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QUALITATIVE ANALYSIS OF A NONAUTONOMOUS NONLINEAR DELAY DIFFERENTIAL EQUATION

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Abstract. This paper is devoted to the systematic study of some qualitative properties of solutions of a nonautonomous nonlinear delay equation, which can be utilized to model single population growths. Various results on the boundedness and oscillatory behavior of solutions are presented. A detailed analysis of the global existence of periodic solutions for the corresponding autonomous nonlinear delay equation is given. Moreover, sufficient conditions are obtained for the solutions to tend to the unique positive equilibrium.

Introduction. Using the adsorption theory of chemical kinetics, Ciu and Lawson [1] established the following equation concerning the growth of single populations

$$\dot{x}(t):=\frac{dx(t)}{dt}=\mu_c x(t)\frac{1-x(t)/x_m}{1-x(t)/x'_m}, \tag{1.1}$$

where $x(t)$ is the population density at time t; x_m is the maximum value of x allowed by the limiting nutrient, which is equivalent to the so-called carrying capacity; x'_m is a parameter which is related to the amount of nutrient and its utilization efficiency by an organism (in units of concentration); μ_c is a parameter related to the growth velocity (in units of time^{-1}), i.e., the so-called intrinsic growth rate.

The ratio of x_m and x'_m is a very important parameter for Equation (1.1). It is assumed that [1–4] $0<x_m/x'_m<1$. When $x_m=x'_m$, (1.1) reduces to the Malthus exponential equation, and when $x'_m \gg x_m$, (1.1) reduces to the well-known logistic equation. In other words, the Malthus and logistic equations are two special cases of Equation (1.1).

Most growth observed in nature seems to support the new equation (1.1) rather than the logistic hypothesis. In the logistic equation, the per capita growth rate is assumed to be linear $[1-x/x_m]$, so that the population growth rate $\mu_c x(t)[1-x(t)/x_m]$ always achieves its maximum at $0.5x_m$. However, as observed by Thompson [5], microorganisms, plants and animals all show a maximum growth velocity when the population density is greater than $0.5x_m$. One can show easily that the maximum growth velocity of (1.1) is achieved at $x_m(1-\sqrt{1-c})c^{-1}$ where $c=x_m/x'_m$. Since $0<c<1$, we have $\sqrt{1-c}<1-c/2$, thus $1-\sqrt{1-c}>c/2$, which implies $(1-\sqrt{1-c})c^{-1}>0.5$. This

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fact is of particular importance in forest management. Since a good estimate of the maximum velocity of the population growth and the time at which this occurs is necessary for the reasonable estimates of indices, such as the cutting age, the cutting intensity and the cutting cycle period.

As pointed out by May in [6], due to the effect of aging, digestion or other factors, it is necessary and more realistic to incorporate time delay into the per capita growth rate of the population, which may result in the following delay differential equation

$$\dot{x}(t)=\mu_c x(t)\frac{1-x(t-\tau(t))/x_m}{1-x(t-\tau(t))/x'_m}, \tag{1.2}$$

where $\tau(t)$ is continuous and positive. If one takes into account the death rate of the population, then μ_c may be an arbitrary real function. In the follwing, we denote it by $r(t)$, which is assumed to be continuous. By letting $u(t)=x(t)/x_m$, $c=x_m/x'_m$, Equation (1.2) reduces to

$$\dot{u}(t)=r(t)u(t)\frac{1-u(t-\tau(t))}{1-cu(t-\tau(t))}. \tag{1.3}$$

From an ecological point of view, we will restrict our attention to the bounded positive solutions of Equation (1.2). Denote

$$E_0=\{t-\tau(t)\colon t-\tau(t)\leq 0,\, t\geq 0\}\cup\{0\}. \tag{1.4}$$

For $\theta\in E_0$, we assume

$$c^{-1}>u(\theta)=\phi(\theta)\geq 0\,, \quad \phi(0)>0\,, \tag{1.5}$$

where $\phi(\theta)$ is continuous on E_0. In the rest of the paper, we assume $\tau(t)\geq\tau>0$, for all $t\geq 0$. It is well-known that Equation (1.3) with initial condition (1.5) always has a unique positive solution (locally) (see Hale [7]).

For any given constant K, we say the solution $u(t)$ of (1.3) is *oscillatory with respect to K*, if there is a positive sequence $\{t_m\}$, $\lim_{n\to\infty} t_n=+\infty$, such that $u(t_n)=K$, $n=1, 2, \cdots$. Otherwise, we say $u(t)$ is *nonoscillatory with respect to K*. When $K=0$, we simply say the solution is *oscillatory* or *nonoscillatory*. It is easy to see that all the positive solutions of Equation (1.1) are monotone and tend to x_m. Due to the effect of the introduced time delay, it is natural to study the possible oscillatory behavior of the solutions of Equation (1.3) and the existence of periodic solutions. This will be our principle theme in the following sections.

For convenience, we would like to introduce the following change of variable by letting

$$y(t)=\frac{1-u(t)}{1-cu(t)}. \tag{1.6}$$

Then, Equation (1.3) reduces to

$$\dot{y}(t) = -\frac{r(t)}{1-c}(1-y(t))(1-cy(t))y(t-\tau(t)) . \tag{1.7}$$

The initial function becomes

$$y(\theta) = [1-\phi(\theta)]/[1-c\phi(\theta)] , \quad \theta \in E_0 . \tag{1.8}$$

Obviously, $-\infty < y(\theta) \leq 1$, $y(0) < 1$. Thus, the study of the oscillatory behaviour of solutions of (1.3) with respect to 1 is equivalent to the study of the oscillatory behavior of solutions of (1.7) with respect to zero.

It is easy to see that the solution of (1.3) with initial condition (1.5) is positive, which implies that the solution of (1.7) and (1.8) satisfies $1-y(t) > 0$, and $1-cy(t) > 0$.

In the next section, we establish some results on the boundedness of solutions for Equations (1.3) and (1.7). These results are essential for the proof of the existence of periodic solutions of these equations in Section 3. Section 4 and Section 5 contain oscillatory results for Equation (1.7) under various assumptions. Section 6 presents sufficient conditions for the solutions to tend to the unique positive equilibrium $u(t) \equiv 1$ in Equation (1.3). We complete the paper by a brief discussion.

2. Boundedness of solutions. Generally speaking, solutions of (1.7) and (1.8) may not be bounded. In fact, they may not exist for all $t > 0$. For example, let $\tau(t) = 1$ and $r(t) > 0$. Then, for $0 < t \leq 1$, it is easy to obtain the solution of (1.7) and (1.8) by direct calculation, which takes the form

$$y(t) = 1-(1-c)\cdot\left[\frac{1-cy(0)}{1-y(0)}\exp\left(-(1-c)\int_{-1}^{t-1} r(\theta+1)y(\theta)d\theta\right)-c\right]^{-1} . \tag{2.1}$$

Obviously, for some choices of $r(t)$, c and $y(\theta)$, $-1 \leq \theta \leq 0$, there is a $t^* \in (0, 1)$, such that

$$c = \frac{1-cy(0)}{1-y(0)}\exp\left(-(1-c)\int_{-1}^{t^*-1} r(\theta+1)y(\theta)d\theta\right) , \tag{2.2}$$

which implies

$$\lim_{t\to t^*} y(t) = -\infty . \tag{2.3}$$

This is due to the fact that for some initial function $\phi(\theta)$, $-1 \leq \theta \leq 0$, the solution $u(t)$ of Equation (1.3) may achieve the value c^{-1} at some time t^*, $0 < t^* < 1$, which will render the Equation (1.3) meaningless at time t^*+1. For this sake, it is necessary for us to consider first the boundedness of the solution of these equations.

