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GORENSTEIN TORIC SINGULARITIES AND CONVEX POLYTOPES

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Introduction. Let X be a normal projective variety over a field k and D an *ample $\boldsymbol{Q}$-divisor*, i.e., a rational coefficient Weil divisor such that bD is an ample Cartier divisor for some positive integer b. We consider a normal graded ring $R(X, D)$ defined by $R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))t^n$. Here t is an indeterminate and $\mathcal{O}_X(nD)$ are the sheaves defined by $\Gamma(U, \mathcal{O}_X(nD)) = \{f \in K(X); \mathrm{div}_U(f) + nD|_U \geq 0\}$ for each open set U of X. We are interested in finding a criterion for a normal projective variety X to have an ample $\boldsymbol{Q}$-divisor D with $R(X, D)$ Gorenstein. Concerning this problem, see also [1, Chapter 5], [10]. Here we discuss this problem when X is a projective torus embedding defined over k.

Our main results are the following:

COROLLARY 2.5. *Let X be a projective torus embedding and D an ample Cartier divisor. Then $R(X, D)$ is Gorenstein if and only if the canonical sheaf ω_X on X is isomorphic to an invertible sheaf $\mathcal{O}_X(-\delta D)$ for a positive integer δ.*

THEOREM 2.6. *Every projective torus embedding X has an ample $\boldsymbol{Q}$-divisor D stable under the torus action such that $R(X, D)$ is Gorenstein.*

To obtain these results, we proceed as follows: First, given a *rational convex r-polytope P* in $\boldsymbol{R}^r$ (i.e., an r-dimensional convex polytope whose vertices have rational coordinates in $\boldsymbol{R}^r$), we construct a pair of projective torus embedding $X(P)$ over k and an ample $\boldsymbol{Q}$-divisor $D(P)$ (Proposition 1.3) following [7, Chapter 2], so that $R(X(P), D(P))$ is isomorphic to the normal semigroup k-algebra

$$R(P) = \bigoplus_{n \geq 0} \left\{ \sum_{m \in nP \cap \boldsymbol{Z}^r} k\boldsymbol{e}(m) \right\} t^n .$$

Here t is an indeterminate and $\boldsymbol{e}$ is the isomorphism from $\boldsymbol{Z}^r$ ($\subset \boldsymbol{R}^r$) into the Laurent polynomial ring $k[X_1^{\pm 1}, \dots, X_r^{\pm 1}]$ sending $(m_1, \dots, m_r)$ to $X_1^{m_1} \cdots X_r^{m_r}$. Thus $X(P)$ is isomorphic to $\mathrm{Proj}(R(P))$ (Proposition 1.5). Conversely, it turns out that every pair of projective torus embedding X and a T-stable ample $\boldsymbol{Q}$-divisor D on X is obtained from a rational convex r-polytope in $\boldsymbol{R}^r$ in this way (Proposition 1.3). On the other hand, since $R(X(P), D(P))$ ($\simeq R(P)$) is Cohen-Macaulay (cf. [4]), we can apply the criterion [10, Corollary 2.9] for the Gorenstein property to $R(X(P), D(P))$. Therefore we have Proposition 2.2, which is a criterion for $R(X(P), D(P)) \simeq R(P)$ to be Gorenstein in terms of $D(P)$ on $X(P)$ as well as the maximal faces of P. This yields another proof for a theorem of Hibi [3]. As immediate consequences of Proposition 2.2, we get our main results.

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Preliminaries.

(0.1) $[a]$ denotes the greatest integer not greater than $a \in \boldsymbol{R}$. $\lceil a \rceil$ denotes $-[-a]$ for $a \in \boldsymbol{R}$.

