toric variety with $\rho=1$. Recently Kleinschmidt [4] has classified all smooth complete d-dimensional toric varieties with $\rho=2$. It turns out that all such varieties are projectivizations of a decomposable bundle over a projective space of a smaller dimension. In this paper we give two generalizations of this result of Kleinschmidt. First in §4 we give a criterion for a smooth complete d-dimensional toric variety V to be produced from a projective space by a sequence of projectivizations of decomposable bundles. On the other hand, in §§5–6 we give the classification of all smooth complete d-dimensional toric varieties with $\rho=3$.

In §5 we prove strong combinatorial restrictions on a d-dimensional fan Σ with $d+3$ generators which generalize the result of Gretenkort, Kleinschmidt and Sturmfels [2]. After that in §6 we find all primitive relations describing Σ. Finally, in §7 we state some open questions.

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2. Basic definitions. We first recall some standard definitions used in the geometry of toric varieties (see [1]).

2.1. DEFINITION. A convex subset $\sigma\subset\boldsymbol{R}^d$ is called a *regular d-dimensional cone* if there exists a $\boldsymbol{Z}$-basis $\{e_1,\ldots,e_d\}$ of the integer lattice $\boldsymbol{Z}^d\subset\boldsymbol{R}^d$ such that

$$\sigma=\{\lambda_1e_1+\cdots+\lambda_de_d\,|\,\lambda_i\in\boldsymbol{R},\ \lambda_i\geq 0\}\,.$$

In this case the elements $e_1,\ldots,e_d$ are called *generators* of Σ.

2.2. DEFINITION. Let $\sigma\in\boldsymbol{R}^d$ be an arbitrary regular d-dimensional cone with generators $e_1,\ldots,e_d\in\boldsymbol{Z}^d$. For any subset $E\subset\{e_1,\ldots,e_d\}$ we denote by $L(E)$ the linear hull of E (if $E=\varnothing$, we let $L(E)=0$). Then we call $\sigma'=L(E)\cap\sigma$ a *face* of σ and we write $\sigma'\prec\sigma$.

2.3. DEFINITION. A convex subset $\sigma'\in\boldsymbol{R}^d$ is called a *regular k-dimensional cone* if there exist a regular d-dimensional cone $\sigma\in\boldsymbol{R}^d$ and a subset E of its generators such that $k = \dim L(E)$ and $\sigma' = L(E) \cap \sigma$ is a face of σ. In this case we call E the set of generators of σ'.

2.4. DEFINITION. A finite system $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ of regular cones in $\boldsymbol{R}^d$ is called a *complete regular d-dimensional fan* if the following conditions hold:

(i) if $\sigma \in \Sigma$ and $\sigma' \prec \sigma$ then $\sigma' \in \Sigma$;

(ii) if σ, σ' are in Σ, then $\sigma \cap \sigma' \prec \sigma$ and $\sigma \cap \sigma' \prec \sigma'$;

(iii) $\boldsymbol{R}^d = \sigma_1 \cup \cdots \cup \sigma_s$.

We call any generator of a cone $\sigma \in \Sigma$ a *generator* of Σ.

Every complete regular d-dimensional fan Σ is associated with a smooth complete d-dimensional toric variety $V(\Sigma)$. Moreover, two smooth complete d-dimensional toric varieties $V(\Sigma)$ and $V(\Sigma')$ are isomorphic algebraic varieties if and only if the corresponding fans Σ and Σ' are isomorphic up to unimodular transformation of $\boldsymbol{Z}^d$.

2.5. DEFINITION. A complete regular d-dimensional fan Σ in $\boldsymbol{R}^d$ is said to be *projective* if there exists a function $\varphi : \boldsymbol{R}^d \to \boldsymbol{R}$ such that

(i) $\varphi(\boldsymbol{Z}^d) \subset \boldsymbol{Z}$;

(ii) φ is a linear function on each cone of Σ;

(iii) for two arbitrary distinct d-dimensional cones σ and σ' in Σ the restrictions $\varphi|_\sigma$ and $\varphi|_{\sigma'}$ are different linear functions;

(iv) φ is a convex function: $\varphi(x) + \varphi(y) \geq \varphi(x+y)$ for all $x, y \in \boldsymbol{R}^d$.

We call such a function φ a *support function* on Σ.

It is well-known that a smooth complete d-dimensional toric variety $V(\Sigma)$ is a projective variety if and only if the corresponding fan Σ has a support function φ (see [1], [7]).

We introduce now our new definitions.

Let Σ be a complete regular d-dimensional fan and Let $G(\Sigma)$ be the set of all generators of Σ.

2.6. DEFINITION. A nonempty subset $\mathscr{P} = \{x_1, \ldots, x_k\} \subset G(\Sigma)$ is called a *primitive collection* if for each generator $x_i \in \mathscr{P}$ the elements of $\mathscr{P} \setminus \{x_i\}$ generate a $(k-1)$-dimensional cone in Σ, while $\mathscr{P}$ does not generate any k-dimensional cone in Σ.

2.7. DEFINITION. Let $\mathscr{P} = \{x_1, \ldots, x_k\}$ be a primitive collection in $G(\Sigma)$. Let $S(\mathscr{P})$ denote $x_1 + \cdots + x_k$. The *focus* $\sigma(\mathscr{P})$ of $\mathscr{P}$ is the cone in Σ of the smallest dimension containing $S(\mathscr{P})$. (It follows from 2.4 (iii) that such $\sigma(\mathscr{P})$ exists.)

2.8. DEFINITION. Let $\mathscr{P} = \{x_1, \ldots, x_k\}$ be a primitive collection in $G(\Sigma)$ and $\sigma(\mathscr{P})$ its focus. Let $y_1, \ldots, y_m$ be generators of $\sigma(\mathscr{P})$. It follows from 2.1–2.3 that there exists a unique linear combination $n_1 y_1 + \cdots + n_m y_m$ with positive integer coefficients n_i which is equal to $x_1 + \cdots + x_k$. Then the linear relation

$$x_1 + \cdots + x_k - n_1 y_1 - \cdots - n_m y_m = 0$$

is called the *primitive relation associated with* $\mathscr{P}$ and is denoted by $\mathscr{R}(\mathscr{P})$.

Suppose that Σ is a projective regular d-dimensional fan with a support function φ.

2.9. DEFINITION. Let $\mathscr{P}=\{x_1, \dots, x_k\}$ be a primitive collection in $G(\Sigma)$ and let

$$x_1+\cdots+x_k-n_1y_1-\cdots-n_my_m=0$$

be the associated primitive relation. The integer

$$\begin{aligned} D_\varphi(\mathscr{P}) &= \varphi(x_1)+\cdots+\varphi(x_k)-n_1\varphi(y_1)-\cdots-n_m\varphi(y_m) \\ &= \varphi(x_1)+\cdots+\varphi(x_k)-\varphi(x_1+\cdots+x_k) \end{aligned}$$

is called the *degree of* $\mathscr{P}$ *relative to* φ. (It follows from 2.5 (iii), and 2.5 (iv) that $D_\varphi(\mathscr{P})$ is always a positive integer.)