THEOREM 2.1. *In Equation* (1.3), *suppose* $t-\tau(t)$ *is continuous and nondecreasing*, $\lim_{t\to\infty}(t-\tau(t)) = +\infty$, *and there is a* $\tau > 0$ *such that* $\tau(t) \geq \tau$ *for* $t \geq 0$. *Assume further* *that* $r(t) \geq r_0 > 0$ *is continuous,*

$$\sigma = \sup\left\{\int_{t-\tau(t)}^{t} r(\theta)d\theta,\ t \geq 0\right\}$$

is finite, $e^{\sigma} < c^{-1}$, *and that the initial function* $\phi(0)$ *satisfies* $0 \leq \phi(\theta) < c^{-1}$, $-\tau(0) \leq \theta < 0$, $\phi(0) < c^{-1}e^{-\sigma}$. *Then the solution* $u(t)$ *of* (1.3) *is bounded, and* $\limsup_{t\to\infty} u(t) \leq e^{\sigma}$.

PROOF. It is easy to see that there exists a $\bar{t} > 0$ such that $\bar{t} = \tau(\bar{t})$. For $0 \leq t \leq \bar{t}$, we have

$$u(t) = u(0)\exp\left(\int_0^t r(\theta)\frac{1 - u(\theta - \tau(\theta))}{1 - cu(\theta - \tau(\theta))}d\theta\right). \tag{2.4}$$

Since $0 \leq \phi(\theta) < c^{-1}$, $1 - u(\theta - \tau(\theta))/(1 - cu(\theta - \tau(\theta))) \leq 1$, thus, for $0 \leq t \leq \bar{t}$

$$\int_0^t r(\theta)\frac{1 - u(\theta - \tau(\theta))}{1 - cu(\theta - \tau(\theta))}d\theta \leq \int_0^t r(\theta)d\theta \leq \sigma\ . \tag{2.5}$$

Obviously, (2.5) implies that for $0 \leq t \leq \bar{t}$,

$$u(t) \leq u(0)e^{\sigma} < c^{-1}\ . \tag{2.6}$$

Hence, we have shown $0 \leq u(t) < c^{-1}$, for $-\tau(0) \leq t \leq \bar{t}$.

Assume t_1 satisfies $t_1 - \tau(t_1) = \bar{t}$. Then for $t \leq t_1$, we have

$$\dot{u}(t) = r(t)u(t)\left(1 - \frac{(1-c)u(t - \tau(t))}{1 - cu(t - \tau(t))}\right) \leq r(t)u(t)\,, \tag{2.7}$$

since $(1-c)u(t-\tau(t))/(1 - cu(t - \tau(t))) \geq 0$. From (2.7), we have

$$u(t) \leq u(t_0)\exp\left(\int_{t_0}^t r(\theta)d\theta\right). \tag{2.8}$$

In particular, we have

$$u(t) \leq u(t - \tau(t))\exp\left(\int_{t-\tau(t)}^t r(\theta)d\theta\right), \tag{2.9}$$

which yields

$$u(t - \tau(t)) \geq u(t)\exp\left(-\int_{t-\tau(t)}^t r(\theta)d\theta\right). \tag{2.10}$$

Hence

$$u(t - \tau(t)) \geq u(t)e^{-\sigma}\ . \tag{2.11}$$

This implies for $\bar{t} \leq t \leq t_1$,

$$\dot{u}(t) \le r(t)u(t)\left(\frac{1-u(t)e^{-\sigma}}{1-cu(t)e^{-\sigma}}\right). \tag{2.12}$$

Since $r(t) \ge r_0 > 0$, we see that all positive solutions of

$$\dot{u}(t) = r(t)u(t)\left(\frac{1-u(t)e^{-\sigma}}{1-cu(t)e^{-\sigma}}\right) \tag{2.13}$$

with initial values less than $\delta > 0$, $e^{\sigma} < \delta < c^{-1}e^{\sigma}$ will be bounded by δ and have e^{σ} as their limit. Therefore, for $\bar{t} \le t \le t_1$, $0 \le u(t) < c^{-1} < c^{-1}e^{\sigma}$. By repeating this argument (assume that $t_2 - \tau(t_2) = t_1$, we can show that (2.12) holds for $t_1 \le t \le t_2$. Define $t_{i+1} - \tau(t_{i+1}) = t_i$, then $t_i \to +\infty$, as $i \to +\infty$), we see $u(t)$ is bounded and satisfies (2.12) for all $t \ge \bar{t}$, thus the solution $u(t)$ of (1.3) is bounded by $u(0)e^{\sigma} < c^{-1}$, and

$$\limsup_{t \to +\infty} u(t) \le e^{\sigma}.$$

This completes the proof of the theorem.

The following theorem is equivalent to Theorem 2.1 for Equation (1.7). It is obtained by applying the transformation (1.6).

THEOREM 2.2. *Suppose $r(t)$ and $\tau(t)$ are the same as described in Theorem 2.1, and the initial function $y(\theta)$, $\theta \in [-\tau(0), 0]$ satisfies $y(\theta) \le 1$, $y(0) > (1-c^{-1}e^{-\sigma})/(1-e^{-\sigma})$. Then the solution $y(t)$ of (1.7) is bounded, and $\liminf_{t\to\infty} y(t) \ge (1-e^{\sigma})/(1-ce^{\sigma})$.*

We call a solution of a differential equation *global*, if it exists for all $t \ge 0$. *In the rest of this paper, we will consider only global solutions of the concerned differential equations.*

THEOREM 2.3. *Suppose $r(t)$ and $\tau(t)$ are positive and continuous, $\lim_{t\to+\infty}(t-\tau(t)) = +\infty$ and $\int_0^{\infty} r(\theta)d\theta = +\infty$. Then every global solution of (1.3) is either oscillatory with respect to 1 or tends to 1 as $t \to +\infty$. If we assume further that $t-\tau(t)$ is increasing, and there is a $t_1 > 0$, such that for $t \ge t_1$*

$$0 < \int_{t-\tau(t)}^{t} r(\theta)d\theta < \sigma, \quad \textit{and} \quad e^{\sigma} < c^{-1}.$$

Then every oscillatory solution of (1.3) (with respect to 1) has maximum value less than e^{σ}, for $t \ge \bar{t} + t_1$, where $\bar{t} = \tau(\bar{t})$.

PROOF. Suppose that $1 < u(t) < c^{-1}$ for $t \ge T$. Since $u(t)$ is global, we see that $\dot{u}(t) < 0$ for $t > T^*$, where $T^* - \tau(T^*) > T$, and hence

$$\lim_{t \to +\infty} u(t) = \mu \ge 1 \quad \text{exists}. \tag{2.14}$$

If $\mu > 1$, then we have

(2.15) $$\dot{u}(t) \leq r(t) \cdot \mu \frac{1-\mu}{1-c\mu} \quad \text{for} \quad t \geq T^* .$$

(2.15) together with the assumption $\int_0^\infty r(\theta)d\theta = +\infty$ yields

(2.16) $$\lim_{t \to +\infty} u(t) = -\infty .$$

This contradicts (2.14). Therefore, we have shown that μ must be 1.

