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CLASSICAL SCHOTTKY GROUPS OF REAL TYPE OF GENUS TWO, II

Dedicated to Professor Tatsuo Fujiie on his sixtieth birthday

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Abstract. We consider three types of Schottky spaces which consist of non-Fuchsian classical Schottky groups of real type of genus two. This paper has the following two aims: (1) to represent the shape of the spaces by using multipliers and cross ratios of the fixed points of two generators of marked Schottky groups; (2) to determine fundamental regions for the Schottky modular group of genus two acting on the spaces.

Introduction. In spite of works by Akaza, Bers, Brooks, Chuckrow, Marden, Maskit, Rodriguez, Sato, Zarrow, and others, much less are known on Schottky spaces and Schottky groups in comparison with Teichmüller spaces. For example, the shape of Schottky spaces is hardly known even in simple cases (cf. Keen [11], [12], Sato [28]). It is important to consider Schottky groups and Schottky spaces in the following simple cases: (1) classical Schottky groups and classical Schottky spaces (cf. Brooks [4], Jørgensen, Marden and Maskit [10], Marden [14], Phillips and Sarnak [20], Sato [28] and Zarrow [31]); (2) Schottky groups and Schottky spaces of genus two related to discrete two-generator groups (cf. Matelski [17], Maskit [16], Purzitsky [21], Rosenberger [24] and Sato [30]).

In this paper we will consider classical Schottky groups and classical Schottky spaces of real type of genus two as a sequel to our previous paper [28], in which we classified the groups and spaces into eight types, and considered the groups and spaces of the first and fourth types. Schottky groups of the first and fourth types are called Fuchsian Schottky groups. Conversely, a Fuchsian Schottky group of genus two is either a group of the first type or of the fourth type (Marden [14], Sato [30]). Schottky groups of these two types were studied by Rosenberger [24], Purzitski [21], Matelski [17] and others in connection with discrete two-generator groups.

As far as we know, Schottky groups and Schottky spaces of the other types are hardly studied. In this paper we will consider the groups and spaces of the second, fifth and seventh types, which are related to each other. This paper has the following two

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aims: (1) to represent the shape of the spaces of the second, fifth and seventh types by using the coordinates introduced in Sato [26] (Theorem 3); (2) to determine fundamental regions for the Schottky modular group of genus two acting on the above spaces (Theorem 4). Here the Schottky modular group is the group of all equivalence classes of orientation preserving automorphisms of the Schottky space (see §§1 and 7), which corresponds to the Teichmüller modular group. That is, the Schottky space modulo the Schottky modular group is the same space as the Teichmüller space modulo the Teichmüller modular group, and is called the Riemann space.

As applications of the previous paper [28] and this paper, we mention the following two results: (1) the non-classical Schottky group constructed by Zarrow [31] is a group of the second type. Namely, the group is a classical Schottky group but not a non-classical Schottky group (cf. Sato [29]); (2) On Jørgensen's inequality for purely hyperbolic groups (cf. Jørgensen [9], Martin [15]). We can show the following: Let $G=\langle A, B\rangle$ be a purely hyperbolic group generated by A and B. Then

$$(*) \qquad |\mathrm{tr}^2 A-4|+|\mathrm{tr}(ABA^{-1}B^{-1})-2|>4\,.$$

To be more precise, a purely hyperbolic two-generator group is either a group of the first or fourth type. If we denote by I the left hand side of $(*)$, then $I>16$ or $I>4$ according as G is of the first type or of the fourth type. Furthermore, both of the lower bounds are the best possible. This gives a complete answer to the problem on Jørgensen's inequality for purely hyperbolic two-generator groups studied by Gilman [6], [7]. The second result will appear elsewhere.

The second result will appear elsewhere. (Notes added on August 27, 1991: The same result with a different proof appeared recently in J. Gilman: A geometrie approach to Jørgensen's inequality, Adv. in Math. 85 (1991), 193–197.)

In §1 we will consider automorphisms of a free group on two generators and list properties of the automorphisms in a series of lemmas. In §2 we will consider the relationship among the spaces of the second, fifth and seventh types. In §3 we will introduce some surfaces and consider the relationship among them in §3 through §5. In §6 we will represent the shape of the classical Schottky spaces of the second, fifth and seventh types. In §7 we will determine fundamental regions for the Schottky modular group acting on the spaces of the above types. The references listed in the previous paper [28] are repeated here for the convenience of the reader.

Finally, we note the following: Schottky groups of real type are also studied in Bobenko [2] and Bobenko and Bordag [3]; our results in §3 are related with Gilman [8].

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1. Automorphisms of a free group on two generators.

1.1. In this section we will state some definitions and list properties of automorphisms of a free group on two generators. See the previous paper [28] for the definitions of Schottky groups, classical Schottky groups, Schottky spaces $\mathfrak{S}_g$ and classical Schottky spaces $\mathfrak{S}_g^0$.

Let Möb be the group of all Möbius transformations. We say two marked subgroups $G=\langle A_1, A_2\rangle$ and $\hat{G}=\langle \hat{A}_1, \hat{A}_2\rangle$ of Möb to be *equivalent* if there exists a Möbius transformation T such that $\hat{A}_j=TA_jT^{-1}$ for $j=1, 2$. We denote by $\mathfrak{M}_2$ the set of all equivalence classes $[\langle A_1, A_2\rangle]$ of marked groups $\langle A_1, A_2\rangle$ generated by loxodromic transformations A_1 and A_2 whose fixed points are all distinct.

