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INFINITESIMAL TORELLI THEOREM FOR COMPLETE INTERSECTIONS IN CERTAIN HOMOGENEOUS KÄHLER MANIFOLDS, III

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Introduction. This is the third part of a study of the infinitesimal Torelli problem for complete intersections X in Kähler C-spaces Y with $b_2(Y)=1$. The preceding papers [6] will be referred to as Parts I and II, where we showed that the problem has an affirmative answer for almost all cases [Part II, Main Theorem]. However, it can never be answered completely as in the case of projective complete intersections [3, Theorem (3.1)]. In view of Flenner's criterion [3] which should be most workable in our context, this is probably because it is technically hard to know precisely when $H^q(Y, \Omega_Y^p(m))$ vanishes. So, in this article, we restrict ourselves to the case where Y is the Grassmannian of lines in $\boldsymbol{P}^l$ in order to get a more accurate result.

In §1, we briefly review Flenner's result for the later use. In §2, we study the infinitesimal Torelli problem for complete intersections in the Grassmannian Y of lines. Unfortunately, in our main result (Theorem 2.6), a few cases are still left unsettled. In order to apply Flenner's criterion, we need the vanishing theorem for $H^q(Y, \Omega_Y^p(m))$ (Theorem 2.1) which will be shown in §4. In §3, we study annoying exceptions in Theorem 2.6, i.e., the case where X is of type (1^c), $2 \leq c \leq 4$. It eventually turns out that some of them are counterexamples to the Torelli problem (Proposition 3.4): They depend on some moduli whereas their Hodge structures have no variations, like a cubic surface in $\boldsymbol{P}^3$ or an even-dimensional projective complete intersections of type (2, 2). We remark here that the Hodge structure of X with $\operatorname{codim} X=2$, 3 was previously studied by Donagi [2]. §4 will be devoted to the proof of Theorem 2.1.

The vanishing theorem for $H^q(Y, \Omega_Y^p(m))$ was obtained by Kimura [4], when Y is an irreducible Hermitian symmetric space of type E_{III} or E_{VII}. In §5, we state the corresponding infinitesimal Torelli theorems which can be shown as in §2.

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1. Flenner's criterion.

In this section, we recall and recast Flenner's criterion [3] for the infinitesimal Torelli theorem.

NATATION 1.1. Let Y be a Kähler C-space with $b_2=1$, and put $N=\dim Y$. The Picard group of Y is isomorphic to $\boldsymbol{Z}$ and we let $\mathcal{O}_Y(1)$ denote its ample generator. There exists a positive integer $k(Y)$ with $K_Y=\mathcal{O}_Y(-k(Y))$. If a global section x of the vector bundle

$$E=\bigoplus_{i=1}^{N-n}\mathcal{O}_Y(d_i)\,,\qquad d_i\in\boldsymbol{N}\,,$$

defines an irreducible nonsingular subvariety X of dimension n, we call it a nonsingular complete intersection of type $(d_1,\cdots,d_{N-n})$ in Y. We put $d=\sum d_i$ and assume that $d_1\geq d_2\geq\cdots\geq d_{N-n}$. We sometimes write, for example, $(2, 1^3)$ instead of $(2, 1, 1, 1)$. We say that X is of *hyperplane-section type* if $d_i=1$ for $1\leq i\leq N-n$, i.e., X is of type (1^{N-n}).

For the fundamental properties of Kähler C-spaces with $b_2=1$, see [6, Part I].

1.2. Let Y be a Kähler C-space with $b_2(Y)=1$, and let X be a nonsingular complete intersection of type $(d_1,\cdots,d_{N-n})$. By using the exact sequence

$$0\longrightarrow N_X^*\longrightarrow \Omega_Y^1|_X\longrightarrow \Omega_X^1\longrightarrow 0\,,$$

we can construct an exact Koszul sequence

$$(\mathrm{KS})_p:\qquad 0\longrightarrow S^pN_X^*\longrightarrow S^{p-1}N_X^*\otimes\Omega_Y^1\longrightarrow\cdots\longrightarrow\Omega_Y^p|_X\longrightarrow\Omega_X^p\longrightarrow 0$$

for any $p>0$. Tensoring $(\mathrm{KS})_{n-1}$ with K_X^{-1}, we get an exact Koszul sequence

$$(1.1)\qquad 0\longrightarrow S^{n-1}N_X^*\otimes K_X^{-1}\longrightarrow\cdots\longrightarrow S^iN_X^*\otimes\Omega_Y^{n-i-1}\otimes K_X^{-1}\longrightarrow \cdots\longrightarrow\Omega_Y^{n-1}\otimes K_X^{-1}\longrightarrow\Theta_X\longrightarrow 0\,.$$

