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A RIGIDITY FOR REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

Dedicated to Professor Tadashi Nagano on his sixtieth birthday

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Abstract. The purpose of this paper is to give a rigidity theorem for real hypersurfaces in $P_n(C)$ satisfying a certain geometric condition.

Introduction. Let $P_n(C)$ denote an $n(\geq 2)$-dimensional complex projective space with the metric of constant holomorphic sectional curvature $4c$.

We proved in [4] that two isometric immersions of a $(2n-1)$-dimensional Riemannian manifold M into $P_n(C)$ are congruent if their second fundamental forms coincide. In general, the type number is defined as the rank of the second fundamental form. In this paper we shall give another rigidity theorem of the same type:

THEOREM A. *Let M be a $(2n-1)$-dimensional Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(C)$ $(n\geq 3)$. Assume that ι and $\hat{\iota}$ have a principal direction in common at each point of M, and that the type number of (M, ι) or $(M, \hat{\iota})$ is not equal to 2 at each point of M. Then ι and $\hat{\iota}$ are congruent, that is, there is a unique isometry φ of $P_n(C)$ such that $\varphi\circ\iota=\hat{\iota}$.*

We shall say that an isometry φ of a real hypersurface M in $P_n(C)$ is *principal* if for each point p of M there exists a principal vector v at p such that the vector $\varphi_*(v)$ is also principal at $\varphi(p)$, where φ_* denotes the differential of φ at p. Then as an application of Theorem A we have:

THEOREM B. *Let M be a homogeneous real hypersurface in $P_n(C)$ $(n\geq 3)$. Assume that each isometry of M is principal. Then M is an orbit under an analytic subgroup of the projective unitary group $PU(n+1)$.*

Note that all orbits in $P_n(C)$ under analytic subgroups of the projective unitary group $PU(n+1)$ are completely classified in [4].

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1. **Preliminaries.** Let M be a $(2n-1)$-dimensional Riemannian manifold, and ι be

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an isometric immersion of M into $P_n(C)$. In this section, let the indices i, j, k, l run from 1 through $2n-1$. Choose a field $\{e_1, \dots, e_{2n-1}\}$ of local orthonormal frame on M, and denote its dual 1-forms by θ_i. Then the connection forms θ_{ij} and the curvature forms Θ_{ij} are defined by

$$\theta_{ij}+\theta_{ji}=0 \quad \text{and} \quad d\theta_i+\sum_j \theta_{ij}\wedge\theta_j=0\,, \tag{1.1}$$

$$\Theta_{ij}=d\theta_{ij}+\sum_k \theta_{ik}\wedge\theta_{kj}\,, \tag{1.2}$$

respectively. We denote the second fundamental tensor of (M, ι) by (H_{ij}), and put $\phi_i=\sum_j H_{ij}\theta_j$. Moreover, we denote the almost contact structure of (M, ι) by (J_{ij}, f_k). Then we have the equations of Gauss and Codazzi,

$$\Theta_{ij}=\phi_i\wedge\phi_j+c\theta_i\wedge\theta_j+c\sum_{k,l}(J_{ik}J_{jl}+J_{ij}J_{kl})\theta_k\wedge\theta_l\,, \tag{1.3}$$

$$d\phi_i+\sum\phi_j\wedge\theta_{ji}=c\sum_{j,k}(f_jJ_{ik}+f_iJ_{jk})\theta_j\wedge\theta_k\,. \tag{1.4}$$

The three tensors $H=(H_{ij})$, $J=(J_{ij})$ and $f=(f_i)$ satisfy

$$H_{ij}=H_{ji}\,, \qquad J_{ij}=-J_{ji}\,, \tag{1.5}$$

$$\sum_k J_{ik}J_{kj}-f_if_j=-\delta_{ij}\,, \quad \sum_j J_{ij}f_j=0\,, \quad \sum_i f_i^2=1\,, \tag{1.6}$$

$$dJ_{ij}=\sum_k J_{ik}\theta_{kj}-\sum_k J_{kj}\theta_{ik}-f_i\phi_j+f_j\phi_i\,, \tag{1.7}$$

$$df_i=\sum_j f_j\theta_{ji}-\sum_j J_{ji}\phi_j\,. \tag{1.8}$$

We denote by t the rank of the matrix (H_{ij}), which is called the *type number* of (M, ι).