Let $[\langle A_1, A_2\rangle]\in\mathfrak{M}_2$. For $j=1, 2$, let λ_j ($|\lambda_j|>1$), p_j and p_{2+j} be the multipliers, the repelling and the attracting fixed points of A_j, respectively. We define t_j by setting $t_j=1/\lambda_j$. Thus $t_j\in D^*=\{z\,|\,0<|z|<1\}$. We determine a Möbius transformation T by $T(p_1)=0$, $T(p_3)=\infty$ and $T(p_2)=1$, and define ρ by $\rho=T(p_4)$. Thus $\rho\in \boldsymbol{C}-\{0, 1\}$. We can define a mapping α of the space $\mathfrak{M}_2$ into $(D^*)^2\times(\boldsymbol{C}-\{0, 1\})$ by setting $\alpha([\langle A_1, A_2\rangle])=(t_1, t_2, \rho)$. Then we say $[\langle A_1, A_2\rangle]$ represents (t_1, t_2, ρ), and (t_1, t_2, ρ) corresponds to $[\langle A_1, A_2\rangle]$ or $\langle A_1, A_2\rangle$. Conversely, λ_1, λ_2 and p_4 are uniquely determined from a given point $\tau=(t_1, t_2, \rho)\in(D^*)^2\times(\boldsymbol{C}-\{0, 1\})$ under the normalization condition $p_1=0$, $p_3=\infty$ and $p_2=1$; we define λ_j ($j=1, 2$) and p_4 by setting $\lambda_j=1/t_j$ and $p_4=\rho$, respectively. We determine $A_1(z)$, $A_2(z)\in$ Möb from τ as follows: The multiplier, the repelling and the attracting fixed points of $A_j(z)$ are λ_j, p_j and p_{2+j}, respectively. Thus we obtain a mapping β of $(D^*)^2\times(\boldsymbol{C}-\{0, 1\})$ into $\mathfrak{M}_2$ by setting $\beta(\tau)=[\langle A_1(z), A_2(z)\rangle]$. Then we note that $\beta\alpha=\alpha\beta=\mathrm{id}$. Therefore we identify $\mathfrak{M}_2$ with $\alpha(\mathfrak{M}_2)$. Similarly we can define the mapping α^* of $\mathfrak{S}_2$ or $\mathfrak{S}_2^0$ into $(D^*)^2\times(\boldsymbol{C}-\{0, 1\})$ by restricting α to this space (cf. Sato [26]), and identify $\mathfrak{S}_2$ (resp. $\mathfrak{S}_2^0$) with $\alpha^*(\mathfrak{S}_2)$ (resp. $\alpha^*(\mathfrak{S}_2^0)$). From now on we denote $\alpha(\mathfrak{M}_2)$, $\alpha^*(\mathfrak{S}_2)$ and $\alpha^*(\mathfrak{S}_2^0)$ by $\mathfrak{M}_2$, $\mathfrak{S}_2$ and $\mathfrak{S}_2^0$ respectively.

Definition 1.1 (cf. [28]). Let (t_1, t_2, ρ) be the point in $\mathfrak{M}_2$ corresponding to $[G]=[\langle A_1, A_2\rangle]$.

(1) G is of the second type (Type II) if $t_1>0$, $t_2<0$ and $\rho>0$.

(2) G is of the fifth type (Type V) if $t_1<0$, $t_2>0$ and $\rho>0$.

(3) G is of the seventh type (Type VII) if $t_1<0$, $t_2<0$ and $\rho<0$.

For each $k=$ II, V, VII, we call the set of all equivalence classes of marked groups (resp. marked Schottky groups and marked classical Schottky groups) of Type k the real space (resp. the real Schottky space and the real classical Schottky space) of Type k, and denote them by $R_k\mathfrak{M}_2$ (resp. $R_k\mathfrak{S}_2$ and $R_k\mathfrak{S}_2^0$).

1.2. Let $G=\langle A_1, A_2\rangle$ be a marked free group on two generators.

Theorem A (Neumann [18]). *The group Φ_2 of automorphisms of G has the following presentation*:

$$\Phi_2=\langle N_1, N_2, N_3\,|\,(N_2N_1N_2N_3)^2=1,\quad N_3^{-1}N_2N_3N_2N_1N_3N_1N_2N_1=1,\ N_1N_3N_1N_3=N_3N_1N_3N_1\rangle,$$

where $N_1:(A_1, A_2)\to(A_1, A_2^{-1})$, $N_2:(A_1, A_2)\to(A_2, A_1)$ *and* $N_3:(A_1, A_2)\to(A_1, A_1A_2)$.

We call the mappings N_1, N_2, and N_3 the *Nielsen transformations*.

DEFINITION 1.2. Let $\phi_1, \phi_2 \in \Phi_2$. We say ϕ_1 and ϕ_2 are *equivalent* if $\phi_1(G)$ is equivalent to $\phi_2(G)$ (cf. §1.1), and denote by $\phi_1 \sim \phi_2$.

REMARKS. (1) We can regard N_j $(j=1, 2, 3)$ and so $\phi \in \Phi_2$ as automorphisms of the space of all equivalence classes of marked free groups on two generators (cf. [28]).