(0.2) For notion of torus embeddings, we refer the reader to [7]. All torus embeddings will be defined over a fixed field k. Let T be an r-dimensional algebraic torus $\mathrm{Spec}(k[X_1^{\pm 1}, \ldots, X_r^{\pm 1}])$ over k. Let M, N be the group of characters and one-parameter subgroups, respectively. By $e(m)$, we denote the Laurent monomial corresponding to a character m. Namely $e(m) = X_1^{m_1} \cdots X_r^{m_r}$ for $m = (m_1, \ldots, m_r)$. Set $M_{\boldsymbol{R}} = M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ and $N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Let $\langle\ ,\ \rangle: M_{\boldsymbol{R}} \times N_{\boldsymbol{R}} \to \boldsymbol{R}$ represent the natural non-degenerate pairing. For a complete fan Δ of N, $\Delta(i)$ denotes the i-dimensional cones of Δ. A one-dimensional cone $\rho \in \Delta(1)$ is generated by a unique primitive integral vector $n(\rho)$ ([7, p. 24]). We denote by $\mathrm{SF}(N, \Delta)$ the additive group consisting of Δ-linear support functions ([7, p. 66]). Set $\mathrm{SF}(N, \Delta, \boldsymbol{Q}) = \mathrm{SF}(N, \Delta) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Its elements are also called Δ-linear support functions. Then we have two injections $M \to \mathrm{SF}(N, \Delta)$ sending m to $\langle m, \ \rangle$, and $\mathrm{SF}(N, \Delta) \to \boldsymbol{Z}^{\Delta(1)}$ sending h to $(h(n(\rho)))_{\rho \in \Delta(1)}$. Let X be a complete torus embedding $T_N\mathrm{emb}(\Delta)$. By $T\mathrm{Div}(X)$, $T\mathrm{CDiv}(X)$ and $\mathrm{PDiv}(X)$, we denote the groups of T-stable Weil divisors, T-stable Cartier divisors and principal divisors on X, respectively. Also, by $T\mathrm{Div}(X, \boldsymbol{Q})$ (resp. $T\mathrm{CDiv}(X, \boldsymbol{Q})$), we denote the group of T-stable $\boldsymbol{Q}$-divisors (resp. T-stable $\boldsymbol{Q}$-Cartier divisors). Namely $T\mathrm{Div}(X, \boldsymbol{Q}) = T\mathrm{Div}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ and $T\mathrm{CDiv}(X, \boldsymbol{Q}) = T\mathrm{CDiv}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. The one-dimensional cones ρ of $\Delta(1)$ are in one-to-one correspondence with the irreducible T-stable closed subvarieties $V(\rho)$ of codimension one in X. Therefore the map $\boldsymbol{Z}^{\Delta(1)} \to T\mathrm{Div}(X)$ sending g to $D_g = -\sum_{\rho \in \Delta(1)} g_\rho \cdot V(\rho)$ is a bijection, and induces two isomorphisms of groups, $\mathrm{SF}(N, \Delta) \to T\mathrm{CDiv}(X)$ and $M \to \mathrm{PDiv}(X) \cap T\mathrm{CDiv}(X)$. As a result, we have two commutative diagrams (cf. [7, §2.1]):

$$
\begin{array}{ccccc}
M & \longrightarrow & \mathrm{SF}(N, \Delta) & \longrightarrow & \boldsymbol{Z}^{\Delta(1)} \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathrm{PDiv}(X) \cap T\mathrm{CDiv}(X) & \longrightarrow & T\mathrm{CDiv}(X) & \longrightarrow & T\mathrm{Div}(X)
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{SF}(N, \Delta, \boldsymbol{Q}) & \longrightarrow & \boldsymbol{Q}^{\Delta(1)} \\
\downarrow\wr & & \downarrow\wr \\
T\mathrm{CDiv}(X, \boldsymbol{Q}) & \longrightarrow & T\mathrm{Div}(X, \boldsymbol{Q}) .
\end{array}
$$

1. Rational polytopes and projective torus embeddings.

LEMMA 1.1. *Let $X = T_N\mathrm{emb}(\Delta)$ be an r-dimensional complete torus embedding. For $g \in \boldsymbol{Q}^{\Delta(1)}$, the set $\square_g = \{m \in M_{\boldsymbol{R}};\ \langle m, n(\rho)\rangle \geq g_\rho$ for all $\rho \in \Delta(1)\}$ is a (possibly empty) convex polytope in $M_{\boldsymbol{R}}$. The set $H^0(X, \mathcal{O}_X(D_g))$ of global sections of the divisorial $\mathcal{O}_X$-module $\mathcal{O}_X(D_g)$ is the finite dimensional k-vector space with $\{\boldsymbol{e}(m);\ m \in M \cap \square_g\}$ as a basis. Let $\mathrm{div}(\boldsymbol{e}(m)) + D_g = \sum_{\rho \in \Delta(1)} a_\rho \cdot V(\rho)$ for an element $m \in M$. Then m is in $\mathrm{int}(\square_g)$ if and only if the coefficient a_ρ for each $\rho \in \Delta(1)$ is a positive rational number. Here $\mathrm{int}(\square_g)$ denotes the interior of the convex polytope $\square_g$.*