For another isometric immersion $\hat{\iota}$ of M into $P_n(C)$ we shall denote the differential forms and tensor fields of $(M, \hat{\iota})$ by the same symbol but with a hat.

2. Key lemmas. Let M be a $(2n-1)$-dimensional Riemannian manifold, and ι, $\hat{\iota}$ be two isometric immersions into the complex projective space $P_n(C)$. In the remainder of this paper, the index α stands for the special index 1 to avoid confusion, and the indices i, j, k, l run from 2 through $2n-1$, unless otherwise stated.

In this section, we assume that at each point of M, ι and $\hat{\iota}$ have a *principal direction in common*. Then we can set $\phi_\alpha=\lambda_\alpha\theta_\alpha$ and $\hat{\phi}_\alpha=\hat{\lambda}_\alpha\theta_\alpha$.

LEMMA 2.1. $\quad J_{\alpha i}J_{jk}=\hat{J}_{\alpha i}\hat{J}_{jk}\,.$

PROOF. From (1.3) we have

$$\phi_\alpha \wedge \phi_i + c\sum(J_{\alpha j}J_{ik} + J_{\alpha i}J_{jk})\theta_j \wedge \theta_k = \hat\phi_\alpha \wedge \hat\phi_i + c\sum(\hat J_{\alpha j}\hat J_{ik} + \hat J_{\alpha i}\hat J_{jk})\theta_j \wedge \theta_k .$$

Taking account of the coefficients of $\theta_j \wedge \theta_k$, we have

$$J_{\alpha j}J_{ik} - J_{\alpha k}J_{ij} + 2J_{\alpha i}J_{jk} = \hat J_{\alpha j}\hat J_{ik} - \hat J_{\alpha k}\hat J_{ij} + 2\hat J_{\alpha i}\hat J_{jk} . \tag{2.1}$$

Putting $j=i$ in (2.1), we have

$$J_{\alpha i}J_{ij} = \hat J_{\alpha i}\hat J_{ij} . \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$(\hat J_{\alpha j}J_{\alpha i} - \hat J_{\alpha i}J_{\alpha j})J_{jk} + (-\hat J_{\alpha j}J_{\alpha k} + \hat J_{\alpha k}J_{\alpha j})J_{ji} + 2(J_{\alpha j}J_{ik} - \hat J_{\alpha j}\hat J_{ik})\hat J_{\alpha j} = 0 , \tag{2.3}$$

$$(J_{\alpha j}J_{ik} - \hat J_{\alpha j}\hat J_{ik})\hat J_{\alpha j} + (\hat J_{\alpha k}J_{\alpha j} - \hat J_{\alpha j}J_{\alpha k})J_{ij} + 2(\hat J_{\alpha j}J_{\alpha i} - \hat J_{\alpha i}J_{\alpha j})J_{jk} = 0 . \tag{2.4}$$

Adding (2.3) to (2.4), we have

$$(J_{\alpha j}J_{ik} - \hat J_{\alpha j}\hat J_{ik})\hat J_{\alpha j} + (\hat J_{\alpha j}J_{\alpha i} - \hat J_{\alpha i}J_{\alpha j})J_{jk} = 0 . \tag{2.5}$$

Exchanging the role of J and $\hat J$, we have

$$(J_{\alpha j}J_{ik} - \hat J_{\alpha j}\hat J_{ik})J_{\alpha j} + (\hat J_{\alpha j}J_{\alpha i} - \hat J_{\alpha i}J_{\alpha j})\hat J_{jk} = 0 . \tag{2.6}$$

Multiplying (2.5) by $J_{\alpha k}$, (2.6) by $\hat J_{\alpha k}$, and then taking their difference, we find

$$(\hat J_{\alpha j}J_{\alpha k} - \hat J_{\alpha k}J_{\alpha j})(J_{\alpha j}J_{ik} - \hat J_{\alpha j}\hat J_{ik}) = 0 ,$$

and hence

$$(\hat J_{\alpha j}J_{\alpha i} - \hat J_{\alpha i}J_{\alpha j})(J_{\alpha j}J_{ik} - \hat J_{\alpha j}\hat J_{ik}) = 0 . \tag{2.7}$$