(2) From the above (1) and Definition 1.2, we have the following: If $\langle A_1, A_2 \rangle \sim \langle \hat{A}_1, \hat{A}_2 \rangle$ and $\phi_1 \sim \phi_2$ $(\phi_1, \phi_2 \in \Phi_2)$, then $\phi_1(\langle A_1, A_2 \rangle) \sim \phi_2(\langle \hat{A}_1, \hat{A}_2 \rangle)$.

DEFINITION 1.3. Let $\phi \in \Phi_2$ and let m_j $(j=1, 2)$ be the numbers of N_j contained in ϕ. If m_1+m_2 is even, we say that ϕ is an orientation preserving automorphism. The *Schottky modular group* of genus two, which is denoted by $\mathrm{Mod}(\mathfrak{S}_2)$, is the set of all equivalence classes of orientation preserving automrophisms of $\mathfrak{S}_2$. We denote by $[\Phi_2(\mathfrak{S}_2)]$ the set of all equivalence classes of automorphisms of $\mathfrak{S}_2$.

1.3. Let (t_1, t_2, ρ) be the point in $\mathfrak{S}_2$ corresponding to a marked Schottky group $G=\langle A_1, A_2 \rangle$. Let $(t_1(j), t_2(j), \rho(j))$ be the images of (t_1, t_2, ρ) under the Nielsen transformations N_j $(j=1, 2, 3)$. We set $X=\rho-t_2-\rho t_1 t_2+t_1$ and $Y=\rho-t_2+\rho t_1 t_2-t_1$. Then by straightforward calculations, we have the following.

LEMMA 1.1 (Sato [28, Lemma 2.1]).

(1) $t_1(1)=t_1$, $t_2(1)=t_2$ *and* $\rho(1)=1/\rho$.

(2) $t_1(2)=t_2$, $t_2(2)=t_1$ *and* $\rho(2)=\rho$.

(3) $t_1(3)=t_1$, $t_2(3)+(1/t_2(3))=Y^2/t_1 t_2(\rho-1)^2-2$ *and* $\rho(3)+(1/\rho(3))=X^2/t_1\rho(1-t_2)^2-2$.

LEMMA 1.2. *Let* N_j $(j=1, 2, 3)$ *be the Nielsen transformations. Then* (1) $N_3N_1N_3N_1 \sim 1$, (2) $N_1^2=1$ *and* $N_2^2=1$, (3) $N_1N_2 \sim N_2N_1$, (4) $N_3^2N_1N_2N_3N_2N_3^{-1}N_2 \sim N_3$, (5) $N_3=N_2N_1N_2N_3^{-1}N_2N_1N_2$, (6) $N_2N_3N_2 \sim N_3N_1N_2N_3$.

2. Relationship among real Schottky spaces.

2.1. In this spection we will consider relationship among $R_{\mathrm{II}}\mathfrak{S}_2$, $R_{\mathrm{V}}\mathfrak{S}_2$ and $R_{\mathrm{VII}}\mathfrak{S}_2$. Throughout this section, let N_j $(j=1, 2, 3)$ be the Nielsen transformations defined in §1.

PROPOSITION 2.1. *Let* $\tau=(t_1, t_2, \rho) \in R_{\mathrm{II}}\mathfrak{M}_2$. *Then* (1) $N_1(\tau) \in R_{\mathrm{II}}\mathfrak{M}_2$, (2) $N_2(\tau) \in R_{\mathrm{V}}\mathfrak{M}_2$ *and* (3) $N_3^{\delta}(\tau) \in R_{\mathrm{II}}\mathfrak{M}_2$, *where* $\delta=+1$ *or* -1.

PROOF. (1) and (2) are easily seen from Lemma 1.1 and the definitions of $R_{\mathrm{II}}\mathfrak{M}_2$ and $R_{\mathrm{V}}\mathfrak{M}_2$. We only prove (3). Set

$$A_1=\frac{1}{t_1^{1/2}}\begin{pmatrix}1 & 0\\ 0 & t_1\end{pmatrix} \quad \text{and} \quad A_2=\frac{1}{t_2^{1/2}(\rho-1)}\begin{pmatrix}\rho-t_2 & \rho(t_2-1)\\ 1-t_2 & t_2\rho-1\end{pmatrix}.$$

Then $\langle A_1, A_2 \rangle$ represents (t_1, t_2, ρ). We set $N_3(\tau)=(t_1^*, t_2^*, \rho^*)$. Let p and q be two solutions of the equation

$$(*) \qquad t_1(1-t_2)z^2-(\rho-t_2-\rho t_1t_2+t_1)z+\rho(1-t_2)=0\,.$$

Then p and q are the fixed points of A_1A_2. We assume that p and q are the repelling and the attracting fixed points of A_1A_2, respectively. Since $pq=\rho/t_1>0$ and $\rho^*=q/p$ we have $\rho^*>0$. Furthermore, since

$$t_2^*+1/t_2^*+2=(\rho-t_2+t_1t_2\rho-t_1)^2/t_1t_2(\rho-1)^2<0\,,$$

we have $t_2^*<0$. Noting that $t_1^*=t_1$, we have $N_3(\tau)\in R_{\mathrm{II}}\mathfrak{M}_2$. By the same method as above, we easily see that $N_3^{-1}(\tau)\in R_{\mathrm{II}}\mathfrak{M}_2$. q.e.d.