Dualizing $(\mathrm{KS})_{n-p}$ and tensoring it with K_X, we get another exact Koszul sequence

$$(1.2)\quad 0\longrightarrow\Omega_X^p\longrightarrow\bigwedge^{n-p}\Theta_Y\otimes K_X\longrightarrow\cdots\longrightarrow S^jN_X\otimes\bigwedge^{n-p-j}\Theta_Y\otimes K_X\longrightarrow \cdots\longrightarrow S^{n-p}N_X\otimes K_X\longrightarrow 0\,.$$

We break (1.1), (1.2) and $(\mathrm{KS})_{p-1}$ into short exact sequences as follows:

$$0\longrightarrow L_{i+1}\rightarrow S^iN_X^*\otimes\Omega_Y^{n-1-i}\otimes K_X^{-1}\longrightarrow L_i\longrightarrow 0\,,$$
$$0\longrightarrow K^j\rightarrow S^jN_X\otimes\bigwedge^{n-p-j}\Theta_Y\otimes K_X\longrightarrow K^{j+1}\longrightarrow 0\,,$$
$$0\longrightarrow K_{k+1}\longrightarrow S^kN_X^*\otimes\Omega_Y^{p-1-k}\longrightarrow K_k\longrightarrow 0\,.$$

Considering the cohomology long exact sequences for these, we have coboundary maps

$$\partial_i\colon H^{i+1}(L_i)\longrightarrow H^{i+2}(L_{i+1})\,,$$
$$\delta^j\colon H^{n-p-j-1}(K^{j+1})\longrightarrow H^{n-p-j}(K^j)\,,$$
$$\partial'_k\colon H^{n-p+k+1}(K_k)\longrightarrow H^{n-p+k+2}(K_{k+1})\,.$$

Note that the natural pairing

$$(S^iN_X^*\otimes\Omega_Y^{n-1-i}\otimes K_X^{-1})\otimes(S^jN_X\otimes\bigwedge^{n-p-j}\Theta_Y\otimes K_X)\longrightarrow S^{i-j}N_X^*\otimes\Omega_Y^{p-1-(i-j)}$$

induces a pairing $H^s(L_i)\otimes H^t(K^j)\to H^{s+t}(K_{i-j})$. In particular, we denote by μ_i ($0\le i\le n-1$) the following pairings:

$$\mu_i\colon H^{i+1}(L_i)\otimes H^{n-p}(\Omega_X^p)\longrightarrow H^{n-p+i+1}(K_i) \qquad \text{for}\quad 0\le i\le p-2\,,$$

$$\mu_i\colon H^{i+1}(L_i)\otimes H^{n-i-1}(K^{i+1-p})\longrightarrow H^n(S^{p-1}N_X^*) \qquad \text{for}\quad p-1\le i\le n-1\,.$$

The following can be found in [3, (2.10)].

LEMMA 1.3. (1) *The diagram*

$$\begin{array}{ccccc} H^{i+1}(L_i) & \otimes & H^{n-p}(\Omega_X^p) & \xrightarrow{\mu_i} & H^{n-p+(i+1)}(K_i) \\ \partial_i\downarrow & & \| & & \downarrow\partial_i' \\ H^{i+2}(L_{i+1}) & \otimes & H^{n-p}(\Omega_X^p) & \xrightarrow{\mu_{i+1}} & H^{n-p+(i+2)}(K_{i+1}) \end{array}$$

commutes up to sign for $0\le i\le p-2$.

(2) *For* $p-1\le i\le n-2$, *then diagram*

$$\begin{array}{ccccc} H^{i+1}(L_i) & \otimes & H^{n-i-1}(K^{i+1-p}) & \xrightarrow{\mu_i} & H^n(S^{p-1}N_X^*) \\ \partial_i\downarrow & & \uparrow\delta^{i+1-p} & & \| \\ H^{i+2}(L_{i+1}) & \otimes & H^{n-i-2}(K^{i+2-p}) & \xrightarrow{\mu_{i+1}} & H^n(S^{p-1}N_X^*) \end{array}$$

commutes up to sign in the sense that $\mu_{i+1}(\partial_i\alpha\otimes\beta)=\pm\mu_i(\alpha\otimes\partial^{i+1-p}\beta)$ *for* $\alpha\in H^{i+1}(L_i)$ *and* $\beta\in H^{n-i-2}(K^{i+2-p})$.

A simple diagram chasing shows the following:

LEMMA 1.4. *Let i_0 and i_1 be integers satisfying $0\le i_0<i_1\le n-1$, and suppose that the cup-product map μ_{i_1} is non-degenerate in the first factor. Then so is μ_{i_0} provided that the composite of the coboundary maps $\partial_{i_1-1}\circ\cdots\circ\partial_{i_0}$ is injective.*

The following is a special case of a more general result due to Flenner [3, Theorem (1.1)].