If there were indices i, j, k such that

$$J_{\alpha j}J_{ik} - \hat J_{\alpha j}\hat J_{ik} \neq 0 , \tag{2.8}$$

then from (2.7) we have $\hat J_{\alpha j}J_{\alpha i} - \hat J_{\alpha i}J_{\alpha j} = 0$. This, together with (2.5) and (2.6), implies $\hat J_{\alpha j} = 0$ and $J_{\alpha j} = 0$, which contradicts (2.8). q.e.d.

LEMMA 2.2. $J = \pm \hat J$.

PROOF. We need to consider three cases.

Case I: $\hat J_{\alpha i} \neq 0$ for some i. Put $\varepsilon = J_{\alpha i}/\hat J_{\alpha i}$. Then from Lemma 2.1 we have

$$\hat J_{ij} = \varepsilon J_{ij} . \tag{2.9}$$

Since $n \geq 3$ and rank $\hat J = 2n-2$, we have $\varepsilon \neq 0$ and so

$$\hat J_{\alpha i} = \frac{1}{\varepsilon} J_{\alpha i} \qquad \text{for all} \quad i . \tag{2.10}$$

From (1.6) we have

(2.11) $$\varepsilon^2\sum_j J_{ij}^2+\frac{1}{\varepsilon^2}J_{\alpha i}^2+\hat{f}_i^2=1\ ,$$

(2.12) $$\sum_j J_{ij}^2+J_{\alpha i}^2+f_i^2=1\ ,$$

(2.13) $$\frac{1}{\varepsilon^2}\sum_i J_{\alpha i}^2+\hat{f}_\alpha^2=1\ ,$$

(2.14) $$\sum_i J_{\alpha i}^2+f_\alpha^2=1\ .$$

Now we put $a=\sum f_i^2=1-f_\alpha^2$ and $\hat{a}=\sum \hat{f}_i^2=1-\hat{f}_\alpha^2$. Then it follows from (2.13) and (2.14) that

(2.15) $$a=\varepsilon^2\hat{a}\ .$$

On the other hand, from (1.6) and (2.2) we have that

$$f_\alpha^2\sum_j f_j^2=\sum_j\Big(\sum_i J_{\alpha i}J_{ij}\Big)^2=\sum_j\Big(\sum_i \hat{J}_{\alpha i}\hat{J}_{ij}\Big)^2=\hat{f}_\alpha^2\sum_j \hat{f}_j^2\ ,$$

which means $(1-a)a=(1-\hat{a})\hat{a}$, and so $a=\hat{a}$ or $a+\hat{a}=1$.

From (2.11) and (2.12), we have

$$\left(\frac{1}{\varepsilon^2}-\varepsilon^2\right)J_{\alpha i}^2+\hat{f}_i^2-\varepsilon^2 f_i^2=1-\varepsilon^2\ .$$

This and (2.15) imply

(2.16) $$\hat{a}-\varepsilon^2 a=(n-1)(1-\varepsilon^2)\ .$$

Regardless of whether $a=\hat{a}$ or $a+\hat{a}=1$, from (2.15) and (2.16) we have $\varepsilon^2=1$ since $n\geq 3$, which shows $J=\pm\hat{J}$.

Case II: $\hat{J}_{\alpha i}=0$ for all i and $\lambda_\alpha^2+\hat{\lambda}_\alpha^2>0$. Then Lemma 2.1 gives $J_{\alpha i}=0$. It follows from the equation of Gauss (1.3) that

$$\phi_\alpha\wedge\phi_i=\hat{\phi}_\alpha\wedge\hat{\phi}_i\ ,$$

and so

(2.17) $$(\hat{\lambda}_\alpha\hat{\phi}_i-\lambda_\alpha\phi_i)\wedge\theta_\alpha=0\ .$$

Thus we can write

(2.18) $$\hat{\lambda}_\alpha\hat{\phi}_i-\lambda_\alpha\phi_i=c_i\theta_\alpha\ .$$