Similarly, we have the following:

PROPOSITION 2.2. *Let* $\tau=(t_1, t_2, \rho)\in R_{\mathrm{V}}\mathfrak{M}_2$. *Then* (1) $N_1(\tau)\in R_{\mathrm{V}}\mathfrak{M}_2$, (2) $N_2(\tau)\in R_{\mathrm{II}}\mathfrak{M}_2$ *and* (3) $N_3^{\delta}(\tau)\in R_{\mathrm{VII}}\mathfrak{M}_2$, *where* $\delta=+1$ *or* -1.

PROPOSITION 2.3. *Let* $\tau=(t_1, t_2, \rho)\in R_{\mathrm{VII}}\mathfrak{M}_2$. *Then* (1) $N_1(\tau)\in R_{\mathrm{VII}}\mathfrak{M}_2$ (2) $N_2(\tau)\in R_{\mathrm{VII}}\mathfrak{M}_2$ *and* (3) $N_3^{\delta}(\tau)\in R_{\mathrm{V}}\mathfrak{M}_2$, *where* $\delta=+1$ *or* -1.

2.2. The following can be seen easily:

LEMMA 2.1. *Let* $G=\langle A_1, A_2\rangle$ *be a marked two-generator group. Then the image* $\phi(G)$ *of* G *under a mapping* $\phi\in\Phi_2$ *is the same group as* G *except for marking, where* Φ_2 *is the group of automorphisms of* G *defined in* § 1.

COROLLARY. *If* $\tau=(t_1, t_2, \rho)\in\mathfrak{S}_2$ (*resp.* $\mathfrak{S}_2^0$), *then* $\phi(\tau)\in\mathfrak{S}_2$ (*resp.* $\mathfrak{S}_2^0$) *for any* $\phi\in\Phi_2$.

Noting that $N_1^2=1$ and $N_2^2=1$, we have the following from Propositions 2.1 through 2.3 and the above corollary.

THEOREM 1. *Let* N_j ($j=1, 2, 3$) *be the Nielsen transformations defined in* § 1. *Then*

(1) $N_1(R_{\mathrm{II}}\mathfrak{S}_2)=R_{\mathrm{II}}\mathfrak{S}_2$, $N_2(R_{\mathrm{II}}\mathfrak{S}_2)=R_{\mathrm{V}}\mathfrak{S}_2$ *and* $N_3(R_{\mathrm{II}}\mathfrak{S}_2)=R_{\mathrm{II}}\mathfrak{S}_2$.

(2) $N_1(R_{\mathrm{V}}\mathfrak{S}_2)=R_{\mathrm{V}}\mathfrak{S}_2$, $N_2(R_{\mathrm{V}}\mathfrak{S}_2)=R_{\mathrm{II}}\mathfrak{S}_2$ *and* $N_3(R_{\mathrm{V}}\mathfrak{S}_2)=R_{\mathrm{VII}}\mathfrak{S}_2$.

(3) $N_1(R_{\mathrm{VII}}\mathfrak{S}_2)=R_{\mathrm{VII}}\mathfrak{S}_2$, $N_2(R_{\mathrm{VII}}\mathfrak{S}_2)=R_{\mathrm{VII}}\mathfrak{S}_2$ *and* $N_3(R_{\mathrm{VII}}\mathfrak{S}_2)=R_{\mathrm{V}}\mathfrak{S}_2$.

REMARK. For $R_{\mathrm{II}}\mathfrak{S}_2^0$, $R_{\mathrm{V}}\mathfrak{S}_2^0$ and $R_{\mathrm{VII}}\mathfrak{S}_2^0$, the same results as above hold.

3. Shapes of $R_{\mathrm{II}}\mathfrak{S}_2^{00}$, $R_{\mathrm{V}}\mathfrak{S}_2^{00}$ and $R_{\mathrm{VII}}\mathfrak{S}_2^{00}$.

3.1. We recall that the sapce $\mathfrak{S}_g^{00}$ consists of all equivalence classes of the following marked classical Schottky groups $G=\langle A_1, \cdots, A_g\rangle$ of genus g: G has defining curves $C_1, C_{g+1}; \cdots; C_g, C_{2g}$ such that all C_j ($j=1, 2, \cdots, 2g$) are circles and $A_j(C_j)=C_{g+j}$, that is, $A_1, \cdots, A_g$ is a set of classical generators (see [28]). In this section we will determine the shapes of the spaces $R_{\mathrm{II}}\mathfrak{S}_2^{00}:=\mathfrak{S}_2^{00}\cap R_{\mathrm{II}}\mathfrak{S}_2^0$, $R_{\mathrm{V}}\mathfrak{S}_2^{00}:=\mathfrak{S}_2^{00}\cap R_{\mathrm{V}}\mathfrak{S}_2^0$ and $R_{\mathrm{VII}}\mathfrak{S}_2^{00}:=\mathfrak{S}_2^{00}\cap R_{\mathrm{VII}}\mathfrak{S}_2^0$.

Let $\tau=(t_1, t_2, \rho)\in(D^*)^2\times(C-\{0, 1\})$. Throughout this section we let

$$A_1(z) := z/t_1$$

and

$$A_2(z) := \{(\rho - t_2)z + \rho(t_2 - 1)\}/\{(1 - t_2)z + (\rho t_2 - 1)\} .$$

Then we note that $\langle A_1(z), A_2(z)\rangle$ represents $\tau = (t_1, t_2, \rho)$.