Similarly, we can prove if $0 < u(t) < 1$, for $t \geq T$, then $\lim_{t \to +\infty} u(t) = 1$.

Now, suppose $u(t)$ is oscillatory with respect to 1. Since $t - \tau(t)$ is increasing and $\lim_{t \to +\infty}(t - \tau(t)) = +\infty$, we see there is at least one $\bar{t}$, $\bar{t} > 0$, such $\bar{t} = \tau(\bar{t})$. Assume $u(t)$ achieves its local maximum at t^*, $t^* \geq \bar{t} + t_1$. Then $\dot{u}(t^*) = 0$, which implies $u(t^* - \tau(t^*)) = 1$, since $u(t) > 0$ for all $t > 0$. Thus

$$u(t^*) = u(t^* - \tau(t^*)) \exp\left(\int_{t^* - \tau(t^*)}^{t^*} r(\theta) \frac{1 - u(\theta - \tau(\theta))}{1 - cu(\theta - \tau(\theta))} d\theta \right) < e^\sigma .$$

Since $e^\sigma < c^{-1}$, we see that $u(t)$ must be bounded and has maximum less than e^σ for $t \geq \bar{t} + t_1$. This proves our theorem.

Obviously, Theorem 2.3 indicates that the last statement of Theorem 2.1 can be strengthened as $u(t) < e^\sigma$ for $t \geq \bar{t}$. Again, by applying the transformation (1.6), we have the following theorem for Equation (1.7), which is equivalent to the above theorem.

THEOREM 2.4. *Suppose $r(t)$ and $\tau(t)$ are positive and continuous,* $\lim_{t \to +\infty}(t - \tau(t)) = +\infty$, *and* $\int_0^\infty r(\theta)d\theta = +\infty$. *Then every global solution of* (1.7) *is either oscillatory or tends to zero as $t \to +\infty$. If we assume further that*

$$0 < \sigma = \sup\left\{ \int_{t-\tau(t)}^{t} r(\theta)d\theta : \, t \geq 0 \right\} < +\infty , \quad e^\sigma < c^{-1} ,$$

and $t - \tau(t)$ is increasing, then every oscillatory solution of (1.7) *has minimum value greater than* $(1 - e^\sigma)/(1 - ce^\sigma)$, *for $t \geq \bar{t}$, where $\bar{t} = \tau(\bar{t})$.*

3. Existence of periodic solutions. In this section, we assume $r(t) = r$, $\tau(t) = \tau$, where r, τ are two positive constants. Under this assumption, we can consider the existence of periodic solutions for the autonomous equations (1.3) and (1.7). Clearly, the local stability of the steady state $u(t) \equiv 1$ in (1.3) is the same as the local stability of the steady state $y(t) \equiv 0$, and the existence of positive periodic solutions of (1.3) is equivalent to the existence of a periodic solution $y(t)$ of (1.7), such that $-\infty < y(t) < 1$. For this sake, we will restrict our attention to Equation (1.7).

Our analysis will be based on the Hopf bifurcation theorem (Hale [7, pp. 245–249]) and a general fixed-poiont theorem due to Nussbaum [8] (also in Hale [7, p. 249]).

Letting $z(t) = -y(t)$, (1.7) becomes

$$\dot{z}(t) = \frac{-r}{1-c}(1+z(t))(1+cz(t))z(t-\tau)\,, \tag{3.1}$$

and (1.8) becomes

$$-1 \le z(\theta) = -[1-\phi(\theta)]/[1-c\phi(\theta)] < +\infty, \quad \theta \in [-\tau, 0]\,. \tag{3.2}$$

Let $t' = \tau^{-1}t$, $\alpha = r\tau/(1-c)$ in (3.1), and then drop the primes on t. We have

$$\dot{z}(t) = -\alpha(1+z(t))(1+cz(t))z(t-1)\,, \tag{3.3}$$

and

$$-1 \le z(\theta) < +\infty\,, \quad \theta \in [-1, 0]\,, \quad z(0) > -1\,. \tag{3.4}$$

The linearized equation of (3.3) at $z(t) \equiv 0$ is

$$\dot{z}(t) = -\alpha z(t-1)\,. \tag{3.5}$$

Its characteristic equation is

$$\lambda e^{\lambda} + \alpha = 0\,. \tag{3.6}$$

The following lemma is proved in Hale [7, pp. 254–255]. It can also be proved easily by other methods (e.g. see Freedman and Kuang [9]).

LEMMA 3.1. *If* $0 < \alpha < \pi/2$, *every root of* (3.6) *has a negative real part. If* $\alpha > e^{-1}$, *there is a root* $\lambda(\alpha) = \gamma(\alpha) + i\beta(\alpha)$ *of* (3.6) *which is continuous together with its first derivative in* α *and satisfies* $0 < \beta(\alpha) < \pi$, $\beta(\pi/2) = \pi/2$, $\gamma(\pi/2) = 0$, $\gamma'(\pi/2) > 0$, *and* $\gamma(\alpha) > 0$ *for* $\alpha > \pi/2$.

As an immediate consequence of this lemma and the Hopf bifurcation theorem (Theorem 1.1 in Hale [7, p. 246]), we have the following theorem. For more details, see Hale [7, pp. 245–249].

THEOREM 3.1. *Equation* (3.3) *has a Hopf bifurcation at* $\alpha = \pi/2$.

From Lemma 3.1, we see if $0 < \alpha < \pi/2$, then the zero solution of (3.3) is locally asymptotically stable (see Hale [7]), and when $\alpha > \pi/2$, the zero solution of (3.3) is unstable. This is equivalent to saying that $u(t) \equiv 1$ is locally asymptotically stable if $r\tau < \pi(1-c)/2$, and is unstable if $r\tau > \pi(1-c)/2$.

Similarly to Lemmea 4.2 in Hale [7, p. 255], we have the following results for Equation (3.3). As pointed out earlier, all solutions considered here are assumed to be global.

LEMMA 3.2. *For Equation* (3.3), *the following statements are true.*

(i) *If* $z(0) > -1$ *and* $z(t)$ *is nonoscillatory, then* $z(t) \to 0$ *as* $t \to +\infty$.

(ii) *If* $-1 < z(0) < [c^{-1}e^{(c-1)\alpha} - 1]/[1 - e^{(c-1)\alpha}]$ *and* $e^{(1-c)\alpha} < c^{-1}$, *then* $z(t)$ *is bounded. Furthermore, if* $z(t)$ *is oscillatory, then* $z(t) < [e^{(1-c)\alpha} - 1]/[1 - ce^{(1-c)\alpha}]$, *for* $t \ge 1$.

(iii) *If* $z(0) > -1$ *and* $\alpha > e^{-1}$, *then* $z(t)$ *is oscillatory.*

(iv) *If* $z(\theta) > 0$, $-1 \leq \theta \leq 0$ [*or if* $z(\theta) > -1$, $z(\theta) < 0$, $-1 \leq \theta \leq 0$], *then the zeroes (if any) of* $z(t)$ *are simple and the distance from a zero of* $z(t)$ *to the next maximum or minimum is* ≥ 1.