2.10. DEFINITION. Let σ be an arbitrary d-dimensional cone in Σ with generators $x_1, \dots, x_d$ and let x be an element of $G(\Sigma)$. There exists a unique linear combination $a_1x_1+\cdots+a_dx_d$ with integer coefficients $a_1, \dots, a_d$ which is equal to x. The integer

$$d_\varphi(x, \sigma)=\varphi(x)-a_1\varphi(x_1)-\cdots-a_d\varphi(x_d)$$

is called the *distance between* x *and* σ. (It follows from 2.5 (iii), and 2.5 (iv) that $d_\varphi(x, \sigma)\geq 0$, and $d_\varphi(x, \sigma)=0$ if and only if $x\in\sigma$.)

2.11. DEFINITION. Let σ be an arbitrary d-dimensional cone in Σ with generators $x_1, \dots, x_d$ and let x be an element of $G(\Sigma)$. We call x a *nearest generator of* Σ *relative to* σ if $x\notin\sigma$ and for any generator $x'\notin\sigma$, one has $d_\varphi(x, \sigma)\leq d_\varphi(x', \sigma)$. (It is possible that σ has several nearest generators.)

We recall the computation of the Picard group $\mathrm{Pic}(V(\Sigma))$ of a smooth toric variety $V(\Sigma)$ associated with a regular fan Σ (see [1], [6], [7]).

2.12. PROPOSITION. *There exists a short exact sequence*

$$0\longrightarrow \boldsymbol{Z}^d \xrightarrow{\ \psi\ } F\longrightarrow \mathrm{Pic}(V(\Sigma))\longrightarrow 0\,,$$

where F *is the free abelian group whose generators are the elements of* $G(\Sigma)$, *and the map* ψ *is defined by the integer matrix* Ψ *whose rows consist of coordinates of the corresponding elements of* $G(\Sigma)$.

2.13. COROLLARY. *If* Σ *is a complete regular fan, then the dual group*

$$\mathrm{Pic}(V(\Sigma))^*=\mathrm{Hom}(\mathrm{Pic}(V(\Sigma)), \boldsymbol{Z})$$

can be identified with the group $A_1(V(\Sigma))$ *of algebraic* 1*-cycles modulo numerical equivalence, and it consists of all possible linear relations with integer coefficients among the elements of* $G(\Sigma)\subset\boldsymbol{Z}^d$.

2.14. REMARK. The group $\mathrm{Pic}(V(\Sigma))$ consists of all functions $\delta\colon \boldsymbol{R}^d \to \boldsymbol{R}$ which satisfy 2.5 (i), (ii) modulo integral linear functions. If

$$a_1x_1 + \cdots + a_kx_k = 0$$

is an integral linear relation among generators of Σ, which is an element R of $A_1(V(\Sigma))$, then

$$\langle R, \delta\rangle = a_1\delta(x_1) + \cdots + a_k\delta(x_k)$$

is the corresponding intersection number. Obviously, this number does not change its value if we replace δ by a sum $\delta + f$, where $f\colon \boldsymbol{R}^d \to \boldsymbol{R}$ is an integral linear function. In particular, the degree of a primitive collection relative to a support function φ is also an intersection number.

We finish this paragraph by the following important theorem.

2.15. THEOREM. *Let Σ be a projective regular d-dimensional fan of cones in $\boldsymbol{R}^d$ and let $\mathrm{Pr}(\Sigma)$ be the cone generated in $A_1(V(\Sigma))\otimes\boldsymbol{R}$ by all primitive relations. Then $\mathrm{Pr}(\Sigma)$ coincides with Mori's cone $\overline{NE}(V(\Sigma))$ of effective 1-cycles (see [9]).*

The proof of this theorem is contained in [6], [8], [9].

3. Some properties. Let Σ be a complete regular d-dimensional fan of cones in $\boldsymbol{R}^d$.

3.1. PROPOSITION. *Let $\mathscr{P} = \{x_1, \ldots, x_k\}$ be a primitive collection in $G(\Sigma)$ with the focus $\sigma(\mathscr{P})$. Then $\mathscr{P} \cap \sigma(\mathscr{P}) = \varnothing$.*

PROOF. Let $\{y_1, \ldots, y_m\}$ be the generators of $\sigma(\mathscr{P})$. It is sufficient to prove that $\{x_1, \ldots, x_k\} \cap \{y_1, \ldots, y_m\} = \varnothing$. Assume, for instance, that $x_1 = y_1$. It follows from the definition of primitive collections that the element $x = x_2 + \cdots + x_k$ is in the interior of the $(k-1)$-dimensional cone σ' generated by $x_2, \ldots, x_k$. On the other hand, it follows from the equality $x_1 = y_1$ and the primitive relation

$$x_1 + \cdots + x_k - n_1y_1 - \cdots - n_my_m = 0$$

that

$$x_2 + \cdots + x_k = (n_1 - 1)y_1 + \cdots + n_my_m\,,$$

and the element $x = x_2 + \cdots + x_k$ is in the interior of the cone σ'' generated by $y_1, \ldots, y_m$ (if $n_1 > 1$), or by $y_2, \ldots, y_m$, (if $n_1 = 1$). By 2.4 (ii), one has $\sigma' = \sigma''$. The last equality is possible only if $\{x_2, \ldots, x_k\} = \{y_1, \ldots, y_m\}$ and $n_1 = 2$, $n_2 = \cdots = n_m = 1$, or if $\{x_2, \ldots, x_k\} = \{y_2, \ldots, y_m\}$ and $n_1 = n_2 = \cdots = n_m = 1$.

If σ'' is generated by $\{y_1, \ldots, y_m\}$, then y_1 must coincide with one $x_2, \ldots, x_k$. This contradicts the assumption that $x_1, \ldots, x_k$ are different generators of Σ.

If σ'' is generated by $\{y_2, \ldots, y_m\}$, then $\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_m\}$. This contradicts the fact that $y_1, \ldots, y_m$ are generators of $\sigma(\mathscr{P})$.

Now we assume that Σ is a projective regular d-dimensional fan of cones in $\boldsymbol{R}^d$ with a support function φ.

3.2. PROPOSITION. *There exists a primitive collection $\mathscr{P}=\{x_1, \ldots, x_k\}$ in $G(\Sigma)$ such that the associated primitive relation is of the form*

$$x_1+\cdots+x_k=0 .$$

In the other words, the focus $\sigma(\mathscr{P})=\{0\}$.

PROOF. Since Σ is a complete fan, there exist generators $x_1, \ldots, x_m \in G(\Sigma)$ and positive integers $a_1, \ldots, a_m$ such that

$$a_1x_1+\cdots+a_mx_m=0 .$$

We can assume that the sum

$$a_1\varphi(x_1)+\cdots+a_m\varphi(x_m)$$

has the smallest possible value r (by 2.5 (iii), (iv), r is a positive integer).

Now we shall prove that in fact $a_1=\cdots=a_m=1$ and $\{x_1, \ldots, x_m\}$ is a primitive collection in $G(\Sigma)$.