THEOREM 1.5. *Assume that the multiplication map*

$$\mu\colon H^0(S^{n-p}N_X\otimes K_X)\otimes H^0(S^{p-1}N_X\otimes K_X)\longrightarrow H^0(S^{n-1}N_X\otimes K_X^2)$$

is surjective. If the map $\partial:=\partial_{n-2}\circ\cdots\circ\partial_0$ is injective, then the infinitesimal period map

$$H^1(X,\Theta_X)\longrightarrow \mathrm{Hom}_{\mathbf{C}}(H^{n-p}(X,\Omega_X^p),\,H^{n-p+1}(X,\Omega_X^{p-1}))$$

is injective.

PROOF. It is clear that the infinitesimal period map is injective if and only if μ_0, which is nothing but the cup-product map

$$H^1(\Theta_X)\otimes H^{n-p}(\Omega_X^p)\longrightarrow H^{n-p+1}(\Omega_X^{p-1}),$$

is non-degenerate in the first factor. Therefore, by Lemma 1.4, we get the desired result if

$$\mu_{n-1}: H^n(S^{n-1}N_X^*\otimes K_X^{-1})\otimes H^0(S^{n-p}N_X\otimes K_X)\longrightarrow H^n(S^{p-1}N_X^*)$$

is non-degenerate in the first factor, which is equivalent to saying that μ is surjective. q.e.d.

We clearly have the following:

LEMMA 1.6. *The map ∂ in Theorem 1.5 is injective if*

$$(\mathrm{V})_i: \qquad H^{i+1}(X, S^iN_X^*\otimes\Omega_Y^{n-1-i}\otimes K_X^{-1})=0$$

holds for $0\le i\le n-2$.

COROLLARY 1.7. *The condition $(\mathrm{V})_i$ is satisfied if the condition*

$$(\mathrm{V})_{i,j}: \qquad H^{i+j+1}(Y, \textstyle\bigwedge^jE^*\otimes S^iE^*\otimes\Omega_Y^{n-1-i}(k(Y)-d))=0$$

is satisfied for $0\le j\le N-n$.

PROOF. We use a spectral sequence associated to the resolution

$$0\longrightarrow\textstyle\bigwedge^{N-n}E^*\longrightarrow\cdots\longrightarrow\bigwedge^2E^*\longrightarrow E^*\longrightarrow\mathcal{O}_Y\longrightarrow\mathcal{O}_X\longrightarrow 0.$$

For details, we refer the reader to [Part II, 2.4].

REMARK 1.8. The result [Part II, Theorem 1.7] follows from Theorem 1.5 and Lemma 1.6.

2. Infinitesimal Torelli theorem. From this section up to §4, Y is the Grassmannian of lines in $\boldsymbol{P}^l$, $l\ge 4$. Therefore $N=2l-2$ and $k(Y)=l+1$. The following will be shown in §4.

THEOREM 2.1. *Let Y be the Grassmannian of lines in $\boldsymbol{P}^l$, $l\ge 4$. Then group $H^q(Y, \Omega_Y^p(m))$ vanishes except in the following cases.*

(1) $q=0$ *and* $m>[(p+3)/2]$.
(2) $p=q$ *and* $m=0$.
(3) $q=2l-2$ *and* $m<[(p+2)/2]-l-1$.
(4) $q-l<p<3q-4l+5$ *and* $m=p-2q+2l-3$.
(5) $3q-1<p<q+l$ *and* $m=p-2q+1$.

Here, the symbol $[s]$ denotes the greatest integer not exceeding $s\in\boldsymbol{Q}$.

LEMMA 2.2. *Let Y be as above and X an n-dimensional nonsingular complete intersection in Y. Put $p=n/2$ if n is even, and $p=(n+1)/2$ if n is odd. Then the multiplication map*

$$H^0(X, S^{n-p}N_X\otimes K_X)\otimes H^0(X, S^{p-1}N_X\otimes K_X)\longrightarrow H^0(X, S^{n-1}N_X\otimes K_X^2)$$

is surjective.

PROOF. As in the proof of [Part II, Lemma 2.3], we can check that each direct summand of $S^{p-1}N_S \otimes K_X$ has nonnegative degree except when X is of hyperplane-section type with codim $X \leq 4$ or of type (2, 1). Since $p-1 \leq n-p$, our assertion follows for those which are not the exceptions. If X is of hyperplane-section type with codim $X \leq 4$, then $H^0(X, S^{n-1}N_X \otimes K_X^2)$ vanishes, since each direct summand of $S^{n-1}N_X \otimes K_X^2$ has negative degree. If X is of type (2, 1), we have $K_X = \mathcal{O}_X(2-l)$ and $p = l-2$. Then

$$S^{p-1}N_X \otimes K_X \simeq S^{p-1}(\mathcal{O}_X(1) \oplus \mathcal{O}_X) \otimes \mathcal{O}_X(-1),$$
$$S^{n-p}N_X \otimes K_X \simeq S^{n-p}(\mathcal{O}_X(1) \oplus \mathcal{O}_X),$$
$$S^{n-1}N_X \otimes K_X^2 \simeq S^{n-1}(\mathcal{O}_X(1) \oplus \mathcal{O}_X) \otimes \mathcal{O}_X(-1).$$