PROOF. (i) is contained in the first half of Theorem 2.4.

(ii) is a combination of Theorem 2.2 and the second half of Theorem 2.4.

(iii) can be viewed as a special case of a more general result to be proved in Section 5. We omit its proof here to avoid repetition.

(iv) Suppose $z(t_0) = 0$ and $z(t) > 0$, $t_0 - 1 \leq t < t_0$. For $t_0 \leq t < t_0 + 1$, $\dot{z}(t) < 0$. Similarly, if $z(t) < 0$ for $t_0 - 1 \leq t < t_0$ and $z(t_0) = 0$, then $\dot{z}(t) > 0$, for $t_0 \leq t < t_0 + 1$. Thus, the assertions of (iv) are obvious and the lemma is proved. ■

Let K be the class of all functions $\phi \in C$ [where C is the space of real continuous functions defined on $[-1, 0]$, with the norm defined as $|\phi| = \max_{-1 \leq \theta \leq 0} |\phi(\theta)|$, for $\phi \in C$], such that $0 \leq \phi(\theta) \leq [c^{-1}e^{(c-1)\alpha} - 1]/[1 - e^{(c-1)\alpha}]$, $-1 \leq \theta \leq 0$, $\phi(-1) = 0$, ϕ is nondecreasing. Then K is a bounded, closed and convex set in C. If $\alpha > 1$, $\phi \in K$, $\phi \neq 0$, we denote $z(\phi, \alpha)(t)$ as the solution of Equation (3.3) with initial function as ϕ. Let

$$p(\phi, \alpha) = \min\{t : z(\phi, \alpha)(t) = 0, \dot{z}(\phi, \alpha)(t) > 0\}\,. \tag{3.7}$$

This minimum exists from Lemma 3.2, part (iii). Also $p(\phi, \alpha) > 1$. Furthermore, Lemma 3.2, part (iv) implies $z(\phi, \alpha)(t)$ is positive and nondecreasing on $(p(\phi, \alpha), p(\phi, \alpha) + 1)$. Suppose $ce^{2(1-c)\alpha} < 1$. Then $e^{(1-c)\alpha} < c^{-1}e^{(c-1)\alpha}$, which implies

$$[e^{(1-c)\alpha} - 1]/[1 - ce^{(1-c)\alpha}] < [c^{-1}e^{(c-1)\alpha} - 1]/[1 - e^{(c-1)\alpha}]\,. \tag{3.8}$$

Consequently, if $\tau(\phi, \alpha) = p(\phi, \alpha) + 1$, then the mapping

$$A(\alpha)0 = 0$$

$$A(\alpha)\phi = z_{\tau(\phi,\alpha)}(\phi, \alpha)\,, \quad \phi \neq 0\,,$$

is a mapping of K into itself, where $z_{\tau(\phi,\alpha)}(\phi, \alpha)(\theta) = z(\phi, \alpha)(\tau(\phi, \alpha) + \theta)$, for $-1 \leq \theta \leq 0$. Since $\dot{z}(\phi, \alpha)(\tau(\phi, \alpha) - 1) > 0$, the continuity of $z(\phi, \alpha)(t)$ in t, ϕ, α implies that $\tau(\phi, \alpha)$ is continuous in $(K \setminus \{0\}) \times (1, \infty)$. In fact, the following stronger conclusion is true.

LEMMA 3.3. $\tau(\phi, \alpha)$ *is completely continuous in* $(K \setminus \{0\}) \times (1, \infty)$.

PROOF. First of all, we claim a solution $z(t) = z(\phi, \alpha)(t)$, $\phi \in K$, cannot take a time longer than 2 to become negative because, if $z(1) = \eta > 0$, then $z(t) \geq \eta$ for $0 \leq t \leq 1$. Since Equation (3.3) is equivalent to

$$\frac{1 + z(t)}{1 + cz(t)} = \frac{1 + z(t_0)}{1 + cz(t_0)} \exp\left(-(1 - c)\alpha \int_{t_0}^{t} z(\theta - 1)\,d\theta\right), \tag{3.9}$$

we have

$$\frac{1+z(2)}{1+cz(2)} \le \frac{1+\eta}{1+c\eta} e^{-(1-c)\alpha\eta} . \tag{3.10}$$

This implies

$$\frac{(1-c)z(2)}{1+cz(2)} < \frac{1+\eta}{1+c\eta} e^{-(1-c)\alpha\eta} - 1 . \tag{3.11}$$

Let $q(\eta) = ((1+\eta)/(1+c\eta))e^{-(1-c)\alpha\eta} - 1$. Then $q(0)=0$, and $q'(\eta) = [((1+\eta)/(1+c\eta))(c-1)\alpha + (1-c)/(1+c\eta)^2]e^{-(1-c)\alpha\eta} < 0$ for $\eta > 0$. Thus $z(2) < 0$.

For any bounded set $B \subseteq K$ and any $\phi \in B$, $\alpha \in (1, \infty)$, let $t_0(\phi, \alpha) \le 3$ denote the point where the solution $z(\phi, \alpha)$ has a minimum. Since $t_0(\phi, \alpha) \ge 1$, it is well-known (Hale [7]) that the closure $H(\alpha) = \mathrm{Cl} \bigcup_{\phi \in B} z_{t_0(\phi,\alpha)}(\phi, \alpha)$ is compact and

$$H(\alpha) \subset K_1 := \{\psi \in C: \ -1 < \psi(\theta) \le 0, \ -1 \le \theta \le 0 \ , \ \psi \text{ nonincreasing}\} .$$

For any $\psi \in K_1$, define a continuous function τ_1: $[K_1 \setminus \{0\}] \times (1, \infty) \to (0, \infty)$ by the relation $\tau_1(\psi, \alpha) = \min\{t > 0: \ z(\psi, \alpha)(t) = 0\}$. Clearly, if we prove $\tau_1(H(\alpha) \setminus \{0\}, \alpha)$ is bounded for each $\alpha \in (1, \infty)$, then $\tau(B \setminus \{0\}, \alpha)$ is bounded for each α, thus the lemma.

Since $H(\alpha)$ is compact, it is therefore only necessary to prove that $\tau_1(\psi, \alpha)$ is bounded on a neighborhood of zero in K_1. This can be proved in the following manner. If $\psi \in K_1 \setminus \{0\}$ and $z(\psi, \alpha)(1) = \beta < 0$, that is, $\tau_1(\psi, \alpha) > 1$, then $z(\psi, \alpha)(\theta - 1) \le \beta$, $1 \le \theta \le 2$, and

$$\frac{1+z(\psi, \alpha)(2)}{1+cz(\psi, \alpha)(2)} \ge \frac{1+\beta}{1+c\beta} \exp(-(1-c)\alpha\beta) . \tag{3.12}$$

This implies

$$\frac{(1-c)z(\psi, \alpha)(2)}{1+cz(\psi, \alpha)(2)} \ge q(\beta) := \frac{1+\beta}{1+c\beta} \exp(-(1-c)\alpha\beta) - 1 .$$

Since $q(0) = 0$, $q'(\beta) = (1+c\beta)^{-2}[1 - \alpha(1+\beta)(1+c\beta)](1-c)\exp(-(1-c)\alpha\beta)$, we see that $q(\beta) > 0$ for small and negative β. Thus $\tau_1(H(\alpha) \setminus \{0\}, \alpha)$ is bounded for each $\alpha \in (1, \infty)$. ■

Lemma 3.3 implies (for details, see Hale [7, p. 257]) that $A(\alpha)$ is completely continuous. Obviously, the Lemma 4.4 in Hale [7, p. 256], is true for Equation (3.3) as well.