Obviously, $x_1, \ldots, x_m$ cannot be generators of a cone $\sigma\in\Sigma$. So, there exists a subset in $\{x_1, \ldots, x_m\}$ (e.g. $\{x_1, \ldots, x_q\}$) which is a primitive collection. Let

$$x_1+\cdots+x_q-b_1y_1-\cdots-b_py_p=0$$

be the corresponding primitive relation. One has

$$\begin{aligned} r&=a_1\varphi(x_1)+\cdots+a_m\varphi(x_m)\\ &=(a_1-1)\varphi(x_1)+\cdots+(a_q-1)\varphi(x_q)\\ &\quad+a_{q+1}\varphi(x_{q+1})+\cdots+a_m\varphi(x_m)+\varphi(x_1)+\cdots+\varphi(x_q)\\ &>(a_1-1)\varphi(x_1)+\cdots+(a_q-1)\varphi(x_q)\\ &\quad+a_{q+1}\varphi(x_{q+1})+\cdots+a_m\varphi(x_m)+b_1\varphi(y_1)+\cdots+b_p\varphi(y_p) . \end{aligned}$$

On the other hand,

$$(a_1-1)x_1+\cdots+(a_q-1)x_q+a_{q+1}x_{q+1}+\cdots+a_mx_m+b_1y_1+\cdots+b_py_p=0 .$$

This contradicts the choice of r unless $a_1=\cdots=a_m=1$, $q=m$ and the subset of generators $\{x_1, \ldots, x_m\}$ is a primitive collection in $G(\Sigma)$.

3.3. PROPOSITION. *Let σ be a d-dimensional cone in Σ and let $x_1, \ldots, x_d$ be the generators of σ. Consider two generators $x, x'\in G(\Sigma)$ which do not belong to σ. By* 2.6, *there exists a primitive collection $\mathscr{P}\subset\{x, x_1, \ldots, x_d\}$. Then the following hold*:

(i) *if $\sigma(\mathscr{P})$ contains x', then $d_\varphi(x, \sigma)>d_\varphi(x', \sigma)$;*

(ii) *if all generators of $\sigma(\mathscr{P})$ are in σ, then $d_\varphi(x, \sigma)=D_\varphi(\mathscr{P})$;*

(iii) *there exists at most one primitive collection $\mathscr{P}\subset\{x, x_1, \cdots, x_d\}$ such that the* *focus* $\sigma(\mathscr{P})\subset\sigma$;

(iv) *if* x *is a nearest generator in* $G(\Sigma)$ *relative to* σ, *then* $\mathscr{P}$ *is a unique primitive collection in* $\{x, x_1, \ldots, x_d\}$, *and* $d_\varphi(x, \sigma)=D_\varphi(\mathscr{P})$.

PROOF. (i) We first prove that if a primitive collection $\mathscr{P}$ (e.g., $\mathscr{P}=\{x, x_1, \ldots, x_k\}$, $k<d$), gives rise to a primitive relation

$$x+x_1+\cdots+x_k-n_1y_1-\cdots-n_my_m=0\ ,$$

then

$$d_\varphi(x, \sigma)>n_1d_\varphi(y_1, \sigma)+\cdots+n_md_\varphi(y_m, \sigma)\ . \tag{1}$$

Let $y_i=b_{i,1}x_1+\cdots+b_{i,d}x_d$ ($b_{i,j}\in\boldsymbol{Z}$), and $x=a_1x_1+\cdots+a_dx_d$. Then

$$\begin{aligned} a_1&=n_1b_{11}+\cdots+n_mb_{m,1}-1\ ,\\ &\ \ \cdots\\ a_k&=n_1b_{1,k}+\cdots+n_mb_{m,k}-1\ ,\\ a_{k+1}&=n_1b_{1,k+1}+\cdots+n_mb_{m,k+1}\ ,\\ &\ \ \cdots\\ a_d&=n_1b_{1,d}+\cdots+n_mb_{m,d}\ . \end{aligned}$$

By 2.5 (iii), (iv), we get

$$\varphi(x_1)+\cdots+\varphi(x_k)+\varphi(x)>\varphi(n_1y_1+\cdots+n_my_m)\ .$$

It follows from 2.5 (ii) that

$$\varphi(n_1y_1+\cdots+n_my_m)=n_1\varphi(y_1)+\cdots+n_m\varphi(y_m)\ .$$

Hence,

$$\begin{aligned} &\varphi(x_1)+\cdots+\varphi(x_k)+d_\varphi(x, \sigma)\\ &\quad=\varphi(x_1)+\cdots+\varphi(x_k)+\varphi(x)-a_1\varphi(x_1)-\cdots-a_d\varphi(x_d)\\ &\quad>n_1\varphi(y_1)+\cdots+n_m\varphi(y_m)-a_1\varphi(x_1)-\cdots-a_d\varphi(x_d)\\ &\quad=n_1(\varphi(y_1)-b_{11}\varphi(x_1)-\cdots-b_{1,d}\varphi(x_d))+\cdots\\ &\qquad+n_m(\varphi(y_m)-b_{m,1}\varphi(x_1)-\cdots-b_{m,d}\varphi(x_d))+\varphi(x_1)+\cdots+\varphi(x_k)\\ &\quad=\varphi(x_1)+\cdots+\varphi(x_k)+n_1d_\varphi(y_1, \sigma)+\cdots+n_md_\varphi(y_m, \sigma). \end{aligned}$$

This inequality implies (1). Thus, $d_\varphi(x, \sigma)>d_\varphi(x', \sigma)$, if $x'=y_i$ for some i ($1<i<m$).

(ii) Let

$$x+x_1+\cdots+x_k-n_1y_1-\cdots-n_my_m=0$$

be a primitive relation associated with the primitive collection $\mathscr{P}$. Then

$$D_\varphi(\mathscr{P})=\varphi(x)+\varphi(x_1)+\cdots+\varphi(x_m)-n_1\varphi(y_1)-\cdots-n_m\varphi(y_m)\ .$$

Let $y_1, \ldots, y_m$ be generators of σ (i.e. $\{y_1, \ldots, y_m\} \subset \{x_1, \ldots, x_d\}$). Using 2.5 (ii), we get

$$a_1\varphi(x_1) + \cdots + a_d\varphi(x_d) = n_1\varphi(y_1) + \cdots + n_m\varphi(y_m) - \varphi(x_1) - \cdots - \varphi(x_k),$$

where $x = a_1x_1 + \cdots + a_dx_d$. Hence, $d_\varphi(x, \sigma) = D_\varphi(\mathscr{P})$.

(iii) Assume that there exist two different primitive collections $\mathscr{P}_1$ and $\mathscr{P}_2$ in

$$\{x, x_1, \ldots, x_d\},$$

such that $\sigma(\mathscr{P}_1) \subset \sigma$ and $\sigma(\mathscr{P}_2) \subset \sigma$. Then, from the corresponding primitive relations, we get two different linear combinations of $x_1, \ldots, x_d$ which are equal to x. This is impossible, since $x_1, \ldots, x_d$ form a basis of $\boldsymbol{Z}^d$.

(iv) This statement is a corollary of (i), (ii) and (iii).

3.4. T-invariant Divisors. Every generator $x \in G(\Sigma)$ of a complete regular d-dimensional fan Σ in $\boldsymbol{R}^d$ gives rise to a complete rugular $(d-1)$-dimensional fan Σ_x in $\boldsymbol{R}^{d-1}$ corresponding to a smooth T-invariant divisor on $V(\Sigma)$. The fan Σ_x consists of images of all cones in Σ containing x via the natural pojection $\boldsymbol{R}^d \to \boldsymbol{R}^{d-1} = \boldsymbol{R}^d/\boldsymbol{R}\langle x\rangle$. The following easy statement describes all primitive collections for Σ_x.