Therefore, the multiplication map in question is nothing but

$$\left(\bigoplus_{i=0}^{p-2} H^0(\mathcal{O}_X(i))\right) \otimes \left(\bigoplus_{i=0}^{n-p} H^0(\mathcal{O}_X(i))\right) \longrightarrow \bigoplus_{i=0}^{n-2} H^0(\mathcal{O}_X(i)),$$

which is clearly surjective. q.e.d.

By virtue of Theorem 2.1, an easy calculation shows the following:

LEMMA 2.3. *With the above notation, for* $0 \leq i \leq n-2$ *and* $0 \leq j \leq N-n$, *the condition* $(\mathrm{V})_{i,j}$ *in Corollary* 1.7 *is satisfied except in the following cases*:

(1) *X is of hyperplane-section type.*

codim $X=2$	$(i,j)=((l-5)/2, 0), ((l-7)/2, 2)$	(l: *odd*),
	$(i,j)=((l-6)/2, 1)$	(l: *even*),
codim $X=3$	$(i,j)=((l-5)/2, 0), (l-5, 3)$	(l: *odd*),
	$(i,j)=((l-6)/2, 1), (l-5, 3)$	(l: *even*),
codim $X=4$	$(i,j)=((l-5)/2, 0), (l-5, 2), ((3l-13)/2, 4)$	(l: *odd*),
	$(i,j)=(l-5, 2)$	(l: *even*),
codim $X=5$	$(i,j)=(l-5, 1)$,	
codim $X=6$	$(i,j)=(l-5, 0)$.	

(2) *X is not of hyperplane-section type.*

type (2)	$(i,j)=(0, 0)$	$(l=4)$,
type (2, 1)	$(i,j)=(l-3, 0), ((3l-9)/2, 2)$	(l: *odd*),
	$(i,j)=(l-3, 0), ((l-4)/2, 0)$	(l: *even*),
type (3, 1)	$(i,j)=(l-3, 0)$,	
type (2, 1, 1)	$(i,j)=(l-4, 1)$,	
type (2, 1, 1, 1)	$(i,j)=(l-4, 0)$.	

COROLLARY 2.4. *X is rigid, that is,* $H^1(X, \Theta_X)=0$, *in the following cases*:

(1) *X is a hypersurface of degree* 1.

(2) $l=4$ *and X is of hyperplane-section type with* codim $X \leq 4$.

PROOF. By Lemma 2.3, Corollary 1.7 and Lemma 1.6, we see that $H^1(\Theta_X)$ is mapped injectively in $H^n(S^{n-1}N_X^* \otimes K_X^{-1})$. But this latter is zero, since we have

$$H^n(S^{n-1}N_X^* \otimes K_X^{-1}) \simeq H^0(S^{n-1}N_X \otimes K_X^2)^*$$

by Serre duality. q.e.d.

LEMMA 2.5. *Let p be an integer with* $0<2p\leq n=\dim X$. *Then* $H^p(Y, \Omega_Y^p) \simeq H^p(X, \Omega_Y^p|_X)$.

PROOF. Let I_X denote the ideal sheaf of X in Y. Then we have

$$0 \longrightarrow I_X \otimes \Omega_Y^p \longrightarrow \Omega_Y^p \longrightarrow \Omega_Y^p|_X \longrightarrow 0 .$$

Therefore, it suffices to check that $H^p(I_X \otimes \Omega_Y^p)=H^{p+1}(I_X \otimes \Omega_Y^p)=0$. We have the Koszul resolution

$$0 \longrightarrow \bigwedge^{N-n}E^* \otimes \Omega_Y^p \longrightarrow \cdots \longrightarrow E^* \otimes \Omega_Y^p \longrightarrow I_X \otimes \Omega_Y^p \longrightarrow 0 .$$

Therefore, by an easy spectral sequence argument, it suffices to check that

$$H^{p+i-1}(\bigwedge^i E^* \otimes \Omega_Y^p)=H^{p+i}(\bigwedge^i E^* \otimes \Omega_Y^p)=0$$

for $1 \leq i \leq N-n$, which follows from Theorem 2.1. q.e.d.

THEOREM 2.6. *Let Y be the Grassmannian of lines in* $\boldsymbol{P}^l$, $l\geq 4$. *For a nonsingular complete intersection X of type* $(d_1, \cdots, d_{N-n})$ *in Y, the infinitesimal Torelli theorem holds except possibly in the following cases*:

(1) *X is of hyperplane-section type*, $l\geq 5$ *and* $2\leq$ codim $X\leq 5$.