PROPOSITION 3.1. *Fix t_1 with $0 < t_1 < 1$.*

(1) *Let $1 < \rho < 1/t_1$. If $t_2 = (t_1^{1/2}\rho^{1/2} - 1)/(\rho^{1/2} - t_1^{1/2})$, then $A_1^{-1}A_2^2$ is a parabolic transformation whose fixed point is $\rho^{1/2}t_1^{1/2}$. Furthermore $G = \langle A_1, A_2\rangle$ is a discontinuous group.*

(2) *Let $t_1 < \rho < 1$. If $t_2 = (t_1^{1/2} - \rho^{1/2})/(1 - t_1^{1/2}\rho^{1/2})$, then $A_1A_2^2$ is a parabolic transformation whose fixed point is $\rho^{1/2}t_1^{-1/2}$. Furthermore $G = \langle A_1, A_2\rangle$ is a discontinuous group.*

PROOF. By straightforward calculations, we see that $A_1^{-1}A_2^2$ in (1) and $A_1A_2^2$ in (2) are parabolic transformations whose fixed points are $\rho^{1/2}t_1^{1/2}$ and $\rho^{1/2}t_1^{-1/2}$, respectively.

For case (1), the region bounded by the following four circles C_1, C_2, C_3 and C_4 is a fundamental region for G:

$$C_1 : |z| = \rho^{1/2}t_1^{1/2} ,$$
$$C_2 : |z - \rho^{1/2}(1 + t_1^{1/2})/2| = \rho^{1/2}(1 - t_1^{1/2})/2 ,$$
$$C_3 : |z| = \rho^{1/2}/t_1^{1/2} ,$$
$$C_4 : |z - \rho^{1/2}(1 + t_1^{1/2})/2t_1^{1/2}| = \rho^{1/2}(1 - t_1^{1/2})/2t_1^{1/2} .$$

For case (2), the following four circles C_j $(j = 1, 2, 3, 4)$ bound a fundamental region for G:

$$C_1 : |z| = \rho^{1/2}t_1^{1/2} ,$$
$$C_2 : |z - \{(\rho^{1/2}/t_1^{1/2}) + \rho^{1/2}\}/2| = \{(\rho^{1/2}/t_1^{1/2}) - \rho^{1/2}\}/2 ,$$
$$C_3 : |z| = \rho^{1/2}/t_1^{1/2} ,$$
$$C_4 : |z - (\rho^{1/2} + \rho^{1/2}t_1^{1/2})/2| = (\rho^{1/2} - \rho^{1/2}t_1^{1/2})/2 .$$

Hence G is a discontinuous group. q.e.d.

3.2. Similarly, we have the following.

PROPOSITION 3.2. *Fix t_1 with $-1 < t_1 < 0$.*

(1) *Let $1 < \rho < 1/t_1^2$. If $t_2^{1/2} = (1 + t_1\rho^{1/2})/(\rho^{1/2} + t_1)$, then $A_1^{-2}A_2$ is a parabolic transformation whose fixed point is $-\rho^{1/2}t_1$. Furthermore $G = \langle A_1, A_2\rangle$ corresponding to (t_1, t_2, ρ) is a discontinuous group.*

(2) *Let* $t_1^2 < \rho < 1$. *If* $t_2^{1/2} = (\rho^{1/2} + t_1)/(1 + t_1\rho^{1/2})$, *then* $A_1^2 A_2$ *is a parabolic transformation whose fixed point is* $-\rho^{1/2}/t_1$. *Furthermore* $G = \langle A_1, A_2 \rangle$ *corresponding to* (t_1, t_2, ρ) *is a discontinuous group.*

PROOF. (1) The region bounded by the following four circles ($j = 1, 2, 3, 4$) is a fundamental region for G:

$$C_1 : |z + \rho^{1/2}(1 + t_1)/2| = \rho^{1/2}(1 - t_1)/2 ,$$

$$C_2 : |z - (\alpha - \rho^{1/2} t_1)/2| = (\alpha + \rho^{1/2} t_1)/2 ,$$

$$C_3 : |z + \rho^{1/2}(1 + 1/t_1)/2| = \rho^{1/2}(1 - 1/t_1)/2 ,$$

$$C_4 : |z - (\beta - \rho^{1/2}/t_1)/2| = (-\rho^{1/2}/t_1 - \beta)/2 ,$$

where $\alpha = T(-(\rho t_1 + \rho^{1/2})/(t_1 + \rho^{1/2}))$, $\beta = T(-(\rho + t_1\rho^{1/2})/(1 + t_1\rho^{1/2}))$ and $T(z) = \rho(z - 1)/(z - \rho)$.

(2) The region bounded by the following four circles C_j ($j = 1, 2, 3, 4$) is a fundamental region for G: C_1 and C_3 are the same circles as in (1);

$$C_2 : |z - \{\alpha - (\rho^{1/2}/t_1)\}/2| = (-\rho^{1/2}/t_1 - \alpha)/2 ,$$

$$C_4 : |z - (\beta - t_1\rho^{1/2})/2| = (\beta + t_1\rho^{1/2})/2 ,$$

where $\alpha = T(-(\rho + t_1\rho^{1/2})/(1 + t_1\rho^{1/2}))$, $\beta = T(-(\rho t_1 + \rho^{1/2})/(t_1 + \rho^{1/2}))$ and $T(z) = \rho(z - 1)/(z - \rho)$. q.e.d.

PROPOSITION 3.3. *Fix* t_1 *with* $-1 < t_1 < 0$.