PROOF. The first part is the same as that in the case of $g \in \boldsymbol{Z}^{\Delta(1)}$ (cf. [7, Lemma 2.3]). Since $n(\rho)$ is a primitive vector and the pairing $\langle\ ,\ \rangle$ is non-degenerate, we have $\square_g \cap M = \square_{\lceil g \rceil} \cap M$, where $\lceil g \rceil$ denotes the integral vector $(\lceil g_\rho \rceil)_{\rho \in \Delta(1)}$. On the other hand, we have $\mathcal{O}_X(D_g) = \mathcal{O}_X(D_{\lceil g \rceil})$ by definition. Hence we may assume that $g \in \boldsymbol{Z}^{\Delta(1)}$. In this case, the assertion follows from [5, p. 42, Theorem] (cf. [7, Lemma 2.3]). The rest is obvious.

Recall that a Δ-linear support function $h \in \mathrm{SF}(N, \Delta, \boldsymbol{Q})$ is said to be *strictly upper convex* with respect to Δ if h is upper convex, namely $h(n) + h(n') \leq h(n + n')$ for all $n, n' \in N_{\boldsymbol{R}}$, and Δ is the coarsest among the fans Δ' in N for which h is Δ'-linear (cf. [7, p. 82]).

LEMMA 1.2. *Let $X = T_N\mathrm{emb}(\Delta)$ be an r-dimensional complete torus embedding and $h \in \mathrm{SF}(N, \Delta, \boldsymbol{Q})$. Then D_h is an ample $\boldsymbol{Q}$-divisor if and only if h is strictly upper convex with respect to Δ.*

PROOF. This easily follows from [7, Corollary 2.14]. q.e.d.

PROPOSITION 1.3. *Let P be a rational convex r-polytope in $M_{\boldsymbol{R}} = \boldsymbol{R}^r$ and $h_P: N_{\boldsymbol{R}} \to \boldsymbol{R}$ the support function for P defined by $h_P(n) = \inf\{\langle m, n\rangle;\ m \in P\}$. Then there exists a unique finite complete fan Δ_P in N such that h_P is strictly upper convex Δ_P-linear with respect to Δ_P. We denote the corresponding r-dimensional projective torus embedding $T_N\mathrm{emb}(\Delta_P)$ and the ample T-stable $\boldsymbol{Q}$-divisor D_{h_P} by $X(P)$ and $D(P)$. Conversely, every pair of a projective torus embedding and a T-stable ample $\boldsymbol{Q}$-divisor on it is obtained from a rational convex r-polytope in $M_{\boldsymbol{R}}$ in this way.*

PROOF. The first part follows from [7, Theorem A.18 and Corollary A.19]. Then, by (1.2), $D(P)$ is a T-stable ample $\boldsymbol{Q}$-divisor on $X(P)$. Conversely, given a projective torus embedding X with a T-stable ample $\boldsymbol{Q}$-divisor D, there exist a complete fan Δ and a strictly upper convex Δ-linear support function $h \in \mathrm{SF}(N, \Delta, \boldsymbol{Q})$ with $X = T_N\mathrm{emb}(\Delta)$ and $D = D_h$, by (1.2). Set $\square_h = \{u \in M_{\boldsymbol{R}};\ \langle u, n(\rho)\rangle \geq h(n(\rho))$ for all $\rho \in \Delta(1)\}$. By construction and [7, Theorem A.18 and Corollary A.19], we have $X = X(\square_h)$ and $D = D(\square_h)$. q.e.d.