(2) $l=4$ *and X is of type* (2).

(3) *X is of type* $(2, 1^m)$, $1\leq m\leq 2$.

PROOF. Except when X is of type (3, 1), $(2, 1^3)$ or (1^6), the assertion follows from Theorem 1.5 by Lemmas 1.6, 1.7, 2.2 and 2.3.

We assume that X is of type (3, 1), since the other cases can be treated similarly. We put $p=n/2=l-2$. In view of Lemma 2.3, ∂_i is injective unless $i=p-1$. By Lemma 1.3, we have the commutative diagram

$$\begin{array}{ccccc} H^p(L_{p-1}) & \otimes & H^p(\Omega_X^p) & \xrightarrow{\mu_{p-1}} & H^n(S^{p-1}N_X^*) \\ {\scriptstyle\partial_{p-1}}\downarrow & & \uparrow{\scriptstyle\delta^0} & & \| \\ H^{p+1}(L_p) & \otimes & H^{p-1}(K^1) & \xrightarrow{\mu_p} & H^n(S^{p-1}N_X^*) . \end{array}$$

We shall show that μ_{p-1} is non-degenerate in the first factor. For this purpose, since

$$H^p(S^{p-1}N_X^*\otimes\Omega_Y^p\otimes K_X^{-1})\longrightarrow H^p(L_{p-1})\xrightarrow{\partial_{p-1}} H^{p+1}(L_p)$$

and since we already know, by Lemmas 1.4 and 2.2, that μ_p is non-degenerate in the first factor, it suffices to show that

$$H^p(S^{p-1}N_X^*\otimes\Omega_Y^p\otimes K_X^{-1})\otimes H^p(\Omega_X^p)\longrightarrow H^n(S^{p-1}N_X^*) \tag{2.1}$$

is non-degenerate in the first factor. Note that we have $K_X=\mathcal{O}_X(1-p)$ and, furthermore,

$$H^p(S^{p-1}N_X^*\otimes\Omega_Y^p\otimes K_X^{-1})\simeq H^p(S^{p-1}(\mathcal{O}_X\oplus\mathcal{O}_X(-2))\otimes\Omega_Y^p)\simeq H^p(\Omega_Y^p|_X),$$

$$\begin{aligned}H^n(S^{p-1}N_X^*)&\simeq H^n(S^{p-1}N_X^*\otimes\mathcal{O}_X(p-1)\otimes K_X)\\&\simeq H^n(S^{p-1}(\mathcal{O}_X\oplus\mathcal{O}_X(-2))\otimes K_X)\\&\simeq H^n(K_X)\oplus\bigoplus_{i=1}^{p-1}H^n(K_X(-2i)).\end{aligned}$$

Therefore, the pairing (2.1) is non-degenerate in the first factor, if so is

$$H^p(\Omega_Y^p|_X)\otimes H^p(\Omega_X^p)\longrightarrow H^n(K_X). \tag{2.2}$$

Note that we have $H^p(\Omega_Y^p|_X)\simeq H^p(\Omega_Y^p)\subset H^p(\Omega_X^p)$ by Lemma 2.5 and the Lefschetz theorem. Since $H^p(\Omega_X^p)\otimes H^p(\Omega_X^p)\to H^n(K_X)$ is non-degenerate, we see that (2.2) is non-degenerate in the first factor. q.e.d.

REMARK 2.7. If l is odd, a nonsingular hypersurface of degree 1 is the Kähler C-space of type $(C_{(l+1)/2},\alpha_2)$ (see [9] or [7]). Therefore, the infinitesimal Torelli theorem holds for a general X of type (2, 1) by [Part II, Main Theorem].

3. Complete intersections of hyperplane-section type. In this section, we assume that X is a nonsingular complete intersection of hyperplane-section type with $2\le\operatorname{codim}X\le 4$. We also assume that $l\ge 5$.

LEMMA 3.1. *Let X be as above. If* $2p<n$, *then* $H^{n-p}(\Omega_X^p)=0$ *except in the cases where l is odd and*

(1) $\operatorname{codim}X=3$ *and* $p=l-3$, *or*

(2) $\operatorname{codim}X=4$ *and* $p=l-4$, $l\neq 5$.

Therefore, no variations of Hodge structures exist except in the above cases.

PROOF. By $(\mathrm{KS})_p$, if $H^{n-p+i}(S^iN_X^*\otimes\Omega_Y^{p-i})=0$ for $0\le i\le p$, then $H^{n-p}(\Omega_X^p)=0$. Since X is of hyperplane-section type, it suffices to check $H^{n-p+i+j}(Y,\Omega_Y^{p-i}(-i-j))=0$ for $0\le i\le p$, $0\le j\le\operatorname{codim}X$. By Theorem 2.1, these hold except in the cases:

(a) $2i=l-5, j=3, p=l-3$, n is odd,

(b) $2i=l-7, j=4, p=l-4$, n is even.

q.e.d.