If we now take $M > 0$ such that

$$\frac{e^{(1-c)\alpha} - 1}{1 - ce^{(1-c)\alpha}} < M < \frac{c^{-1}e^{(c-1)\alpha} - 1}{1 - e^{(c-1)\alpha}} , \tag{3.13}$$

then the above lemmas imply that all of the conditions of Theorem 2.2 and Theorem 2.3 in Hale [7, pp. 249–251] are satisfied. (Note, the proof of Theorem 2.3 in [7,

p. 250] may be incomplete. For a complete proof, see Alt [17].) Thus, we have proved the following main result.

THEOREM 3.2. *If* $\alpha > \pi/2$ *and* $ce^{2(1-c)\alpha} < 1$, *then Equation* (3.3) *has a nonzero periodic solution.*

The equivalent result for Equation (1.3) is:

THEOREM 3.3. *If* $\tau(t) = \tau > 0$, $r(t) = r > 0$, $ce^{2r\tau} < 1$, *and* $r\tau > \pi(1-c)/2$, *then Equation* (1.3) *has at least a nonzero positive periodic solution.*

Theorem 3.3 implies that if c is small enough, and $r\tau$ is big enough, then the steady state $u(t) \equiv 1$ is not stable and periodic solutions may exist. This amounts to saying that if the intrinsic growth rate is high, if the time delay is long, and if Equation (1.3) is close to the logistic equation (in which case $c = 0$), then solutions of (1.3) are oscillatory, and periodic solutions exist.

If we define the following two sets for Equation (3.3)

$$S = \mathrm{Cl}\{(\phi, \alpha) \in K \times (1, \infty) : A(\alpha)\phi = \phi,\ \phi \neq 0\}$$

$$S_0 = \text{maximal closed connected component of } S \text{ which contains } (0, \pi/2),$$

then we have the following results similar to Theorem 4.3 and Theorem 4.4 in Hale [7, pp. 259–260]. The proofs of those results can be obtained by modifying properly those arguments presented in Nussbaum [10]. We choose to omit these proofs here in order to avoid repetition.

THEOREM 3.4. *For Equation* (3.3), S_0 *is unbounded and, for any* $\alpha_0 > 1$, *there is an* $\alpha > \alpha_0$ *and* ϕ *such that* $(\phi, \alpha) \in S_0$.

THEOREM 3.5. *For any* $p > 4$, *there is a periodic solution of Equation* (3.3) *of period* p.

4. Oscillatory results when $r(t)$ is nonnegative. In this section, *we always assume* $r(t)$ *and* $\tau(t)$ *are positive and continuous on* $[0, \infty)$, *and* $\lim_{t\to+\infty}(t-\tau(t)) = +\infty$. We will first restrict our attention to Equation (1.7) with initial function satisfying (1.8). In addition to Theorem 2.4, we have the following two results concerning the oscillatory behavior of solutions of (1.7):

THEOREM 4.1. *If* $\liminf_{t\to+\infty}\int_{t-\tau(t)}^{t} r(\theta)d\theta > (1-c)e^{-1}$, *then every solution of Equation* (1.7) *is oscillatory.*

This result is clearly more general then part (iii) of Lemma 3.2, but still can be viewed as a special case of a more general result to be proved in the next section. Thus, we omit the proof here.

THEOREM 4.2. *Suppose* $r(t)$ *is bounded above, and there is a* $t_1 > 0$, *such that*

$$\int_{t-\tau(t)}^{t} r(\theta)d\theta \le \frac{1-c}{e} \qquad \textit{for all } t \ge t_1 .$$

Then Equation (1.7) *has a bounded nonoscillatory solution on* $[T, \infty)$ *for some* $T > 0$.

PROOF. Assume $|r(t)| \le M$, for all $t \ge 0$. Let $C[0, \infty)$ denote the space of all bounded continuous functions defined on $[0, \infty)$, with the norm defined as $|\phi| = \sup\{|\phi(\theta)|, \theta \in [0, \infty)\}$, for $\phi \in C[0, \infty)$. Let

$$S = \left\{ y \in C[0, \infty) \;\middle|\; \begin{array}{l} y(t) \text{ is lipschitzian and nonincreasing on } [0, +\infty), \\ \operatorname{lip} y(t)|_{[0, +\infty)} \le Me(1-a)/(1-c) . \\ y(t) \equiv 1-a , \quad t \in [0, t_1] , \\ ey(t-\tau(t)) \ge y(t)e \ge y(t-\tau(t)) , \quad t \ge t_1 . \\ (1-a)\exp\left(-\dfrac{e}{1-c} \displaystyle\int_{t_1}^{t} r(\theta)d\theta \right) \le y(t) \le 1-\alpha , \quad t \ge t_1 , \\ |y'(t)| \le Me\dfrac{1-a}{1-c} , \quad t \in [0, +\infty) , \quad t \ge t_1 , \end{array} \right\}$$

where $1 > a > 0$ is a constant. Denote

$$y_0(t) = \begin{cases} 1-a & t \in [0, t_1] \\ (1-a)\exp\left(-\dfrac{e}{1-c} \displaystyle\int_{t_1}^{t} r(\theta)d\theta \right) & t \ge t_1 . \end{cases}$$

Then $y_0(t) \in S$ and S is nonempty. It is easy to show that S is convex and compact.

Now, we define an operator F: $S \to C[0, +\infty)$ as

$$F(y)(t) = \begin{cases} 1-a , & t \in [0, t_1] \\ (1-a)\exp\left(\dfrac{-1}{1-c} \displaystyle\int_{t_1}^{t} r(\theta)(1-y(\theta))(1-cy(\theta)) \frac{y(\theta-\tau(\theta))}{y(\theta)} d\theta \right) , & t \ge t_1 . \end{cases}$$

Obviously, $F(y)(t) \le 1-a$, $t \ge 0$ when $y \in S$, and when $t \ge t_1$, $F(y)(t) \ge (1-a)\exp(-e/(1-c)\int_{t_1}^{t} r(\theta)d\theta)$. Thus we have

$$\begin{aligned} \frac{F(y)(t)}{F(y)(t-\tau(t))} &= \exp\left(\frac{-1}{1-c} \int_{t-\tau(t)}^{t} r(\theta)(1-y(\theta))(1-cy(\theta)) \frac{y(\theta-\tau(\theta))}{y(\theta)} d\theta \right) \\ &\ge \exp\left(\frac{-e}{1-c} \int_{t-\tau(t)}^{t} r(\theta)d\theta \right) \ge e^{-1} . \end{aligned}$$

Clearly, $F(y)(t) \le F(y)(t-\tau(t))$, and $|(F(y)(t))'| \le (1-a)r(t)e/(1-c) \le (1-a)Me/(1-c)$.