REMARK 1.4. In (1.3), $D(P)$ is a Cartier divisor if and only if P is an *integral* convex r-polytope, namely, a convex polytope whose vertices have integral coordinates. $D(P)$ is a Weil divisor if and only if P is *facet-reticular*, that is, each supporting hyperplane carried by a *facet* (maximal face) of P contains an element of M.

PROPOSITION 1.5. *Let P be a rational convex r-polytope in $M_{\boldsymbol{R}}$. Then the graded semigroup ring*

$$R(P)=\bigoplus_{n\geq 0}\left\{\sum_{m\in nP\cap M} ke(m)\right\}t^n$$

is isomorphic as a graded k-algebra to the graded ring $R(X(P), D(P))$ associated with the projective torus embedding $X(P)$ and the T-stable ample $\boldsymbol{Q}$-divisor $D(P)$. Consequently, $\mathrm{Proj}(R(P))$ *is isomorphic to $X(P)$ and the sheaf $\mathcal{O}_{X(P)}(n):=R(P)(n)^{\sim}$ on* $\mathrm{Proj}(R(P))$ *corresponds via this isomorphism to $\mathcal{O}_{X(P)}(nD(P))$ for all $n\in\boldsymbol{Z}$.*

PROOF. Since $\square_{nh_P}=nP$ and $D(nP)=D_{nh_P}$ for all $n\in\boldsymbol{N}$, we have

$$H^0(X(P), \mathcal{O}_{X(P)}(nD(P)))=\sum_{m\in nP\cap M} ke(m)$$

by (1.1). This implies that $R(P)\simeq R(X(P), D(P))$. The rest follows from a standard argument in the theory of Demazure's construction (cf. [10, Lemma 2.1]). q.e.d.

COROLLARY 1.6. *For an r-dimensional projective torus embedding $X=T_N\mathrm{emb}(\Delta)$ and a strictly upper convex Δ-linear support function $h\in\mathrm{SF}(N, \Delta, \boldsymbol{Q})$, we have*:

(a) $\dim_k H^0(X, \mathcal{O}_X(nD_h))=\begin{cases}\#(n\square_h\cap M) & \text{if } n\geq 0\\ 0 & \text{if } n<0\,;\end{cases}$

(b) $\dim_k H^i(X, \mathcal{O}_X(nD_h))=0$ *for* $0<i<r$ *and all* $n\in\boldsymbol{Z}$;

(c) $\dim_k H^r(X, \mathcal{O}_X(nD_h))=\begin{cases}0 & \text{if } n\geq 0\\ \#(\mathrm{int}((-n)\square_h)\cap M) & \text{if } n<0\,.\end{cases}$

PROOF. (a) follows from (1.1). Since $R(X, D_h)$ is a normal numerical semigroup ring by (1.3) and (1.5), $R(X, D_h)$ is normal and Cohen-Macaulay by a theorem of Hochster [4]. Therefore, (b) follows from [10, Corollary 2.4]. By the Serre duality, we have $\mathrm{Hom}_k(H^r(X, \mathcal{O}_X(nD_h)), k)\simeq H^0(X, \mathcal{O}_X(-[nD_h]+K_X))$, where K_X denotes a canonical divisor on X. Since $K_X=-\sum_{\rho\in\Delta(1)}V(\rho)$ (cf. [5, p. 29]), (c) follows from (1.1). q.e.d.

REMARK 1.7. Let P be a rational convex r-polytope in $\boldsymbol{R}^r$ and $m=\min\{i\in\boldsymbol{N};\, i>0$ and iP is integral$\}$. By (1.3), (1.5) and (1.6), we have $\#(nP\cap\boldsymbol{Z}^r)=\chi(X(P), \mathcal{O}_{X(P)}(nD(P)))$ for $n\geq 0$ and $\#(\mathrm{int}((-n)P)\cap\boldsymbol{Z}^r)=(-1)^r\chi(X(P), \mathcal{O}_{X(P)}(nD(P)))$ for $n<0$, where $\chi(X(P), \mathcal{O}_{X(P)}(nD(P))):=\sum_{j=0}^{r}(-1)^j\dim_k H^j(X(P), \mathcal{O}_{X(P)}(nD(P)))$. By a result due to Snapper and Kleiman, for every $d\in\boldsymbol{Z}$, there exists a polynomial $P_d(\lambda)$ with coefficients in $\boldsymbol{Q}$ such that $\chi(X(P), \mathcal{O}_{X(P)}((d+m\lambda)D(P)))=P_d(\lambda)$. Thus we recover the reciprocity theorem for Ehrhart quasi-polynomials. (See, for example, [7, Proposition 2.24], [9, (4.6)]).