REMARK 3.2. If $\operatorname{codim}X=3$ and l is odd, then we have $h^{2l-5}=h^{l-2,l-3}+$

$h^{l-3,l-2}=(l-1)(l-3)/4$, see [2, Lemma 2.6].

LEMMA 3.3. *Let X be as above. If* $\operatorname{codim} X=3$ *or* 4, *then* $H^1(\Theta_X)\neq 0$.

PROOF. We consider the cohomology long exact sequence for

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_Y|_X \longrightarrow N_X \longrightarrow 0\,.$$

It is known, and can be shown by using Theorem 2.1, that $H^1(\Theta_Y|_X)=0$. Therefore, we get

$$0 \longrightarrow H^0(\Theta_X) \longrightarrow H^0(\Theta_Y|_X) \longrightarrow H^0(N_X) \longrightarrow H^1(\Theta_X) \to 0\,.$$

Since we have $H^{i-1}(Y, \Theta_Y \otimes \bigwedge^i E^*)=H^i(Y, \Theta_Y \otimes \bigwedge^i E^*)=0$ for $1\le i\le N-n$ by Theorem 2.1, we get $H^0(\Theta_Y)\simeq H^0(\Theta_Y|_X)$. Therefore, we have $h^0(\Theta_Y|_X)=l(l+2)$. Furthermore, we have $h^0(N_X)=c(l(l+1)/2-c)$, where $c=\operatorname{codim} X$. Since $h^0(\Theta_Y|_X)<h^0(N_X)$ for $c\ge 3$, we have $h^1(\Theta_X)\neq 0$. q.e.d.

By Lemmas 3.1 and 3.3, we get:

PROPOSITION 3.4. *Let X be a complete intersection of hyperplane-section type in the Grassmannian of lines in $\boldsymbol{P}^l$, $l\ge 5$. Then the Torelli theorem cannot hold in the following cases*:

(1) $l=5$ *and* $\operatorname{codim} X=4$.

(2) l *is even and* $\operatorname{codim} X=3, 4$.

REMARK 3.5. When X is of type (1^2), Donagi [2, 2.2 and 2.3] showed the following: We have $h^{2l-4}=h^{l-2,l-2}=l-1$ (l is odd), $l/2$ (l is even). Let H and H' be hypersurfaces of degree 1 with $X=H\cap H'$, and consider the pencil L spanned by them. If l is odd, L depends on $(l-5)/2$ parameters, whereas, if l is even, it has no moduli. Since X is the base locus of L, we may have $h^1(\Theta_X)=(l-5)/2$ if l is odd, and $h^1(\Theta_X)=0$ if l is even.

4. Proof of Theorem 2.1. For irreducible Hermitian symmetric spaces of compact type, Kimura [4] gave a method to determine when the cohomology group $H^q(\Omega_Y^p(m))$ vanishes, following Bott [1] and Kostant [8]. In this section, we show Theorem 2.1 using his method.

NATATION 4.1. Let $\{e_i : 1\le i\le l+1\}$ be the standard orthonormal basis of $\boldsymbol{R}^{l+1}$ with respect to the usual inner product $(\cdot,\cdot)$. Put $\Phi=\{e_i-e_j : 1\le i,j\le l+1, i\neq j\}$. Then we can identify it with the root system of the simple Lie algebra of type A_l, and $\Delta=\{\alpha_i := e_i-e_{i+1} : 1\le i\le l\}$ is a basis consisting of positive simple roots. We also put

$$\lambda_i=e_1+e_2+\cdots+e_i-(i/(l+1))\sum_{j=1}^{l+1} e_j\,.$$

Then $(\lambda_i, \alpha_j)=\delta_{ij}$ and the λ_i are the fundamental weights. If δ denotes the half of the sum of all positive roots, then

$$2\delta=le_1+(l-2)e_2+\cdots-(l-2)e_l-le_{l+1}\,.$$

The Weyl group W of Φ can be identified with the permutation group of $l+1$ letters: $\sigma e_i = e_{\sigma(i)}$. Therefore, we can write $\sigma \in W$ as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & l+1 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(l+1) \end{pmatrix}.$$

We set

$$\Phi(n^+) = \{e_1 - e_i \ (3 \le i \le l+1),\ e_2 - e_j \ (3 \le j \le l+1)\}$$

and

$$W^1 = \left\{ \sigma \in W : \sigma^{-1} = \begin{pmatrix} 1 & \cdots & l+1 \\ \sigma^{-1}(1) & \cdots & \sigma^{-1}(l+1) \end{pmatrix},\quad \begin{matrix} \sigma^{-1}(1) < \sigma^{-1}(2) \\ \sigma^{-1}(3) < \cdots < \sigma^{-1}(l+1) \end{matrix} \right\}.$$