(1) *Let* $1/t_1 < \rho \leqq -1$. *If* $(-t_2)^{1/2} = \{1 - (-t_1)^{1/2}(-\rho)^{1/2}\}/\{(-\rho)^{1/2} + (-t_1)^{1/2}\}$, *then* $A_1^{-1} A_2$ *is a parabolic transformation whose fixed point is* $(-\rho)^{1/2}(-t_1)^{1/2}$. *Furthermore* $G = \langle A_1, A_2 \rangle$ *corresponding to* (t_1, t_2, ρ) *is a discontinuous group.*

(2) *Let* $-1 \leqq \rho < t_1$. *If* $(-t_2)^{1/2} = \{(-\rho)^{1/2} - (-t_1)^{1/2}\}/\{1 + (-\rho)^{1/2}(-t_1)^{1/2}\}$, *then* $A_1 A_2$ *is a parabolic transformation whose fixed point is* $(-\rho)^{1/2}/(-t_1)^{1/2}$. *Firthermore* $G = \langle A_1, A_2 \rangle$ *corresponding to* (t_1, t_2, ρ) *is a discontinuous group.*

PROOF. Assume that $\tau = (t_1, t_2, \rho)$ satisfies the condition of (1). Let $\langle A_1, A_2 \rangle$ be the marked group corresponding to τ. Set $\langle A_1^*, A_2^* \rangle = \langle A_1, A_1 A_2 \rangle$. Let $\tau^* = (t_1^*, t_2^*, \rho^*)$ be the point corresponding to $\langle A_1^*, A_2^* \rangle$. Then $\tau^* \in R_V \mathfrak{M}_2$ by Proposition 2.3. By straightforward calculations, we see that (t_1^*, t_2^*, ρ^*) satisfies the equation in Proposition 3.2, (1). Let p^* and q^* be the repelling and the attracting fixed points of A_2^*, respectively. Set $S(z^*) := z^*/p^*$, $\hat{A}_1 := S A_1^* S^{-1}$ and $\hat{A}_2 := S A_2^* S^{-1}$. Then we note that $\langle \hat{A}_1, \hat{A}_2 \rangle$ represents the same point (t_1^*, t_2^*, ρ^*) as $\langle A_1^*, A_2^* \rangle$ does, and that the repelling and the attracting fixed points of $\hat{A}_2$ are 1 and ρ^*, respectively. Let C_4, $\hat{\alpha}$ and $\hat{\beta}$ be the circle and the points for $\langle \hat{A}_1, \hat{A}_2 \rangle$ corresponding to C_4, α and β for $\langle A_1, A_2 \rangle$ in Proposition 3.2, (1). We set $C_4^* = S^{-1}(\hat{C}_4)$, $\alpha^* = S^{-1}(\hat{\alpha})$ and $\beta^* = S^{-1}(\hat{\beta})$. We choose four circles C_j^* ($j = 1, 2, 3, 4$) as follows: $C_2 = C_2^* = A_2^{*-1}(C_4^*)$; $C_4 = A_1^{*-1}(C_4^*)$; $C_3 : |z - \{(\alpha^* + \beta^*)/2 + S^{-1}(-\rho^{*1/2})\}/2| = \{(\alpha^* + \beta^*)/2 - S^{-1}(-\rho^{*1/2})\}/2$; $C_1 = A_1^{*-1}(C_3) = A_1^{-1}(C_3)$. Then C_j

$(j=1, 2, 3, 4)$ bound a fundamental region for $\langle A_1, A_2\rangle$ and so $\langle A_1, A_2\rangle$ is a discontinuous group.

(2) follows similarly from Proposition 3.2, (2). q.e.d.

3.3. We set

$$M_{\mathrm{II}}(1)=\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid (t_1^{1/2}\rho^{1/2}-1)/(\rho^{1/2}-t_1^{1/2})<t_2<0,\ 0<\rho<1/t_1, 0<t_1<1\},$$

$$M_{\mathrm{II}}(-1)=\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid (t_1^{1/2}-\rho^{1/2})/(1-t_1^{1/2}\rho^{1/2})<t_2<0,\ t_1<\rho<1, 0<t_1<1\},$$

$$M_{\mathrm{V}}(1)=\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid 0<t_2^{1/2}<(1+t_1\rho^{1/2})/(\rho^{1/2}+t_1),\ 1<\rho<1/t_1^2-1<t_1<0\},$$

$$M_{\mathrm{V}}(-1)=\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid 0<t_2^{1/2}<(\rho^{1/2}+t_1)/(1+t_1\rho^{1/2}),\ t_1^2<\rho<1, -1<t_1<0\}$$

and

$$M_{\mathrm{VII}}(0)=\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid (1-(-t_1)^{1/2}(-\rho)^{1/2})/((-\rho)^{1/2}+(-t_1)^{1/2}) <(-t_2)^{1/2}<((-\rho)^{1/2}-(-t_1)^{1/2})/(1+(-\rho)^{1/2}(-t_1)^{1/2}),\ 1/t_1<\rho<t_1, -1<t_1<0\}.$$

From Propositions 3.1, 3.2 and 3.3, we have the following:

THEOREM 2.

(1) $R_{\mathrm{II}}\mathfrak{S}_2^{00}=M_{\mathrm{II}}(1)\cup M_{\mathrm{II}}(-1)$.

(2) $R_{\mathrm{V}}\mathfrak{S}_2^{00}=M_{\mathrm{V}}(1)\cup M_{\mathrm{V}}(-1)$.