3.5. Proposition. (i) *The set $G(\Sigma_x)$ of all generators for Σ_x consists of the images $\bar{x}' \in \boldsymbol{R}^d/\boldsymbol{R}\langle x\rangle$ of all generators x' such that $\{x, x'\}$ generate a 2-dimensional cone in Σ.*

(ii) *If $\{\bar{x}_1, \ldots, \bar{x}_k\}$ is a primitive collection in $G(\Sigma_x)$, then*

$$\{x, x_1, \ldots, x_k\}, \quad or \quad \{x_1, \ldots, x_k\}$$

is a primitive collection in $G(\Sigma)$.

Proof. (i) The first statement is an immediate consequence of 3.4.

(ii) Let $\{\bar{x}_1, \cdots, \bar{x}_k\}$ be a primitive collection in $G(\Sigma_x)$. By 3.4, $x, x_1, \cdots, x_k$ are not generators of a cone in Σ. Hence, there exists a primitive collection $\mathscr{P} \subset \{x, x_1, \ldots, x_k\}$. Since $\{x, x_1, \ldots, x_k\} \setminus \{x_i\}$ generates a cone in Σ for all i $(1 \leq i \leq k)$, we get $\{x_1, \ldots, x_k\} \subset \mathscr{P}$. Thus, $\mathscr{P} = \{x, x_1, \ldots, x_k\}$, or $\mathscr{P} = \{x_1, \ldots, x_k\}$.

4. Toric bundles. By [7], using the language of primitive collections and associated primitive relations, we get the following characterization of toric bundles.

4.1. Proposition. *A regular complete d-dimensional fan Σ corresponds to a toric variety $V = V(\Sigma)$ which is a toric $\mathbf{P}^k$-bundle over a smooth $(d-k)$-dimensional toric variety W if and only if there exists a primitive collection $\mathscr{P} = \{x_1, \ldots, x_{k+1}\} \subset G(\Sigma)$ such that*

(i) *the corresponding primitive relation is*

$$x_1 + \cdots + x_{k+1} = 0;$$

(ii) *$\mathscr{P} \cap \mathscr{P}' = \varnothing$ for any primitive collection $\mathscr{P}' \subset G(\Sigma)$ such that $\mathscr{P} \neq \mathscr{P}'$.*

4.2. Definition. We say that a regular complete d-dimensional fan Σ is a *splitting* fan if any two different primitive collections in $G(\Sigma)$ have no common elements.

4.3. THEOREM. *Let Σ be a splitting fan. Then the corresponding toric variety $V(\Sigma)$ is a projectivization of a decomposable bundle over a toric variety W which is associated with a splitting fan of a smaller dimension.*

PROOF. By 4.1, we have only to prove the existence of a primitive collection with zero focus (we cannot use 3.2 without knowing the projectivity of the fan Σ). We prove the last statement by induction of $\#G(\Sigma)$.

By 3.5 (ii), any divisor $D_{x_i} = V(\Sigma)$ corresponding to a generator $x_i \in G(\Sigma)$ on the toric variety $V(\Sigma)$ is also associated with a splitting fan. This allows us to apply the induction hypothesis.

Assume that any primitive collection in $G(\Sigma)$ has no zero focus. Choose a generator $x_0 \in G(\Sigma)$. Let $\{\bar{x}_1, \ldots, \bar{x}_k\}$ be a primitive collection in $G(\Sigma_{x_0})$ having zero focus (by the induction hypothesis, it exists). By 3.5 (ii), we have to consider two cases.

CASE 1. $\mathscr{P} = \{x_0, x_1, \ldots, x_k\}$ is a primitive collection in $G\{\Sigma)$. It follows from our choice of the set $\{\bar{x}_1, \ldots, \bar{x}_k\}$ that the sum $S(\mathscr{P}) = x_0 + x_1 + \cdots + x_k$ is an integral multiple of x_0. By 3.1, $S(\mathscr{P})$ cannot be a *positive* multiple of x_0. Assume that $S(\mathscr{P}) = -ax_0$, where $a \in \boldsymbol{Z}_{>0}$. Then

$$x_1 + \cdots + x_k = -(a+1)x_0 \, .$$

Thus, $S(\mathscr{P})$ is in the interior of the cone $\sigma \in \Sigma$ generated by $\{x_1, \ldots, x_k\}$. By 3.1, $\sigma \cap \sigma\ (\mathscr{P}) = \varnothing$, a contradiction. Hence only the next case is possible.

CASE 2. $\mathscr{P} = \{x_1, \ldots, x_k\}$ is a primitive collection, and the sum $S(\mathscr{P}) = x_1 + \cdots + x_k$ is an integral multiple of x_0.

Since every primitive collection has at least two generators, the number of primitive collections for a splitting fan Σ is not greater than a half of the number of generators of Σ. So, there exist two different generators $x_i, x_j \in G(\Sigma)$ and a primitive collection $\mathscr{P} = \{x_1, \ldots, x_k\}$ such that the sum $S(\mathscr{P}) = x_1 + \cdots + x_k$ is an integral multiple of both x_i and x_j. This is possible only if $x_i = -x_j$. So, $\{x_i, x_j\}$ is a primitive collection with zero focus.

The statement is proved.

4.4. COROLLARY. *A smooth complete toric variety V is produced from a projective space by a sequence of projectivizations of decomposable bundles if and only if the corresponding fan $\Sigma(V)$ is a splitting fan.*

4.5. REMARK. One can notice that any complete smooth toric variety with Picard number 2 is associated with a splitting fan [4].

5. Toric varieties with $\rho = 3$: the number of primitive collections. Kleinschmidt and Sturmfels [5] have proved that an arbitrary smooth complete toric variety V of dimension d with Picard number $\rho = 3$ is projective. Consequently, for any complete regular d-dimensional fan with $d+3$ generators there exists a strictly convex support function $\varphi: \boldsymbol{R}^d \to \boldsymbol{R}$ as in 2.5. Thus, the notions of the degree and the distance introduced in §2 are well-defined.

5.1. Let $X=\{x_1, \dots, x_{d+3}\}$ be an arbitrary set consisting of $d+3$ elements. We divide X into m nonempty subsets $X_0, X_1, \dots, X_{m-1}$ without common elements, where $m=2p+3$ and p is a nonnegative integer. We can assume that

$$X_0=\{x_1, \dots, x_{s_0}\}$$
$$X_1=\{x_{s_0+1}, \dots, x_{s_1}\}$$
$$\dots$$
$$X_{m-1}=\{x_{s_{m-1}+1}, \dots, x_{s_{m-1}}\},$$

where $s_0<s_1\cdots<s_{m-1}=d+3$ and $\#X_i=s_i-s_{i-1}$ for $i>0$. It is more convenient in the sequel to assume that the index i for X_i is an element of the residue ring $\boldsymbol{Z}/m\boldsymbol{Z}$. We denote by $\mathscr{X}_i$ the union

$$X_i \cup X_{i+1} \cup \cdots \cup X_{i+p}.$$

5.2. Proposition. *Let Σ be an arbitrary complete regular d-dimensional fan with $d+3$ generators. Then there exists a nonnegative integer p such that the set*

$$G(\Sigma)=X=\{x_1, \dots, x_{d+3}\}$$

of all generators of Σ can be represented as a union of subsets $X_0, X_1, \dots, X_{m-1}$ without common elements (*see* 5.1) and the corresponding subsets $\mathscr{X}_i$ ($i\in \boldsymbol{Z}/m\boldsymbol{Z}$) *are exactly all primitive collections of the generators of Σ.*

Proof. This statement is a simple translation of the well-known description of combinatorial types of d-polytopes with $d+3$ vertices from the Gale-transform language (see [3], [8]) to the one of primitive collections.