The index $n(\sigma)$ of $\sigma \in W^1$ is given by

$$n(\sigma) = \sigma^{-1}(1) + \sigma^{-1}(2) - 3$$

(cf. Takeuchi [10]) and we set $W^1(p) = \{\sigma \in W^1 ; n(\sigma) = p\}$. We also remark that the following hold:

$$(\lambda_2, \alpha) = 1, \qquad \forall \alpha \in \Phi(n^+),$$

$$(\sigma\delta, e_i - e_j) = \sigma^{-1}(j) - \sigma^{-1}(i), \qquad 1 \le i, j \le l+1, \quad \forall \sigma \in W.$$

For $\sigma \in W^1$, we set $s_i = \sigma^{-1}(i)$ $(i = 1, 2)$ for simplicity. Then,

$$\sigma^{-1}(i) = \begin{cases} i-2, & \text{if } 3 \le i \le s_1 + 1 \\ i-1, & \text{if } s_1 + 2 \le i \le s_2 \\ i, & \text{if } s_2 + 1 \le i \le l+1. \end{cases}$$

If β varies in $\Phi(n^+)$ and σ in W^1, $(\sigma\delta, \beta)$ can take the following values:

$$(\sigma\delta, e_1 - e_i) = \begin{cases} i-2-s_1 & \text{if } 3 \le i \le s_1 + 1, \\ i-1-s_1 & \text{if } s_1 + 2 \le i \le s_2, \\ i-s_1 & \text{if } s_2 + 1 \le i \le l+1. \end{cases}$$

$$(\sigma\delta, e_2 - e_j) = \begin{cases} j-2-s_2 & \text{if } 3 \le j \le s_1 + 1, \\ j-1-s_2 & \text{if } s_1 + 2 \le j \le s_2, \\ j-s_2 & \text{if } s_2 + 1 \le j \le l+1. \end{cases}$$

For the proof of the following fact, see [4, Theorems 1 and 2].

LEMMA 4.2. *Let Y be the Grassmannian of lines in $\boldsymbol{P}^l$. The group $H^q(\Omega_Y^p(m))$ does not vanish if and only if there exists a $\sigma \in W^1(p)$ satisfying the following conditions.*

(1) $m \ne -(\sigma\delta, \beta)$ *for all* $\beta \in \Phi(n^+)$.

(2) $q = \mathrm{card}\{\beta \in \Phi(n^+) : (\sigma\delta, \beta) < -m\}$.

4.3. Now, using Lemma 4.2, our calculation proceeds as follows. Let $\sigma \in W^1(p)$. Then $p = s_1 + s_2 - 3$. We consider the case $1 < s_1 < s_2 - 1 < l$ for simplicity. If m is an integer with $-m \leq -s_2$, $-m=0$ or $-m \geq l+1-s_1$, then Lemma 4.2 (1) is satisfied. Therefore, $H^0(\Omega_Y^p(m))$ (resp. $H^{2l-2}(\Omega_Y^p(m))$) does not vanish if $m \geq s_2$ (resp. $m \leq s_1 - l - 1$), and $H^p(\Omega_Y^p) \neq 0$. Furthermore, note that it may be possible that m with $-m = s_2 - s_1$, $s_1 - s_2$ satisfies (1) of Lemma 4.2. The case $-m = s_2 - s_1$ can occur if and only if $l+1-s_2 < s_2 - s_1$. Then, we have

$$q = \mathrm{card}\{\beta \in \Phi(n^+) : (\sigma\delta, \beta) < -m = s_2 - s_1\} = l - 3 + s_2 .$$

Therefore, we have $s_1 = p - q + l$, $s_2 = q - l + 3$ and $-m = 2q - p - 2l + 3$. The condition $l+1-s_2 < s_2 - s_1$ is equivalent to $3q - p > 4l - 5$. Since we have assumed that $1 < s_1 < s_2 - 1 < l$, we have $p - q + l > 1$ and $q < 2l - 2$ in addition. Therefore, $H^q(\Omega_Y^p(p+2l-3-2q))$ does not vanish if $3q - p > 4l - 4$, $p - q + l > 1$, $q < 2l - 2$. Quite similarly, considering the case $-m = s_1 - s_2$, we know that $H^q(\Omega_Y^p(p+1-2q))$ does not vanish if $3q - p < 1$, $q > 0$, $p - q + 1 < l$.

For the other types of $\sigma \in W^1(p)$, e.g., $s_1 = 1$, the calculation goes similarly. Then, varying $\sigma \in W^1(p)$, we get Theorem 2.1. The details are left to the reader.

5. Complete intersections in E_{III} and E_{VII}. For the irreducible Hermitian symmetric space Y of type E_{III} or E_{VII}, Kimura [4] determined completely when $H^q(\Omega_Y^p(m))$ vanishes. Therefore, as in the case of the Grassmannian of lines, we can show the following theorems.