2. Criteria for Gorenstein property.

LEMMA 2.1. *Let Δ be a complete fan in N and $h\in \mathrm{SF}(N, \Delta, \boldsymbol{Q})$ a strictly upper convex Δ-linear support function. Set $\square_h=\{u\in M_{\boldsymbol{R}};\ \langle u, n(\rho)\rangle \geq h(n(\rho))$ for each $\rho\in\Delta(1)\}$. Suppose that h has negative values except at the origin, or equivalently, $\square_h$ contains the origin in its interior. Then the set of vertices of the* polar convex polyhedral set $(\square_h)^\circ:=\{v\in N_{\boldsymbol{R}};\ \langle u, v\rangle\geq -1$ *for all* $u\in\square_h\}$ *for* $\square_h$ *is* $\{-(1/h(n(\rho)))n(\rho);\ \rho\in\Delta(1)\}$.

PROOF. By [7, Corollary A.19], there exists a bijection from $\Delta(1)$ to the set $\mathscr{F}^{r-1}(\square_h)$ of $(r-1)$-dimensional faces of $\square_h$ sending $\rho\in\Delta(1)$ to $Q_\rho:=\{u\in\square_h;\ \langle u, n(\rho)\rangle = h(n(\rho))\}$. Also, by [7, Proposition A.17], there exists a bijection from $\mathscr{F}^{r-1}(\square_h)$ to the set of vertices of $(\square_h)^\circ$ sending an $(r-1)$-dimensional face Q to $Q^*:=\{v\in(\square_h)^\circ;\ \langle u, v\rangle=-1$ for all $v\in Q\}$. Then we observe that $(Q_\rho)^*$ is $-(1/h(n(\rho)))n(\rho)$. q.e.d.

For a Noetherian graded ring R with the canonical module K_R of R, we consider the integer $a(R)$ defined by $a(R)=-\min\{m\in\boldsymbol{Z};\ (K_R)_m\neq 0\}$. For details concerning this integer, see [1, p. 194].

PROPOSITION 2.2 (cf. [2], [3]). *For a rational convex r-polytope P in $M_{\boldsymbol{R}}=\boldsymbol{R}^r$ with $M=\boldsymbol{Z}^r$ and a positive integer δ, the following are equivalent*:

(a) *The semigroup ring $R(P)$ over k is a Gorenstein ring with $a(R(P))=-\delta$.*

(b) *The projective torus embedding $X(P)=T_N\mathrm{emb}(\Delta_P)$ over k, and the ample $\boldsymbol{Q}$-divisor $D(P)=\sum_{\rho\in\Delta_P(1)}(p_\rho/q_\rho)\cdot V(\rho)$ ($q_\rho>0$, p_ρ and q_ρ are coprime) satisfy the following*:

(b1) *There exist a positive integer r_ρ for each $\rho\in\Delta_P(1)$ and a character $m\in M$ such that*

$$\delta D(P)+\mathrm{div}(\boldsymbol{e}(m))=\sum_{\rho\in\Delta_P(1)}(1/r_\rho)\cdot V(\rho)\,;$$

(b2) δ and *q_ρ are coprime for each $\rho\in\Delta_P(1)$.*

(c) *(Hibi's condition) P satisfies the following*:

(c1) *There exists a character $m\in M$ such that the polar polyhedral set $(\delta P-m)^\circ:=\{v\in N_{\boldsymbol{R}};\ \langle u, v\rangle\geq -1$ for all $u\in\delta P-m\}$ for $\delta P-m:=\{\delta p-m\in M_{\boldsymbol{R}};\ p\in P\}$ is an integral convex r-polytope*;

(c2) *The convex hull $\tilde{P}$ of the set $\{(u, 0)\in M_{\boldsymbol{R}}\times\boldsymbol{R};\ u\in P\}\cup\{(0,\dots,0,1/\delta)\}$ in $M_{\boldsymbol{R}}\times\boldsymbol{R}$ is facet-reticular (cf. (1.4)).*