Therefore, we have shown that $FS \subset S$. It is easy to see that the operator F is completely continuous. Hence, by the well-known Schauder-Tychonov fixed-point theorem, we conclude that F has a fixed point in S. That is, there is a $y^* \in S$, $y^*(t) = F(y^*)(t)$ for all $t \geq 0$. Choose $T > t_1$. Then $y^*(t)$ is differentiable for $t \geq T$. By differentiating both sides of $y^*(t) = F(y^*)(t)$ for $t \geq T$, we see $y^*(t)$ satisfies Equation (1.7) for $t \geq T$. Obviously, $y^*(t)$ is positive, bounded and nonoscillatory. This proves the theorem.

COROLLARY 4.1. *If $r(t) = r$, $\tau(t) = \tau$ where r and τ are positive constants, then all solutions of* (1.7) *are oscillatory if and only if $r\tau e > 1 - c$.*

PROOF. This is obvious from Theorem 4.1 and Theorem 4.2. ∎

In the real system, we should expect that all those parameters that appeared in equation (1.2) are time dependent. For this reason, we may replace x_m by $K(t)$, and x'_m by $K(t)c^{-1}$. This results in the following equation

$$\frac{dx(t)}{dt} = r(t)x(t)\frac{K(t) - x(t - \tau(t))}{K(t) - cx(t - \tau(t))}. \tag{4.1}$$

For convenience, we assume $\tau(t) = \tau > 0$ in the following theorem. Naturally we assume $K(t) > cx(t - \tau)$, for $0 \leq t \leq \tau$.

THEOREM 4.3. *In Equation* (4.1), *we assume $K(t)$ is a nonconstant positive continuous periodic function of period τ and $\liminf_{t \to \infty} r(t) > 0$. Then all global solutions of* (4.1) *are oscillatory with respect to $K(t)$, i.e., there is a positive sequence $\{t_n\}$, $\lim_{n \to \infty} t_n = +\infty$, such that $x(t_n) = K(t_n)$.*

PROOF. Otherwise, there is a global solution $x(t)$ of (4.1), $x(t) > K(t)$ or $x(t) < K(t)$ for all large t. We may assume first that $x(t) > K(t)$ for $t \geq t^* > 0$. Since $x(t)$ is global, we see $K(t) - cx(t - \tau) > 0$ for all $t \geq 0$, and $\dot{x}(t) < 0$, for $t \geq t^*$. Thus there is a $\beta > 0$, such that $\lim_{t \to +\infty} x(t) = \beta$. Denote

$$a = \min\{K(t) \colon 0 \leq t \leq \tau\} > 0, \quad b = \max\{K(t) \colon 0 \leq t \leq \tau\} > 0.$$

Then $\beta \geq b$. From (4.1), we have

$$x(t) = x(t^*)\exp\left(\int_{t^*}^{t} \frac{K(s) - x(s - \tau)}{K(s) - cx(s - \tau)} ds\right), \tag{4.2}$$

which implies

$$x(t) < x(t^*)\exp\left(\frac{1}{b}\int_{t^*}^{t} (K(s) - \beta) ds\right). \tag{4.3}$$

Since $b > a$, we see

$$\lim_{t \to +\infty} \int_{t^*}^{t} (K(s) - \beta) ds = -\infty. \tag{4.4}$$

Thus

$$\lim_{t\to+\infty} x(t)\le 0 . \tag{4.5}$$

This obviously contradicts $\beta\ge b>0$.

Similarly, we can show that there is no $x(t)$ such that $x(t)<K(t)$ for all large time t. This proves the theorem.

5. Oscillatory results when $r(t)$ is arbitrary. As we mentioned before, when $r(t)$ is viewed as the difference of the growth rate and the death rate, then $r(t)$ may not always be positive. Denote

$$r^+(t)=\max(r(t),0)\,,\quad r^-(t)=\max(-r(t),0)\,. \tag{5.1}$$

LEMMA 5.1. *Suppose $\tau(t)$ is positive and continuous,* $\lim_{t\to\infty}(t-\tau(t))=+\infty$, *and*

$$\int_0^\infty r^-(t)dt<\infty\,,\quad \int_0^\infty r^+(t)dt=+\infty\,. \tag{5.2}$$

Then all positive nonoscillatory solutions of Equation (1.7) *bounded by* 1 *tend to zero as* $t\to\infty$.

PROOF. Let $y(t)$ be a bounded positive nonoscillatory solution of Equation (1.7). Suppose $y(t)>0$, $y(t-\tau(t))>0$, for $t\ge T$. Let $t_2>t_1\ge T$. Then

$$\begin{aligned} y(t_2)-y(t_1)=&\frac{-1}{1-c}\int_{t_1}^{t_2} r^+(\theta)(1-y(\theta))(1-cy(\theta))y(\theta-\tau(\theta))d\theta \\ &+\frac{1}{1-c}\int_{t_1}^{t_2} r^-(\theta)(1-y(\theta))(1-cy(\theta))y(\theta-\tau(\theta))d\theta\,. \end{aligned} \tag{5.3}$$

The first integral is negative and decreasing with respect to t_2, while the second one is nonnegative and increasing. Since $\int_0^\infty r^-(t)dt<\infty$, we see the second integral converges as $t_2\to+\infty$. Together with the boundedness of $y(t)$, we see that the first integral must converge as $t_2\to+\infty$. Thus there is an α, $0\le\alpha\le 1$, such that

$$\lim_{t\to+\infty} y(t)=\alpha\,.$$

If $\alpha=1$, then (1.6) implies $\lim_{t\to+\infty}u(t)=0$ in Equation (1.3). Thus there is a $T>0$, when $t\ge T$, $[1-u(t-\tau(t))]/[1-cu(t-\tau(t))]\ge 1/2$. Hence

$$\frac{1}{u(t)}\frac{du(t)}{dt}\ge\frac{1}{2}(r^+(t)-r^-(t))\,. \tag{5.4}$$

Integrating both sides of (5.4) yields

$$\ln\frac{u(t)}{u(T)}\ge\frac{1}{2}\left(\int_T^t r^+(\theta)d\theta-\int_T^t r^-(\theta)d\theta\right). \tag{5.5}$$

Letting $t \to +\infty$ in (5.5), we obtain a contradiction.

If $0<\alpha<1$, then there exists ε, $0<\varepsilon<\min\{\alpha, 1-\alpha\}$, and $T_1 \geq T$ such that when $t \geq T_1$,

$$0<\alpha-\varepsilon<y(t-\tau(t))<\alpha+\varepsilon .$$

Hence, when $t \geq T_1$,

$$-\frac{dy(t)}{dt} \geq \frac{1}{1-c}[1-(\alpha+\varepsilon)][1-c(\alpha+\varepsilon)](\alpha-\varepsilon)r^+(t) - \frac{1}{1-c}[1-(\alpha-\varepsilon)][1-c(\alpha-\varepsilon)](\alpha+\varepsilon)r^-(t) . \tag{5.6}$$

Integrating (5.6) leads to

$$m_1 \int_{T_1}^t r^+(\theta)d\theta \leq y(T_1)-y(t)+m_2 \int_{T_1}^t r^-(\theta)d\theta , \tag{5.7}$$

where

$$m_1 = \frac{1}{1-c}[1-(\alpha+\varepsilon)][1-c(\alpha+\varepsilon)](\alpha-\varepsilon)>0 ,$$

$$m_2 = \frac{1}{1-c}[1-(\alpha-\varepsilon)][1-c(\alpha-\varepsilon)](\alpha+\varepsilon)>0 .$$

In (5.7), letting $t \to +\infty$, we obtain $\int_{T_1}^{\infty} r^+(\theta)d\theta < +\infty$, a contradiction to our assumption that $\int_0^{\infty} r^+(\theta)d\theta = +\infty$. Thus $\alpha=0$. This proves the lemma.