5.3. Corollary. *Let $x_a\in X_\alpha$, $x_b\in X_\beta$, $x_c\in X_\gamma$ be three of $d+3$ generators of a fan Σ as in 5.2. Then the elements of $X\setminus\{x_a, x_b, x_c\}$ generate a d-dimensional cone of Σ if and only if the zero point 0 of the complex plane $\boldsymbol{C}$ is in the interior of the triangle with the vertices $e^{2\pi i\alpha/m}$, $e^{2\pi i\beta/m}$ and $e^{2\pi i\gamma/m}$.*

5.4. Proposition. *In the situation as in 5.2, one has $m\le 7$.*

Proof. Assume that $m>7$. Since m is odd, we have $m>9$. Choose three generators $x_a, x_b, x_c\in X$ such that $x_a\in X_0$, $x_b\in X_t$, $x_c\in X_{2t}$, where $m=3t+t'$,$|t'|<1$. By 5.3, $X\setminus\{x_a, x_b, x_c\}$ generates a d-dimensional cone σ of Σ. By 5.2, for each $x_i\in\{x_a, x_b, x_c\}$ there exist at least two primitive collections which contain only x_i and generators of σ. This contradicts 3.3 (iv), since at least one generator form the set $\{x_a, x_b, x_c\}$ is a nearest generator relative to σ.

5.5. Proposition. *In the situation as in 5.2, one has $m\ne 7$.*

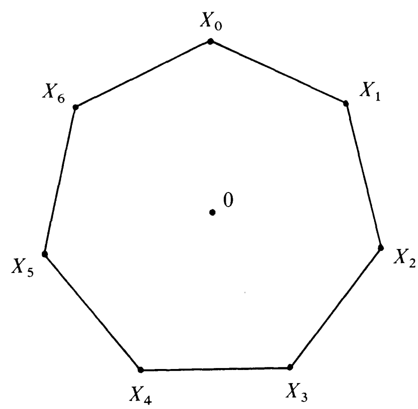

FIGURE 1.

PROOF. Assume that $m=7$. We have seen primitive relations

$$\mathscr{R}(\mathscr{X}_r): \sum_{x_i \in \mathscr{X}_r} x_i - \sum_{x'_j \in \sigma(\mathscr{X}_r)} a_{r,j} x'_j = 0 ,$$

where $a_{r,j}$ are positive integers and $r \in \boldsymbol{Z}/7\boldsymbol{Z}$. It is convenient to use a picture of heptagon with the vertices $ie^{2\pi i r/7} \in \boldsymbol{C}$ (see Figure 1).

5.6. LEMMA. *For any $\alpha \in \boldsymbol{Z}/7\boldsymbol{Z}$, one has*

$$\sigma(X_\alpha) \cap G(\Sigma) \subset X_{\alpha+4} \cup X_{\alpha+5} .$$

PROOF OF LEMMA 5.6. Choose $x_a \in X_{\alpha+1}$, $x_b \in X_{\alpha+3}$, $x_c \in X_{\alpha+6}$. By 5.3, $X \setminus \{x_a, x_b, x_c\}$ generates a d-dimensional cone σ in Σ. By 3.3 (iv), in $\{x_a, x_b, x_c\}$ only x_a can be a nearest generator relative to σ, since $x_b \in \mathscr{X}_{\alpha+2} \cap \mathscr{X}_{\alpha+3}$ and $x_c \in \mathscr{X}_{\alpha+4} \cap \mathscr{X}_{\alpha+5}$. By 3.3 (i), $\sigma(\mathscr{X}_\alpha)$ does not contain x_b and x_c. But we can choose an arbitrary element in $\mathscr{X}_\alpha$ as x_b. So, $\sigma(\mathscr{X}_\alpha) \cap X_{\alpha+3} = \varnothing$. Similarly, $\sigma(\mathscr{X}_\alpha) \cap X_{\alpha+6} = \varnothing$. By 3.1, $\sigma(\mathscr{X}_\alpha) \cap (X_\alpha \cup X_{\alpha+1} \cup X_{\alpha+2}) = \varnothing$. Thus, the lemma is proved.

We return to 5.5.

We can take $\alpha \in \boldsymbol{Z}/7\boldsymbol{Z}$ such that

$$D_\varphi(\mathscr{X}_\alpha) = \max\{D_\varphi(\mathscr{X}_\beta) \mid \beta \in \boldsymbol{Z}/7\boldsymbol{Z}\} .$$

Choose again $x_a \in X_{\alpha+1}$, $x_b \in X_{\alpha+3}$, $x_c \in X_{\alpha+6}$. Using 5.6 and 3.3 (ii), we get $D_\varphi(\mathscr{X}_\alpha) = d_\varphi(x_a, \sigma)$, where σ is generated by $X \setminus \{x_a, x_b, x_c\}$. We have already seen in the proof of 5.6 that in $\{x_a, x_b, x_c\}$ only x_a can be a nearest generator relative to $\sigma \in \Sigma$. So, $d_\varphi(x_a, \sigma) < d_\varphi(x_b, \sigma)$ and $d_\varphi(x_a, \sigma) < d_\varphi(x_c, \sigma)$. Assume, for instance, that $d_\varphi(x_b, \sigma) \leq d_\varphi(x_c, \sigma)$. Applying 5.6 after the cyclic permutation $\alpha \mapsto \alpha+2$, one has $x_a \notin \sigma(\mathscr{X}_{\alpha+2})$. Since $x_b \in \mathscr{X}_{\alpha+2}$, if follows from 3.3 (i) that $x_c \notin \sigma(\mathscr{X}_{\alpha+2})$. Hence, by 3.3 (ii), we have $D_\varphi(\mathscr{X}_{\alpha+2})=d_\varphi(x_b, \sigma)$. Consequently, $D_\varphi(\mathscr{X}_{\alpha+2})>D_\varphi(\mathscr{X}_\alpha)$. This contradicts the choice of $\alpha\in \boldsymbol{Z}/7\boldsymbol{Z}$. Thus, the case $m=7$ is impossible.

Propositions 5.4 and 5.5 imply the following theorem.

5.7. THEOREM. *If Σ is a complete regular d-dimensional fan with $d+3$ generators, then the number of primitive collections of its generators is equal to* 3 *or* 5.

If Σ has exactly three primitive collections in $G(\Sigma)$, then we come to a particular case of 4.3. In this case the associated smooth toric variety $V(\Sigma)$ is isomorphic to a projectivization of a decomposable bundle over a smooth toric variety W of a smaller dimension with Picard number 2. Hence, we have to investigate only the case of five primitive collections in $G(\Sigma)$. This is the object of the next section.

6. Toric varieties with $\rho=3$: the classification of primitive relations. Let Σ be a complete regular d-dimensional fan of cones in $\boldsymbol{R}^d$ with $d+3$ generators and with a support function φ.

We use the notation of the previous section and assume that $G(\Sigma)$ contains exactly five primitive collections $\mathscr{X}_\alpha=X_\alpha\cup X_{\alpha+1}$, where $\alpha\in\boldsymbol{Z}/5\boldsymbol{Z}$. In our investigation it is convenient to use a picture of the pentagon with vertices $ie^{2\pi i\alpha/5}\in\boldsymbol{C}$ (see Figure 2).