Again from the equation of Gauss (1.3) we have

(2.19) $$\phi_i\wedge\phi_j\equiv\hat{\phi}_i\wedge\hat{\phi}_j\ \ (\mathrm{mod}\ \theta_k\wedge\theta_l)\ .$$

Here we may set $\phi_i=\lambda_i\theta_i$. Then cancelling ϕ_i and ϕ_j from (2.18) and (2.19), we have

$$\hat{\lambda}_\alpha^2\lambda_i\lambda_j\theta_i\wedge\theta_j\equiv(\lambda_\alpha\lambda_i\theta_i+c_i\theta_\alpha)\wedge(\lambda_\alpha\lambda_j\theta_j+c_j\theta_\alpha)\pmod{\theta_k\wedge\theta_l}.$$

Taking account of the coefficients of $\theta_\alpha\wedge\theta_i$, we have

$$c_j\lambda_i\lambda_\alpha=0 \qquad \text{for} \quad i\neq j.$$

Since $\lambda_\alpha\neq 0$ or $\hat{\lambda}_\alpha\neq 0$, we may assume $\lambda_\alpha\neq 0$. Then we have $c_j\lambda_i=0$ for $i\neq j$. But it is known that in any non-empty open set U of M there exists a point p where rank $H\geq 2$ (cf. [4]). These facts imply that there exists an index i' such that $c_{i'}=0$, i.e., the vector $e_{i'}$ is a principal direction common to $\imath$ and $\hat{\imath}$. Now, the index i' can play the same role as α. Therefore, since $J_{i'j}\neq 0$ for some j, the present case have been reduced to Case I.

Case III: $\hat{J}_{\alpha i}=0$ for all i and $\lambda_\alpha=\hat{\lambda}_\alpha=0$. Then Lemma 2.1 gives $J_{\alpha i}=0$. It follows from (1.6) that $f_\alpha^2=1$ and $\hat{f}_\alpha^2=1$. We may set $f_\alpha=1$ and $\hat{f}_\alpha=1$.

Denote by K (resp. G) the matrix (H_{ij}) (resp. (J_{ij})) of degree $2n-2$. In such a situation we shall show:

(2.20) *The matrices K, $\hat{K}$, G and $\hat{G}$ are all non-singular.*

(2.21) $$GK=\hat{G}\hat{K} \quad \text{and} \quad KG=\hat{K}\hat{G}.$$

(2.22) $$KGK=cG \quad \text{and} \quad \hat{K}\hat{G}\hat{K}=c\hat{G}.$$

First, the matrices G and $\hat{G}$ are non-singular by (1.6) and $f_i=\hat{f}_i=0$. From $J_{\alpha i}=0$ and (1.7) we have $\phi_i=-\sum_j J_{ji}\theta_{\alpha j}$ or equivalently

(2.23) $$\theta_{\alpha i}=\sum_j \phi_j J_{ji}.$$

Similarly, we have $\theta_{\alpha i}=\sum_j\hat{\phi}_j\hat{J}_{ji}$. Thus these equations show (2.21).

On the other hand, since $J_{\alpha i}=0$, the equation of Codazzi (1.4) implies

(2.24) $$\sum_i\phi_i\wedge\theta_{i\alpha}=c\sum_{i,j}J_{ij}\theta_i\wedge\theta_j.$$

From (2.23) and (2.24) we have (2.22), which shows the non-singularity of K and $\hat{K}$. Thus our assertion was proved.

On the other hand, from the equation of Gauss (1.3) and the fact that $\Theta_{ij}=\hat{\Theta}_{ij}$ it follows that

(2.25) $$\begin{aligned}&H_{ik}H_{jl}-H_{il}H_{jk}+c(J_{ik}J_{jl}-J_{il}J_{jk}+2J_{ij}J_{kl})\\&\qquad=\hat{H}_{ik}\hat{H}_{jl}-\hat{H}_{il}\hat{H}_{jk}+c(\hat{J}_{ik}\hat{J}_{jl}-\hat{J}_{il}\hat{J}_{jk}+2\hat{J}_{ij}\hat{J}_{kl}).\end{aligned}$$

Multiplying (2.25) by J_{jk} and summing up over j and k, we have

(2.26) $$KGK+cG+c\langle G,G\rangle G=\hat{K}G\hat{K}-c\hat{G}G\hat{G}+c\langle G,\hat{G}\rangle\hat{G},$$

where we put $\langle G,G\rangle=\sum_{i,j}J_{ij}J_{ij}$ etc.