THEOREM 5.1. *In Equation* (1.7), *let* $h(t)=t-\tau(t)$, *and choose* $0<\varepsilon<1$, $d>0$, *such that* $\mu=d(1-\varepsilon)(1-c\varepsilon)/(1-c)>e^{-1}$. *Let N be an integer greater than* $2(\ln 2-\ln \mu)(1+\ln \mu)^{-1}$. *Assume*:

(H1) $r(t)$, $\tau(t)$ *are continuous on* $[0, \infty)$ *and* $\int_0^{\infty} r^-(\theta)d\theta<\infty$;

(H2) $\tau(t)>0$, $h(t)$ *is nondecreasing*, $\lim_{t\to+\infty} h(t)=+\infty$.

(H3) *There is a sequence* t_n, $\lim_{n\to\infty} t_n=+\infty$, *such that*: (i) $r(t) \geq 0$ *for* $t \in E_n^{N+1}$, (ii) $\int_{h(t)}^t r(\theta)d\theta>d>(1-c)/e$, *for* $t \in E_n^N$, *where* $n=1, 2, \cdots$, $h^N(t)= h(h(\cdots h(t)\cdots))$, *the n-th composition of* $h(t)$, $E_n^N=[h^N(t_n), t_n]$.

Then all solutions of (1.7) *are oscillatory.*

PROOF. We assume first that (1.7) has an eventually positive solution $y(t)$. From Lemma 5.1, we know $\lim_{t\to+\infty} y(t)=0$.

When $t \in E_n^{N+1}$, $r(t) \geq 0$, thus $\dot{y}(t) \leq 0$. For the given ε, there is a $T_2>0$ such that $y(t)<\varepsilon$ when $t \geq T_2$. By (H2), there exists n_0 such that $h(t) \geq T_2$ for $t \in E_{n_0}^N$. Thus, for $t \in E_{n_0}^N$,

$$-\int_{h(t)}^t \frac{dy(\theta)}{y(\theta)} = \frac{1}{1-c}\int_{h(t)}^t r(\theta)(1-y(\theta))(1-cy(\theta))\frac{y(h(\theta))}{y(\theta)}d\theta$$

$$\geq \frac{(1-\varepsilon)(1-c\varepsilon)}{1-c}\int_{h(t)}^{t} r(\theta)d\theta \geq \mu > e^{-1},$$

which implies

$$y(h(t))/y(t) \geq e^{\mu} \geq e\mu > 1, \qquad \text{for} \quad t \in E_{n_0}^{N}.$$

For $t \in E_{n_0}^{N-1}$,

$$-\int_{h(t)}^{t} \frac{dy(\theta)}{y(\theta)} = \frac{1}{1-c}\int_{h(t)}^{t} r(\theta)(1-y(\theta))(1-cy(\theta))\frac{y(h(\theta))}{y(\theta)}d\theta$$

$$\geq \frac{e\mu}{1-c}(1-\varepsilon)(1-c\varepsilon)\int_{h(t)}^{t} r(\theta)d\theta \geq e\mu^2,$$

where we used the fact that $E_{n_0}^{N-1} \subset E_{n_0}^{N}$. Thus

$$y(h(t))/y(t) \geq e^{e\mu^2} \geq (e\mu)^2 \qquad \text{for} \quad t \in E_{n_0}^{N-1}.$$

By repeating the above argument, we obtain

$$y(h(t))/y(t) \geq (e\mu)^N \qquad \text{for} \quad t \in E_{n_0}^{1}. \tag{5.8}$$

From (H3), we have

$$\int_{h(t_{n_0})}^{t_{n_0}} r(\theta)d\theta > d > \frac{1-c}{e},$$

hence, there is a $t_{n_0}^* \in (h(t_{n_0}), t_{n_0}) = E_{n_0}^{1}$ such that

$$\int_{h(t_{n_0})}^{t_{n_0}^*} r(\theta)d\theta \geq \frac{d}{2}, \quad \int_{t_{n_0}^*}^{t_{n_0}} r(\theta)d\theta \geq \frac{d}{2}.$$

Integrating (1.7) from $h(t_{n_0})$ to $t_{n_0}^*$ yields

$$y(t_{n_0}^*) - y(h(t_{n_0})) = \frac{-1}{1-c}\int_{h(t_{n_0})}^{t_{n_0}^*} r(\theta)(1-y(\theta))(1-cy(\theta))y(h(\theta))d\theta$$

$$\leq -\frac{(1-\varepsilon)(1-c\varepsilon)}{1-c}y(h(t_{n_0}^*))\int_{h(t_{n_0})}^{t_{n_0}^*} r(\theta)d\theta \leq -\frac{\mu}{2}y(h(t_{n_0}^*)).$$

Thus

$$y(h(t_{n_0})) \geq \frac{\mu}{2}y(h(t_{n_0}^*)).$$

Similarly, we have

$$y(t_{n_0}^*) \geq \frac{\mu}{2}y(h(t_{n_0})).$$

Therefore

$$y(h(t^*_{n_0}))/y(t^*_{n_0}) \leq 4/\mu^2 . \tag{5.9}$$

By combining (5.8) and (5.9), we obtain

$$(e\mu)^N \leq 4/\mu^2 . \tag{5.10}$$

This implies

$$N \leq \frac{2(\ln 2 - \ln \mu)}{1 + \ln \mu}, \tag{5.11}$$

which contradicts our assumption on N. This proves that (1.7) cannot have an eventually positive solution.

Similarly, we can show that (1.7) has no eventually negative solution. We omit the details to avoid repetition. ■

REMARK 5.1. Obviously, the proof of the above theorem implies that Theorem 4.1 is true.

REMARK 5.2. Assume (1.7) has $y(t)$ as its eventually negative solution. Without loss of generality, we assume $y(t) < 0$ for $t \geq 0$. Then

$$\begin{aligned} \dot{y}(t) &= -\frac{r^+(t)}{1-c}(1-y(t))(1-cy(t))y(1-\tau(t)) + \frac{r^-(t)}{1-c}(1-y(t))(1-cy(t))y(t-\tau(t)) \\ &\geq -\frac{r^+(t)}{1-c}y(t-\tau(t)) + \frac{r^-(t)}{1-c}(1-y(t))(1-cy(t))y(t-\tau(t)) . \end{aligned} \tag{5.12}$$

We may rewrite (5.12) as

$$\dot{y}(t) + p(t)y(t-\tau(t)) \geq 0 , \tag{5.13}$$

where

$$p(t) = \frac{1}{1-c} r^+(t) - \frac{r^-(t)}{1-c}(1-y(t))(1-cy(t)) .$$

Thus

$$p^+(t) = \frac{1}{1-c} r^+(t) .$$

Under the assumptions of Theorem 5.1, the proof of the nonexistence of an eventually negative solution of the differential inequality (5.13) is essentially contained in the proof of Theorem 2.1 in Erbe and Zhang [11].