6.1. PROPOSITION. *Suppose that $\sigma(X_\alpha)\cap G(\Sigma)\subset X_{\alpha+3}$ for all $\alpha\in\boldsymbol{Z}/5\boldsymbol{Z}$. Then for any $\alpha\in\boldsymbol{Z}/5\boldsymbol{Z}$ at least one of the following statements hold*:

(i) $\sigma(\mathscr{X}_{\alpha+2})\cap G(\Sigma)=X_\alpha$;

(ii) $\sigma(\mathscr{X}_{\alpha+3})\cap G(\Sigma)=X_{\alpha+1}$.

PROOF. It follows from our conditions that $\sigma(\mathscr{X}_{\alpha+2})\cap G(\Sigma)\subset X_\alpha$ and $\sigma(\mathscr{X}_{\alpha+3})\cap G(\Sigma)\subset X_{\alpha+1}$. Assume that there exist $x_a\in X_\alpha$ and $x_b\in X_{\alpha+1}$ such that $x_a\notin\sigma(\mathscr{X}_{\alpha+2})$ and $x_b\notin\sigma(\mathscr{X}_{\alpha+3})$. Choose an arbitrary element $x_c\in X_{\alpha+3}$. By 5.3, $X\setminus\{x_a, x_b, x_c\}$ generates a

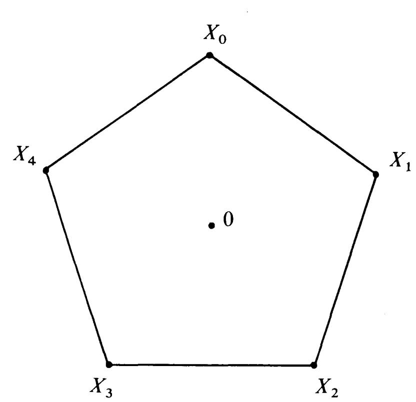

FIGURE 2.

d-dimensional cone $\sigma \in \Sigma$. Thus, we have two primitive collections $\mathscr{X}_{\alpha+2}, \mathscr{X}_{\alpha+3} \subset X \setminus \{x_a, x_b\}$ such that $\sigma(\mathscr{X}_{\alpha+2}), \sigma(\mathscr{X}_{\alpha+3}) \subset \sigma$. This contradicts 3.3 (iii).

The sum $S(\mathscr{X}_\alpha)$ of all generators in $\mathscr{X}_\alpha$ is denoted by S_α. Let P_α be the sum of all generators in X_α.

6.2. PROPOSITION. *Suppose that $\sigma(\mathscr{X}_\alpha) \cap G(\Sigma) \subset X_{\alpha+3}$ for all $\alpha \in \mathbf{Z}/5\mathbf{Z}$. Then up to a cyclic permutation of indices, one has $S_0 = 0$, $S_1 = P_4$, $S_2 = 0$, $S_3 = P_1$, $S_4 = P_2$.*

PROOF. Using 6.1 for all $\alpha \in \mathbf{Z}/5\mathbf{Z}$, one can easily conclude that there exists $\beta \in \mathbf{Z}/5\mathbf{Z}$ such that

$$\sigma(\mathscr{X}_{\beta+2}) \cap G(\Sigma) = X_\beta \quad \text{and} \quad \sigma(\mathscr{X}_\beta) \cap G(\Sigma) = X_{\beta+3}\,.$$

Thus, we have

$$P_{\beta+2} + P_{\beta+3} = S_{\beta+2} = P_\beta + P'_\beta\,, \qquad P_\beta + P_{\beta+1} = S_\beta = P_{\beta+3} + P'_{\beta+3}\,,$$

where $P'_\beta \in \sigma(\mathscr{X}_{\beta+2})$ and $P'_{\beta+3} \in \sigma(\mathscr{X}_\beta)$. It follows from these two equalities that

$$P_{\beta+1} + P_{\beta+2} = P'_\beta + P'_{\beta+3}\,.$$

By 5.3, $X_\beta \cup X_{\beta+3}$ is contained in a d-dimensional cone $\sigma \in \Sigma$. So, the focus $\sigma(\mathscr{X}_{\beta+1})$ is generated by a subset in $X_\beta \cup X_{\beta+3}$. On the other hand, it follows from our conditions that $(\sigma(\mathscr{X}_{\beta+1}) \cap G(\Sigma)) \subset X_{\beta+4}$. Consequently, P'_β and $P'_{\beta+3}$ must be zero and $S_{\beta+2} = P_\beta$, $S_{\beta+2} = P_\beta$, $S_{\beta+1} = 0$. Using again 6.1, we get

$$\sigma(\mathscr{X}_{\beta+4}) \cap G(\Sigma) = X_{\beta+2}\,, \quad \text{or} \quad \sigma(\mathscr{X}_{\beta+3}) \cap G(\Sigma) = X_{\beta+1}.$$

In the first case, we can repeat the above arguments relative to

$$\sigma(\mathscr{X}_{\beta+4}) \cap G(\Sigma) = X_{\beta+2} \quad \text{and} \quad \sigma(X_{\beta+2}) \cap G(\Sigma) = X_\beta\,.$$

As a result, we obtain $S_{\beta+3} = 0$, $S_{\beta+4} = P_{\beta+2}$. In the second case, applying the same arguments to

$$\sigma(\mathscr{X}_{\beta+3}) \cap G(\Sigma) = X_{\beta+1} \quad \text{and} \quad \sigma(X_\beta) \cap G(\Sigma) = X_{\beta+3}\,,$$

we get $S_{\beta+3} = P_{\beta+1}$, $S_{\beta+4} = 0$. Thus, the statement is proved.

6.3. PROPOSITION. *Suppose that a cone $\sigma(\mathscr{X}_\alpha)$ contains a generator $x_a \in X_{\alpha+2}$. Then the following statements hold*:

(i) $X_\alpha \cap (\sigma(\mathscr{X}_{\alpha+1}) \cup \sigma(\mathscr{X}_{\alpha+2}) \cup \sigma(\mathscr{X}_{\alpha+3})) = \varnothing$;

(ii) $S_{\alpha+2} = 0$;

(iii) $\sigma(\mathscr{X}_{\alpha+1}) \cap G(\Sigma) = X_{\alpha+4}$;

(iv) $\sigma(\mathscr{X}_{\alpha+3}) \cap G(\Sigma) = X_{\alpha+1}$;

(v) $S_{\alpha+1} = P_{\alpha+4}$, $S_{\alpha+3} = P_{\alpha+1}$.

PROOF. (i) Choose arbitrary $x_b \in X_\alpha$, $x_c \in X_{\alpha+4}$. By 5.3, $X \setminus \{x_a, x_b, x_c\}$ generates a

d-dimensional cone σ in Σ. By 3.3 (i), $d_\varphi(x_b, \sigma) > d_\varphi(x_a, \sigma)$. By 3.3 (iv), x_a is not a nearest generator relative to σ. Consequently, $d_\varphi(x_a, \sigma) > d_\varphi(x_c, \sigma)$ and $x_b \notin \sigma(\mathcal{X}_{\alpha+1}) \cup \sigma(\mathcal{X}_{\alpha+2}) \cup \sigma(\mathcal{X}_\alpha)$ (see 3.3 (i)). Thus, $X_\alpha \cap (\sigma(\mathcal{X}_{\alpha+1}) \cup \sigma(\mathcal{X}_{\alpha+2}) \cup \sigma(\mathcal{X}_{\alpha+3})) = \emptyset$, since x_b is an arbitrary element of X_α.