THEOREM 5.1. *Let Y be the irreducible Hermitian symmetric space of type E_{III}. The infinitesimal Torelli theorem holds for a nonsingular complete intersection X in Y, except possibly when X is one of the following types.*

(1) *type* (1^m), $2 \leq m \leq 9$,

(2) *type* $(d_1, 1^m)$, $2 \leq d_1 \leq 4$, $1 \leq m \leq 10 - 2d_1$,

(3) *type* $(2^2, 1^m)$, $1 \leq m \leq 3$,

(4) *type* $(3, 2, 1)$.

THEOREM 5.2. *Let Y be the irreducible Hermitian symmetric space of type E_{VII}. The infinitesimal Torelli theorem holds for a nonsingular complete intersection X in Y, except possibly when X is one of the following types.*

(1) *type* (1^m), $2 \leq m \leq 10$,

(2) *type* $(d_1, 1^m)$, $2 \leq d_1 \leq 5$, $1 \leq m \leq 11 - 2d_1$,

(3) *type* $(2^2, 1^m)$, $1 \leq m \leq 4$,

(4) *types* $(3, 2, 1)$, $(3, 2, 1^2)$, $(2^3, 1)$.

OUTLINE OF THE PROOF OF THEOREMS 5.1 AND 5.2. Since the surjectivity of μ in Theorem 1.5 can be shown as in Lemma 2.2, our task is reduced to showing the injectivity of ∂ in Theorem 1.5. Using [4, Theorems 4 and 5], one can check that the condition $(\mathrm{V})_{i,j}$ does not hold for some (i, j) only when the type of X is one of the above and the following:

E_{III}:	(1^{10})	$(i, j)=(2, 0)$,
	$(d_1, 1^{11-2d_1})$, $2 \le d_1 \le 5$,	$(i, j)=(d_1+1, 0)$,
	$(2^2, 1^4)$	$(i, j)=(4, 0)$,
	$(3, 2, 1^2)$	$(i, j)=(5, 0)$,
	$(2^3, 1)$	$(i, j)=(5, 0)$.
E_{VII}:	(1^{11})	$(i, j)=(7, 0)$,
	$(d_1, 1^{12-2d_1})$, $2 \le d_1 \le 5$,	$(i, j)=(d_1+6, 0)$,
	$(2^2, 1^5)$	$(i, j)=(9, 0)$,
	$(3, 2, 1^3)$	$(i, j)=(10, 0)$,
	$(3^2, 1)$	$(i, j)=(11, 0)$,
	$(4, 2, 1)$	$(i, j)=(11, 0)$.

In these cases, however, we can show that ∂ is injective as in the proof of Theorem 2.6.

REMARK 5.3. If Y is of type E_{III}, then a nonsingular hypersurface of degree 1 is the Kähler C-space of type (F_4, α_4) (see [5] or [7]). Therefore, if X is a general complete intersection of type $(d_1, 1)$, $2 \le d_1 \le 4$, $(2^2, 1)$ or $(3, 2, 1)$, then the infinitesimal Torelli theorem holds for X by [Part II, Main Theorem]. If Y is of type E_{VII}, a nonsingular hypersurface of degree 1 is rigid.

REFERENCES

[1] R. BOTT, Homogeneous vector bundles, Ann. of Math. 66 (1957), 203–248.

[2] R. DONAGI, On the geometry of Grassmannians, Duke Math. J. 44 (1977), 795–837.

[3] H. FLENNER, The infinitesimal Torelli problem for zero sets of sections of vector bundles, Math. Z. 193 (1986), 307–322.

[4] Y. KIMURA, On the hypersurfaces of Hermitian symmetric spaces of compact type, Osaka J. Math. 16 (1979), 97–119.

[5] Y. KIMURA, A hypersurface of the irreducible Hermitian symmetric space of type EIII, Osaka J. Math. 16 (1979), 431–438.

[6] K. KONNO, Infinitesimal Torelli theorem for complete intersections in certain homogeneous Kähler manifolds, Tôhoku Math. J. 38 (1986), 609–624; II, ibid. 42 (1990), 333–338, (referred to as Parts I and II).

[7] K. KONNO, Homogeneous hypersurfaces in Kähler C-spaces with $b_2=1$, J. Math. Soc. Japan 40 (1988), 687–703.

[8] B. KOSTANT, Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. of Math. 74 (1961), 329–387.

[9] Y. SAKANE, On non-singular hyperplane sections of some Hermitian symmetric spaces, Osaka J. Math. 22 (1985), 107–121.

[10] M. TAKEUCHI, Cell decompositions and Morse equalities on certain symmetric spaces, J. Fac. Sci. Univ. Tokyo 12 (1965), 81–192.

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