PROOF. (a)$\Leftrightarrow$(b): By (1.5), $R(P)$ is isomorphic to $R(X(P), D(P))$ and, therefore, $R(X(P), D(P))$ is Cohen-Macaulay (cf. [4]). Since a canonical divisor $K_{X(P)}$ on $X(P)$ is $-\sum_{\rho\in\Delta_P(1)}V(\rho)$ (cf. [5, p. 29, Theorem 9, III.d]), it follows from [10, Corollary 2.9] that $R(P)$ is a Gorenstein ring with $a(R(P))=-\delta$ if and only if there exists a character

$m \in M$ such that $\delta D(P) + \mathrm{div}(e(m)) = \sum_{\rho \in \Delta_P(1)} (1/q_\rho) \cdot V(\rho)$. Note that a semi-invariant rational function $f \in K(X(P))^*$ is a scalar multiple of a character $m \in M$.

Suppose (a) holds. By the preceding remark, we have the relation above and, therefore, (b1) holds. Rewriting this relation, we have $\mathrm{div}(e(m)) = \sum_{\rho \in \Delta_P(1)} \{(1 - \delta p_\rho)/q_\rho\} \cdot V(\rho)$. Hence $(1 - \delta p_\rho)/q_\rho$ is an integer and, therefore, δ and q_ρ are coprime for each $\rho \in \Delta_P(1)$.

Conversely, suppose (b) holds. By the preceding remark, we claim that $r_\rho = q_\rho$ for each $\rho \in \Delta_P(1)$. Since r_ρ is a factor of q_ρ, $b_\rho := (q_\rho / r_\rho)$ is a positive integer. Then, by (b1), $(b_\rho - \delta p_\rho)/(r_\rho b_\rho)$ is an integer and, therefore, b_ρ is a factor of δp_ρ. Hence we have $b_\rho = 1$ for each $\rho \in \Delta_P(1)$ as required, because neither δ nor p_ρ has any common factor with q_ρ.

(b1)⇒(c1): Set $g = \delta h_P - m \in \mathrm{SF}(N, \Delta_P, \boldsymbol{Q})$. Since $D_g = \delta D(P) + \mathrm{div}(e(m))$ and D_g is ample, g is strictly upper convex and $g(n(\rho)) = -(1/r_\rho)$ for each $\rho \in \Delta_P(1)$. Therefore, by (2.1), the set of vertices of the polar convex polyhedral set $(\square_g)^\circ$ is $\{r_\rho n(\rho);\ \rho \in \Delta_P(1)\}$ $(= \{-(1/g(n(\rho)))n(\rho);\ \rho \in \Delta_P(1)\})$. On the other hand, we have $\square_g = \delta P - m$ by definition. Therefore $(\delta P - m)^\circ$ is an integral convex polytope.

(c1)⇒(b1): Set $g = \delta h_P - m \in \mathrm{SF}(N, \Delta_P, \boldsymbol{Q})$. Since g is strictly upper convex with respect to Δ_P and $O \in \mathrm{int}(\delta P - m)$, it follows from (2.1) that the vertex set of $(\delta P - m)^\circ$ is $\{-(1/g(n(\rho)))n(\rho);\ \rho \in \Delta_P(1)\}$. Hence, by assumption, $-(1/g(n(\rho)))n(\rho)$ is an integral vector. Since $n(\rho)$ is a primitive integral vector and $g \in \mathrm{SF}(N, \Delta_P, Q)$ is negative-valued, $r_\rho := -1/(g(n(\rho)))$ is a positive integer for each $\rho \in \Delta_P(1)$ and $\delta D(P) + \mathrm{div}(e(m)) = D_g = \sum_\rho (1/r_\rho) \cdot V(\rho)$.