(3) $R_{\mathrm{VII}}\mathfrak{S}_2^{00}=M_{\mathrm{VII}}(0)$.

PROOF. We will only prove this theorem for Type V, since the proof is similarly for the other types.

(i) First we will show that $M_{\mathrm{V}}(1)\subset R_{\mathrm{V}}\mathfrak{S}_2^{00}$. Let $\tau=(t_1, t_2, \rho)\in M_{\mathrm{V}}(1)$. Let C_j $(j=1, 2, 3, 4)$ be circles perpendicular to the real axis such that $A_1(C_1)=C_3$ and $A_2(C_2)=C_4$. For $j=1, 2, 3, 4$, we denote by a_j and b_j $(a_j<b_j)$ the intersection points of the circles C_j with the real axis. It is easily seen that if a_j and b_j satisfy the inequality

$$(*) \qquad a_3<a_1<0<b_1<a_2<1<b_2<a_4<\rho<b_4<b_3,$$

then $\tau\in R_{\mathrm{V}}\mathfrak{S}_2^{00}$. It suffices to show that a_j and b_j $(j=1, 2, 3, 4)$ are chosen in such a way that the above condition $(*)$ are satisfied.

We take a_j and b_j $(j=1, 2, 3, 4)$ as follows: $a_1=-\rho^{1/2}+\varepsilon$, $b_1=-t_1(\rho^{1/2}-\varepsilon/2)$; $a_2=-\rho^{1/2}t_1$, $b_2=(1+\rho)/2$; $a_3=-\rho^{1/2}+\varepsilon/2$, $b_3=(-\rho^{1/2}+\varepsilon)/t_1$; $a_4=A_2((1+\rho)/2)$, $b_4=A_2(-\rho^{1/2}t_1)$, where

$$\varepsilon=\frac{1}{2}\frac{\rho^{1/2}\{t_2(\rho^{1/2}+t_1)^2-(1+\rho^{1/2}t_1)^2\}}{\rho^{1/2}t_2(\rho^{1/2}+t_1)-(1+\rho^{1/2}t_1)}\,.$$

Then we easily see that $A_1(a_1)=b_3$, $A_1(b_1)=a_3$, $A_2(a_2)=b_4$, $A_2(b_2)=a_4$ and that the inequality

$$a_3<a_1<0<b_1<a_2<1<b_2<a_4<\rho<b_4$$

holds. Thus it suffices to show that $b_4<b_3$, that is $A_2(-\rho^{1/2}t_1)<(-\rho^{1/2}+\varepsilon)/t_1$. We note that ε is positive by the condition $0<t_2^{1/2}<(1+t_1\rho^{1/2})/(\rho^{1/2}+t_1)$. The inequality $(-\rho^{1/2}+\varepsilon)/t_1-A_2(-\rho^{1/2}t_1)>0$ is equivalent to

$$\varepsilon<\rho^{1/2}-\frac{t_1^2\rho^{1/2}(\rho-t_2)-t_1\rho(t_2-1)}{(1-t_2)(-\rho^{1/2}t_1)+(\rho t_2-1)}\,.$$

By straightforward calculations we see that the right hand side of the above inequality is equal to 2ε.

Similarly, $M_{\mathrm{V}}(-1)\subset R_{\mathrm{V}}\mathfrak{S}_2^{00}$, hence we have $M_{\mathrm{V}}(1)\cup M_{\mathrm{V}}(-1)\subset R_{\mathrm{V}}\mathfrak{S}_2^{00}$.

(ii) Next we will show that $M_{\mathrm{V}}(1)\cup M_{\mathrm{V}}(-1)\supset R_{\mathrm{V}}\mathfrak{S}_2^{00}$. Let $\tau=(t_1, t_2, \rho)\in R_{\mathrm{V}}\mathfrak{M}_2$. It is easily seen that if $\tau\in R_{\mathrm{V}}\mathfrak{S}_2^{00}$, then $1<\rho<1/t_1^2$ and $1<\rho<1/t_2$ for $\rho>1$, and $t_1^2<\rho<1$ and $t_2<\rho<1$ for $0<\rho<1$. We will show that if $\tau\notin M_{\mathrm{V}}(1)\cup M_{\mathrm{V}}(-1)$, then $\tau\notin R_{\mathrm{V}}\mathfrak{S}_2^{00}$. We only consider the case $\rho>1$, since the case $0<\rho<1$ is similar.

Since $\tau\notin M_{\mathrm{V}}(1)$ and $1<\rho<1/t_1^2$, we have $t_2^{1/2}\geqq(1+t_1\rho^{1/2})/(\rho^{1/2}+t_1)$. If $(1+t_1\rho^{1/2})/(\rho^{1/2}+t_1)=t_2^{1/2}$, then $A_1^{-2}A_2$ is parabolic by Proposition 3.1, (1), and so $\tau\notin R_{\mathrm{V}}\mathfrak{S}_2^{00}$. If $(1+t_1\rho^{1/2})/(\rho^{1/2}+t_1)<t_2^{1/2}<(1-t_1\rho^{1/2})/(\rho^{1/2}-t_1)$, then $A_1^{-2}A_2$ is elliptic and so $\tau\notin R_{\mathrm{V}}\mathfrak{S}_2^{00}$. Furthermore if $1/\rho^{1/2}<t_2^{1/2}$, then $\tau\notin R_{\mathrm{V}}\mathfrak{S}_2^{00}$ by the above remark $1<\rho<1/t_2$. Since $1/\rho^{1/2}<(1-t_1\rho^{1/2})/(\rho^{1/2}-t_1)$, we see that if $\tau\notin M_{\mathrm{V}}(1)$ and $\rho>1$, then $\tau\notin R_{\mathrm{V}}\mathfrak{S}_2^{00}$.

q.e.d.