(ii) Assume that there exists $x_b \in X_{\alpha+1}$ such that $x_b \in \sigma(\mathcal{X}_{\alpha+2})$. Take an element $x_c \in X_{\alpha+4}$. Then $X \setminus \{x_a, x_b, x_c\}$ is the set of generators of a d-dimensional cone $\sigma \in \Sigma$. By 3.3 (i), it follows from $x_a \in \sigma(\mathcal{X}_\alpha)$ that $d_\varphi(x_b, \sigma) > d_\varphi(x_a, \sigma)$. Similarly, $x_b \in \sigma(\mathcal{X}_{\alpha+2})$ implies $d_\varphi(x_a, \sigma) > d_\varphi(x_b, \sigma)$. This is a contradiction. So, $\sigma\ (\mathcal{X}_{\alpha+2}) \cap X_{\alpha+1} = \emptyset$. Using 3.1, one has $\sigma(\mathcal{X}_{\alpha+2}) \cap (X_{\alpha+2} \cup X_{\alpha+3}) = \emptyset$. By 6.3 (i), one has $\sigma(\mathcal{X}_{\alpha+2}) \cap X_\alpha = \emptyset$. It suffices to prove that $\sigma(\mathcal{X}_{\alpha+2}) \cap X_{\alpha+4} = \emptyset$.

Assume that there exists a generator $x_d \in X_{\alpha+4}$ such that $x_d \in \sigma(\mathcal{X}_{\alpha+4})$. Using 6.3 (i) after the cyclic permutation $\alpha \mapsto \alpha+2$, we get

$$X_{\alpha+2} \cap (\sigma(\mathcal{X}_{\alpha+3}) \cup \sigma(\mathcal{X}_{\alpha+4}) \cup \sigma(\mathcal{X}_\alpha)) = \emptyset\ .$$

This contradicts $x_a \in \sigma(\mathcal{X}_\alpha)$.

(iii) By 6.3 (i) and 3.1, $\sigma(\mathcal{X}_{\alpha+1}) \cap (X_{\alpha+1} \cup X_{\alpha+2} \cup X_\alpha) = \emptyset$. Assume that there exists $x_b \in X_{\alpha+3} \cap \sigma(\mathcal{X}_{\alpha+1})$. Using 6.3 (ii) after the cyclic permutation $\alpha \mapsto \alpha+1$, one has $\sigma(\mathcal{X}_{\alpha+3}) = 0$. This contradicts 3.3 (iii), since we have $\sigma(\mathcal{X}_{\alpha+2}) = \sigma(\mathcal{X}_{\alpha+3}) = 0$. Thus, $\sigma(\mathcal{X}_{\alpha+1}) \cap G(\Sigma) \subset X_{\alpha+4}$.

Suppose that there exists $x_b \in X_{\alpha+4}$ such that $x_b \notin \sigma(\mathcal{X}_{\alpha+4})$. Take an element $x_c \in X_\alpha$. Then $X \setminus \{x_a, x_b, x_c\}$ is the set of generators of a d-dimensional cone $\sigma \in \Sigma$. We get two primitive collections $\mathcal{X}_{\alpha+1}$ and $\mathcal{X}_{\alpha+2}$ in $G(\Sigma) \setminus \{x_b, x_c\}$ such that $\sigma(\mathcal{X}_{\alpha+1}) \cup \sigma(\mathcal{X}_{\alpha+2}) \subset \sigma$. This contradicts 3.3 (iii).

(iv) By 6.3 (i) and 3.1, $\sigma(\mathcal{X}_{\alpha+3}) \cap (X_{\alpha+3} \cup X_{\alpha+4} \cup X_\alpha) = \emptyset$. Assume that there exists $x_b \in X_{\alpha+2} \cap \sigma(\mathcal{X}_{\alpha+3})$. Using 6.3 (ii) after the symmetry $\alpha+\beta \mapsto \alpha-\beta$ of pentagon and the cyclic permutation $\alpha \mapsto \alpha+1$, one has $\sigma(\mathcal{X}_{\alpha+1}) = 0$. This contradicts 3.3 (iii), since we have $\sigma(\mathcal{X}_\alpha) = \sigma(\mathcal{X}_{\alpha+1}) = 0$. Thus, $\sigma(\mathcal{X}_{\alpha+3}) \cap G(\Sigma) \subset X_{\alpha+1}$.

Suppose that there exists $x_b \in X_{\alpha+1}$ such that $x_b \notin \sigma(\mathcal{X}_{\alpha+3})$. Take elements $x_c \in X_\alpha$ and $x_d \in X_{\alpha+3}$. Then $X \setminus \{x_b, x_c, x_d\}$ is the set of generators of a d-dimensional cone $\sigma \in \Sigma$. We get two primitive collections $\mathcal{X}_{\alpha+2}$ and $\mathcal{X}_{\alpha+3}$ in $G(\Sigma) \setminus \{x_b, x_c\}$ such that $\sigma(\mathcal{X}_{\alpha+2}) \cup \sigma(\mathcal{X}_{\alpha+3}) \subset \sigma$. This contradicts 3.3 (iii).

(v) By 6.3 (iii) and 6.3 (iv), one has

$$P_{\alpha+1} + P_{\alpha+2} = S_{\alpha+1} = P_{\alpha+4} + P'_{\alpha+4}\ , \quad P_{\alpha+3} + P_{\alpha+4} = S_{\alpha+3} = P_{\alpha+1} + P'_{\alpha+1}\ ,$$

where $P'_{\alpha+4} \in \sigma(\mathcal{X}_{\alpha+1})$ and $P'_{\alpha+1} \in \sigma(\mathcal{X}_{\alpha+3})$. It follows from these two equalities that

$$P_{\alpha+2} + P_{\alpha+3} = P'_{\alpha+1} + P'_{\alpha+4}\ .$$

Thus, $\sigma(\mathcal{X}_{\alpha+2}) \cap G(\Sigma) \subset (X_{\alpha+1} \cup X_{\alpha+4})$. On the other hand, we have $S_{\alpha+2} = 0$ (see 6.3 (ii)). So, $P'_{\alpha+1} = P'_{\alpha+4} = 0$ and $S_{\alpha+1} = P_{\alpha+4}$, $S_{\alpha+3} = P_{\alpha+1}$. The statement is proved.

6.4. Corollary. *Suppose that a cone $\sigma(\mathcal{X}_\alpha)$ contains a generator $x_a \in X_{\alpha+2}$. Then one has*

$$(\sigma(\mathscr{X}_\alpha) \cup \sigma(\mathscr{X}_{\alpha+4})) \cap G(\Sigma) \subset X_{\alpha+2} \cup \mathscr{X}_{\alpha+3} .$$

PROOF. Assume, for instance, that there exists $x_b \in X_{\alpha+1} \cap \sigma(\mathscr{X}_{\alpha+4})$. By 6.3. (ii), after the cyclic permutation $\alpha \mapsto \alpha+4$, one has $\sigma(\mathscr{X}_{\alpha+1})=0$. This contradicts 6.3 (v).

Now we assume that there exists $x_b \in X_{\alpha+4} \cap \sigma(\mathscr{X}_\alpha)$. By 6.3 (ii), after the symmetry $\alpha+\beta \mapsto \alpha-\beta$ and the cyclic permutation $\alpha \mapsto \alpha+1$, one has $\sigma(\mathscr{X}_{\alpha+3})=0$. This again contradicts 6.3 (v).