(b2)⇔(c2): Since the supporting hyperplane carried by the facet of P corresponding to $\rho \in \Delta_P(1)$ is $H_\rho = \{u \in M_{\boldsymbol{R}};\ \langle u, n(\rho) \rangle = h_P(n(\rho))\}$, the supporting hyperplane carried by a facet of $\tilde{P}$ is of the form $\tilde{H}_\rho := \{(u, x) \in M_{\boldsymbol{R}} \times \boldsymbol{R};\ \delta x + (1/h_P(n(\rho)))\langle u, n(\rho) \rangle = 1\}$ or $\{(u, 0) \in M_{\boldsymbol{R}} \times \boldsymbol{R}\}$. Since $h_P(n(\rho)) = -(p_\rho / q_\rho)$ and $n(\rho)$ is a primitive vector, δ and q_ρ are coprime if and only if $\tilde{H}_\rho \cap (M \times \boldsymbol{Z})$ is non-empty. q.e.d.

REMARK 2.3. The equivalence between the conditions (a) and (c) in (2.2) is originally due to Hibi [3]. Combining the equivalence between (a) and (c) in (2.2) and a theorem of Stanley [8, Theorem 4.4], we get another proof for theorems of Hibi [2], [3]. Our proof makes clear why the condition (c2) in (2.2) is needed, in terms of Demazure's construction. Indeed, let $R(X, D)$ be a Cohen-Macaulay graded ring obtained from a normal projective variety X and an ample $\boldsymbol{Q}$-divisor $D = \sum_V (p_V / q_V) \cdot V$, with V running through irreducible subvarieties of codimension 1, where $q_V > 0$ and p_V, q_V are coprime for each V. Then it follows from [10, Corollary 2.9] that $R(X, D)$ is Gorenstein if the Veronese subring $R(X, D)^{(d)}$ of order d is Gorenstein for an integer d such that $a(R(X, D)) \equiv 0 \pmod{d}$ and that d and q_V are coprime for each V.

COROLLARY 2.4 (cf. [7, (2.20)]). *For a rational convex r-polytope P in $M_{\boldsymbol{R}} = \boldsymbol{R}^r$ with $M = \boldsymbol{Z}^r$ and a positive integer δ, the following are equivalent*:

(a) *P is integral and there exists a character $m \in M$ such that the polar polyhedral set $(\delta P - m)^\circ$ for $\delta P - m$ is an integral convex r-polytope*;

(b) *The $\boldsymbol{Q}$-divisor $D(P)$ on the projective torus embedding $X(P)$ is an ample Cartier divisor. The invertible sheaf $\mathcal{O}_{X(P)}(-\delta D(P))$ is isomorphic to the canonical sheaf $\omega_{X(P)}$.*

PROOF. It follows from (1.4) and (2.2) that (a) holds if and only if $D(P)$ is a Cartier divisor and there exists a character $m \in M$ such that $\delta D(P) + \mathrm{div}(e(m)) = \sum_{\rho \in \Delta_P(1)} V(\rho)$. Since a canonical divisor $K_{X(P)}$ on $X(P)$ is $-\sum_{\rho \in \Delta_P(1)} V(\rho)$, (a) is equivalent to (b). q.e.d.

Since every Cartier divisor on a complete torus embedding is linearly equivalent to a T-stable Cartier divisor (cf. [6, Proposition 6.1]), we have:

COROLLARY 2.5. *Let X be a projective torus embedding and D an ample Cartier divisor. Then $R(X, D)$ is Gorenstein if and only if the canonical sheaf ω_X on X is isomorphic to an invertible sheaf $\mathcal{O}_X(-\delta D)$ for a positive integer δ.*

THEOREM 2.6. *Every projective torus embedding X has a T-stable ample $\boldsymbol{Q}$-divisor D such that $R(X, D)$ is a Gorenstein ring with $a(R(X, D)) = -1$.*

PROOF. By assumption, $X = T_N \mathrm{emb}(\Delta)$ has a T-stable ample Cartier divisor E of the form $E = \sum_{\rho \in \Delta(1)} a_\rho \cdot V(\rho)$, $a_\rho > 0$. Set $c = \mathrm{L.C.M.}\{a_\rho;\ \rho \in \Delta(1)\}$ and $D = (1/c)E$. By (1.3), (X, D) corresponds to a rational polytope P in $M_{\boldsymbol{R}}$. Then, by (1.5) and (2.2), $R(X, D)$ is a Gorenstein ring with $a(R(X, D)) = -1$, as required. q.e.d.

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