6. Global stability. Our objective in this section is to derive sufficient conditions for global solutions of Equation (1.3) with initial conditions satisfying (1.5) tend to its unique positive equilibrium $u(t)\equiv 1$. In the following, $u(t)$ denotes a solution of (1.3) and (1.5).

THEOREM 6.1. *We consider Equation* (1.3) *with initial condition* (1.5). *Assume*

(i) *$r(t)$ and $\tau(t)$ are positive and continuous, $t-\tau(t)$ is increasing,*

$$\lim_{t\to+\infty}(t-\tau(t))=+\infty \quad \text{and} \quad \int_0^\infty r(\theta)d\theta=+\infty\,; \tag{6.1}$$

(ii) *there is a $t_1>0$, such that for $t\geq t_1$,*

$$0<\int_{t-\tau(t)}^{t} r(\theta)d\theta\leq\sigma\,, \quad \text{and} \quad e^{\sigma}<c^{-1}\,; \tag{6.2}$$

(iii)

$$\sigma\leq 1-ce^{\sigma}\,. \tag{6.3}$$

Then, every global solution of (1.3) *and* (1.5) *tends to* 1 *as $t\to+\infty$. In particular, if $\phi(0)<c^{-1}e^{-\sigma}$, then $\lim_{t\to+\infty}u(t)=1$.*

PROOF. Clearly, all conditions of Theorem 2.3 are satisfied. Thus, for every oscillatory solution $u(t)$ (with respect to 1), we have

$$u(t)<e^{\sigma}\,, \quad \text{for} \quad t\geq\bar{t}+t_1\,, \quad \text{where} \quad \bar{t}=\tau(\bar{t})\,. \tag{6.4}$$

From (2.3), we know that if $u(t)$ is nonoscillatory with respect to 1, then $\lim_{t\to+\infty}u(t)=1$. Thus, in the following we always assume $u(t)$ is oscillatory with respect to 1.

Let

$$v(t)=u(t)-1\,. \tag{6.5}$$

Then (1.3) reduces to

$$\dot{v}(t)=-r(t)(1+v(t))\frac{v(t-\tau(t))}{1-c-cv(t-\tau(t))}\,. \tag{6.6}$$

Thus, $u(t)$ is oscillatory with respect to 1 if and only if $v(t)$ is oscillatory. Denote

$$p=\limsup_{t\to+\infty}v(t)\,, \quad q=-\liminf_{t\to+\infty}v(t)\,. \tag{6.7}$$

Then, we have $0\leq p<e^{\sigma}-1$, $0\leq q\leq 1$.

Let ε be a small positive constant such that $1-c-c(p+\varepsilon)>0$, and choose $t_2(\varepsilon)\geq\bar{t}+t_1$ such that for $t\geq t_2(\varepsilon)$

$$-q-\varepsilon<v(t)<p+\varepsilon\,. \tag{6.8}$$

Assume $v(t^*)$ is a maximum or a minimum such that $t^*-\tau(t^*)-\tau(t^*-\tau(t^*))\geq t_2(\varepsilon)$. Then

$v'(t^*)=0$, which implies $v(t^*-\tau(t^*))=0$. Thus

$$\ln(1+v(t^*))=-\int_{t^*-\tau(t^*)}^{t^*}\frac{r(t)v(t-\tau(t))}{1-c-cv(t-\tau(t))}dt\,. \tag{6.9}$$

Since $t^*-\tau(t^*)-\tau(t^*-\tau(t^*))\geq\bar{t}+t_1$, we have for $t\in[t^*-\tau(t^*),t^*]$,

$$[1-c-cv(t-\tau(t))]^{-1}<[1-ce^{\sigma}]^{-1}\,, \tag{6.10}$$

and

$$-v(t-\tau(t))<q+\varepsilon\,. \tag{6.11}$$

Hence, from (6.9), we obtain

$$\ln(1+v(t^*))<\frac{q+\varepsilon}{1-ce^{\sigma}}\int_{t^*-\tau(t^*)}^{t^*}r(t)dt\leq\frac{\sigma}{1-ce^{\sigma}}(q+\varepsilon)\leq q+\varepsilon. \tag{6.12}$$

That is

$$v(t^*)<e^{q+\varepsilon}-1\,. \tag{6.13}$$

Similarly, we have

$$v(t^*)>-1+e^{-(p+\varepsilon)}\,. \tag{6.14}$$

By the definition of p and q, we see that there always exist $t_3>t_2(\varepsilon)$, $t_4>t_2(\varepsilon)$ such that

$$v(t_3)>p-\varepsilon\,,\quad v(t_4)<-q+\varepsilon\,. \tag{6.15}$$

Therefore

$$p-\varepsilon<e^{q+\varepsilon}-1\,,\quad q-\varepsilon<1-e^{-(p+\varepsilon)}\,. \tag{6.16}$$

By letting $\varepsilon\to 0$, we obtain

$$p\leq e^{q}-1\,,\quad q\leq 1-e^{-p}\,. \tag{6.17}$$

From (6.17), we see that $p=0$ if and only if $q=0$. Thus we may suppose that

$$p>0\,,\quad 0<q<1\,. \tag{6.18}$$

Clearly, (6.17) leads to

$$1+p\leq e^{q}\leq\exp(1-e^{-p})\,. \tag{6.19}$$

But, since $p>0$, we have

$$1+p-\exp(1-e^{-p})=\int_0^p\int_0^{p_1}(1-e^{-p_2})\exp(1-e^{-p_2}-p_2)dp_2dp_1>0\,,$$

a contradiction to (6.19). This proves the first conclusion of the theorem.

If $\phi(0)<c^{-1}e^{-\sigma}$, then Theorem 2.1 implies that $u(t)$ is global. Thus, by the first

conclusion of the theorem, we have

$$\lim_{t \to +\infty} u(t) = 1 ,$$

proving the theorem.

7. Discussion. Our main finding is that the introduction of delay in Equation (1.2) can destabilize a locally stable steady state, thus causing the solution to oscillate. For large time, together with large intrinsic growth late and small parameter c, the autonomous version of Equation (1.2) may have periodic solutions. This suggests that small growth rate and small delay are essential for the equation to be stable. Large growth rate and/or long time delay may result in the destabilization of the ecological system, thus rendering the system out of control and the prediction hard to make.

Our results on the existence of periodic solutions are similar to the ones obtained by Jones [12]. The present work is distinguished from previous work principally by the fact that the solutions of the equation may not exist for all $t \geq 0$. Thus the bound estimation of the solutions under certain conditions become important. This was accomplished in Section 2.

We note that our sufficient conditions for the oscillation and nonoscillation of Equation (1.7) are sharp in the following sense: if $c = 0$ then (1.7) reduces to

$$\dot{y}(t) = -r(t)[1 + y(t)]y(t - \tau(t)) , \tag{7.1}$$

an equation discussed in detail in Zhang and Gopalsamy [14]. By taking $c = 0$, all our results coincide with those obtained in [14]. Clearly, our work extends many previous ones on the autonomous versions of Equation (1.7), e.g., the work of Wright [15], Kakutani and Markus [16], and some others contained in the recent monograph of Ladde, Lakshmikantham and Zhang [13].

Our analysis of Equation (1.3) can serve as a stepping stone for future works on two-interacting population models, assuming each of the species' growth is governed by Equation (1.3).

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