Using 3.1, we finish our proof.

6.5. PROPOSITION. *Suppose that a cone $\sigma(\mathscr{X}_\alpha)$ contains a generator $x_a \in X_\alpha$. Then at least one and only one of the following statements hold*:

(i) $X_{\alpha+3} \subset \sigma(\mathscr{X}_\alpha) \cap G(\Sigma)$;

(ii) $X_{\alpha+2} \subset \sigma(\mathscr{X}_{\alpha+4}) \cap G(\Sigma)$.

PROOF. We first assume that there exist $x_b \in \mathscr{X}_{\alpha+2}$ and $x_c \in \mathscr{X}_{\alpha+3}$ such that $x_b \notin \sigma(\mathscr{X}_{\alpha+4})$ and $x_c \notin \sigma(\mathscr{X}_\alpha)$. Choose an arbitrary element $x_d \in X_\alpha$. By 5.3, $\mathscr{X} \setminus \{x_b, x_c, x_d\}$ generates a d-dimensional cone $\sigma \in \Sigma$. Thus, we have two primitive collections $\mathscr{X}_{\alpha+4}$, $\mathscr{X}_\alpha \subset \mathscr{X} \setminus \{x_b, x_c\}$ such that $\sigma(\mathscr{X}_{\alpha+4})$, $\sigma(\mathscr{X}_\alpha) \subset \sigma$. This contradicts 3.3 (iii). Hence, the "at least one" part is proved.

Assume then, for instance, that (i) holds. Since $X_{\alpha+2} \cup X_{\alpha+3}$ is a primitive collection, at least one element $x_b \in X_{\alpha+2}$ is not a generator of $\sigma(\mathscr{X}_\alpha)$. So, we have

$$P_\alpha + P_{\alpha+1} = S_\alpha = P_{\alpha+3} + P ,$$

where $P \in \sigma(\mathscr{X}_\alpha)$ is a linear combination of $(X_{\alpha+2} \cup X_{\alpha+3}) \setminus \{x_b\}$ with nonnegative integral coefficients. On the other hand, it follows from 6.3 (v) that

$$P_{\alpha+3} + P_{\alpha+4} = P_{\alpha+1} .$$

These two equalities imply

$$P_{\alpha+4} + P_\alpha = P .$$

Hence, $\sigma(\mathscr{X}_{\alpha+4}) \subset \sigma(\mathscr{X}_\alpha)$. This shows that $x_b \notin \sigma(\mathscr{X}_{\alpha+4})$ and $X_{\alpha+2} \not\subset \sigma(\mathscr{X}_{\alpha+4}) \cap G(\Sigma)$.

We can now finish our classification of primitive relations.

6.6. THEOREM. *Let us assume that $\mathscr{X}_\alpha = X_\alpha \cup X_{\alpha+1}$, where $\alpha \in \boldsymbol{Z}/5\boldsymbol{Z}$,*

$$X_0 = \{v_1, \dots, v_{p_0}\} , \quad X_1 = \{y_1, \dots, y_{p_1}\} , \quad X_2 = \{z_1, \dots, z_{p_2}\} ,$$
$$X_3 = \{t_1, \dots, t_{p_3}\} , \quad X_4 = \{u_1, \dots, u_{p_4}\} ,$$

and $p_0 + p_1 + p_2 + p_3 + p_4 = d+3$. Then any complete regular d-dimensional fan Σ with the set of generators $G(\Sigma) = \bigcup X_\alpha$ and five primitive collections $\mathscr{X}_\alpha$ can be described up to a symmetry of the pentagon by the following primitive relations with nonnegative integral coefficients $c_2, \dots, c_{p_2}, b_1, \dots, b_{p_3}$:

$$v_1 + \cdots + v_{p_0} + y_1 + \cdots + y_{p_1} - c_2 z_2 - \cdots - c_{p_2} z_{p_2} - (b_1+1)t_1 - \cdots - (b_{p_3}+1)t_{p_3} = 0 ,$$

$$y_1+\cdots+y_{p_1}+z_1+\cdots+z_{p_2}-u_1-\cdots-u_{p_4}=0\,,$$
$$z_1+\cdots+z_{p_2}+t_1+\cdots+t_{p_3}=0\,,$$
$$t_1+\cdots+t_{p_2}+u_1+\cdots+u_{p_3}-y_1-\cdots-y_{p_1}=0\,,$$
$$u_1+\cdots+u_{p_4}+v_1+\cdots+v_{p_0}-c_2z_2-\cdots-c_{p_2}z_{p_2}-b_1t_1-\cdots-b_{p_3}t_{p_3}=0\,.$$

PROOF. One of the following two conditions hold:

(i) $\sigma(\mathscr{X}_\alpha)\cap G(\Sigma)\subset X_{\alpha+3}$ for all $\alpha\in \boldsymbol{Z}/5\boldsymbol{Z}$,

(ii) up to a symmetry of the pentagon there exists a cone $\sigma(\mathscr{X}_\alpha)$ containing a generator $x_a\in\mathscr{X}_{\alpha+2}$.

In the first case, we can use 6.1 and get the above primitive relations for $\alpha=0$, where $c_1=\cdots=c_{p_2}=b_2=\cdots=b_{p_3}=0$.

In the second case, we can use 6.3–6.5 and get the above primitive relations, where $z_1=x_b$, $\alpha=0$,

$$P=c_2z_2+\cdots+c_{p_2}z_{p_2}+b_1t_1+\cdots+b_{p_3}t_{p_3}$$

(We use the notation in the "only one" part in the proof of 6.5).

We can take the set

$$\{v_1,\ldots,v_{p_0},y_2,\ldots,y_{p_1},z_2,\ldots,z_{p_2},t_1,\ldots,t_{p_3},u_2,\ldots,u_{p_4}\}$$

as a basis of $\boldsymbol{Z}^d$. Thus, t_1, y_1, v_1 are defined by

$$z_1=-z_2-\cdots-z_{p_2}-t_1-\cdots-t_{p_3}\,,$$
$$y_1=-y_2-\cdots-y_{p_1}+z_1+\cdots+z_{p_2}-u_1-\cdots-u_{p_4}\,,$$
$$u_1=-u_2-\cdots-u_{p_4}-v_1-\cdots-v_{p_0}+c_2z_2+\cdots+c_{p_2}z_{p_2}+b_1t_1+\cdots+b_{p_3}t_{p_3}\,.$$

7. Open questions. The most interesting problem related to smooth complete projective toric varieties seems to me the following:

7.1. MAIN CONJECTURE. *For any d-dimensional smooth complete toric variety with Picard number ρ defined by a complete regular fan Σ, there exists a constant $N(\rho)$ depending only on ρ such that the number of primitive collections in $G(\Sigma)$ is always not more than $N(\rho)$.*

It is easy to see that $N(1)=1$, $N(2)=2$. Using our result in §5, we get $N(3)=5$. For 2-dimensional toric variety with $\rho+2$ generators the number of primitive collections equals $(\rho-1)(\rho+2)/2$. In connection with the conjecture, it is interesting to ask the following:

7.2. QUESTION. *Does there exist for $\rho>1$ a complete regular d-dimensional fan Σ with $\rho+d$ generators such that the set $G(\Sigma)$ contains more than*

$$(\rho-1)(\rho+2)/2$$

primitive collections?

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