Multiply (2.26) by $K\hat{G}$ from the left. Then, since $K\hat{G}\hat{K}G\hat{K}=KGKG\hat{K}=cG^2\hat{K}=-c\hat{K}$ etc. by (2.21) and (2.22), we have

$$(2+\langle G, G\rangle)K\hat{G}G=-2\hat{K}-\langle G, \hat{G}\rangle K, \quad \text{and hence}$$

$$(2+\langle G, G\rangle)K\hat{G}=2\hat{K}G+\langle G, \hat{G}\rangle KG\,. \tag{2.27}$$

Exchanging the roles of ι and $\hat{\iota}$, we have

$$(2+\langle \hat{G}, \hat{G}\rangle)\hat{K}G=2K\hat{G}+\langle G, \hat{G}\rangle \hat{K}\hat{G}\,. \tag{2.28}$$

Subtracting (2.28) from (2.27), we have $K\hat{G}=\hat{K}G$. It follows from this and (2.27) that

$$\hat{G}=\varepsilon G\,,$$

where $\varepsilon=\langle G, \hat{G}\rangle/\langle G, G\rangle$. Consequently we have $\varepsilon^2=1$ since $\hat{G}^2=G^2=-I$. q.e.d.

3. The proof of the theorems. We adopt the notation in §1. From Lemma 2.2 and $\Theta_{ij}=\hat{\Theta}_{ij}$ we have

$$\phi_i\wedge\phi_j=\hat{\phi}_i\wedge\hat{\phi}_j\,. \tag{3.1}$$

Then, by a well-known lemma of E. Cartan [1], we have at each point of M,

$$\text{if } t\geq 3 \text{ or } \hat{t}\geq 3, \text{ then } \phi_i=\varepsilon\hat{\phi}_i \ (\varepsilon=\pm 1), \text{ for } i=1,\dots,2n-1, \tag{3.2}$$

$$t+\hat{t}\leq 1 \quad \text{or} \quad t=\hat{t}\,. \tag{3.3}$$

On the other hand, it is known that in any non-empty open subset of M there exists a point p such that $t(p)\geq 2$ (cf. [4]). Thus from (3.2) we have $H=\pm\hat{H}$ everywhere on M. Now Theorem A is reduced to a result in [4, Theorem 3.2]. q.e.d.

PROOF OF THEOREM B. Since M is complete, it follows from a theorem of the first author of the present paper [3] that there exists a point p_0 on M such that $t(p_0)\geq 3$. Let p be an arbitary point on M. Then, since M is homogeneous, there exists an isometry g of M such that $g(p_0)=p$. Since ι is principal by assumption, two isometric immersions ι and $\hat{\iota}=\iota\circ g$ of M into $P_n(C)$ have a *principal direction in common*. Then by Lemma 2.2 we have $J=\pm\hat{J}$. Hence from (3.3) we have $3\leq t(p_0)=\hat{t}(p_0)$.

Since the differential g_* of g is a linear isomorphism, we have

$$t(p)=t(g(p_0))=t(p_0)\,,$$

in particular, $t\geq 3$ on M. Now by Theorem A there exists a unique isometry φ_g of $P_n(C)$ such that $\varphi_g\circ\iota=\iota\circ g$, and $\iota(M)$ is just an orbit under the analytic subgroup $\{\varphi_g;\ g\in I(M)\}$ of $PU(n+1)$, where $I(M)$ denotes the group of all isometries of M.

q.e.d.

REMARK. The present authors think that Theorms A and B are also valid for complex hyperbolic spaces $H_n(C)$ with negative constant holomorphic sectional curvature $4c$, $c<0$. The details will be discussed in a forthcoming paper.

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