4. Surfaces.

4.1. In this section we will introduce some surfaces in $\boldsymbol{R}^3$. We set

$$T_n(t_1, \rho, \mathrm{II})=(t_1^{(2n-1)/2}\rho^{1/2}-1)/(\rho^{1/2}-t_1^{(2n-1)/2})$$

for $0<t_1<1$, $\rho>0$ and $n=\pm1, \pm2, \cdots$. Let $n\geqq2$ be an integer. For fixed t_1 with $0<t_1<1$, we denote by $P_n(t_1:\mathrm{II})=(t_1, t_{2,n}(t_1:\mathrm{II}), \rho_n(t_1:\mathrm{II}))$ the intersection point of the following two curves $K_{\mathrm{II}}^+(n)$ and $K_{\mathrm{II}}^-(n)$ in $-1<t_2<0$:

$$K_{\mathrm{II}}^+(n): t_2=T_n(t_1, \rho:\mathrm{II})\,,$$

$$K_{\mathrm{II}}^-(n): t_2=-T_{n-1}(t_1, \rho:\mathrm{II})\,.$$

We set $P_1(t_1:\mathrm{II})=(t_1, t_{2,1}(t_1:\mathrm{II}), \rho_1(t_1:\mathrm{II}))$, where $t_{2,1}(t_1:\mathrm{II})=-1$ and $\rho_1(t_1:\mathrm{II})=1$.

Let $n\geqq2$ be an integer. For fixed t_1 with $0<t_1<1$, we denote by $P_{-n}(t_1:\mathrm{II})=(t_1, t_{2,-n}(t_1:\mathrm{II}), \rho_{-n}(t_1:\mathrm{II}))$ the intersection point of the following two curves

$K_{\mathrm{II}}^{+}(-n)$ and $K_{\mathrm{II}}^{-}(-n)$ in $-1<t_2<0$:

$$K_{\mathrm{II}}^{+}(-n): t_2=T_n(t_1, 1/\rho:\mathrm{II}),$$
$$K_{\mathrm{II}}^{-}(-n): t_2=-T_{n-1}(t_1, 1/\rho:\mathrm{II}).$$

We set $P_{-1}(t_1:\mathrm{II})=P_1(t_1:\mathrm{II})$. We note that $t_{2,n}(t_1:\mathrm{II})=t_{2,-n}(t_1:\mathrm{II})$ and $\rho_n(t_1:\mathrm{II})\rho_{-n}(t_1:\mathrm{II})=1$.

We define the following sets in $\boldsymbol{R}^3$. For the sake of simplicity, we write τ for a point $(t_1, t_2, \rho)\in\boldsymbol{R}^3$ in the following definitions. We set

$$H_{\mathrm{II}}(n)=\{\tau \mid t_2=0,\ t_1^{-(2n-3)}<\rho<t_1^{-(2n-1)},\ 0<t_1<1\} \qquad (n\geqq 2)$$
$$H_{\mathrm{II}}(-n)=\{\tau \mid t_2=0,\ t_1^{2n-1}<\rho<t_1^{2n-3},\ 0<t_1<1\} \quad (n\geqq 2)$$
$$H_{\mathrm{II}}(1)=\{\tau \mid t_2=0,\ 1<\rho<1/t_1,\ 0<t_1<1\}$$
$$H_{\mathrm{II}}(-1)=\{\tau \mid t_2=0,\ t_1<\rho<1,\ 0<t_1<1\}$$
$$F_{\mathrm{II}}^{+}(n)=\{\tau \mid t_2=T_n(t_1, \rho:\mathrm{II}),\ \rho_n(t_1:\mathrm{II})<\rho<1/t_1^{2n-1},\ 0<t_1<1\} \qquad (n\geqq 2)$$
$$F_{\mathrm{II}}^{+}(1)=\{\tau \mid t_2=T_1(t_1, \rho:\mathrm{II}),\ 1<\rho<1/t_1,\ 0<t_1<1\}$$
$$F_{\mathrm{II}}^{+}(-1)=\{\tau \mid t_2=T_1(t_1, 1/\rho:\mathrm{II}),\ t_1<\rho<1,\ 0<t_1<1\}$$
$$F_{\mathrm{II}}^{+}(-n)=\{\tau \mid t_2=T_n(t_1, 1/\rho:\mathrm{II}),\ t_1^{2n-1}<\rho<\rho_{-n}(t_1:\mathrm{II}),\ 0<t_1<1\} \qquad (n\geqq 2)$$
$$F_{\mathrm{II}}^{-}(n)=\{\tau \mid t_2=-T_{n-1}(t_1, \rho:\mathrm{II}),\ 1/t_1^{2n-3}<\rho<\rho_n(t_1:\mathrm{II}),\ 0<t_1<1\} \qquad (n\geqq 2)$$
$$F_{\mathrm{II}}^{-}(-n)=\{\tau \mid t_2=-T_{n-1}(t_1, 1/\rho:\mathrm{II}),\ \rho_{-n}(t_1:\mathrm{II})<\rho<t_1^{2n-3},\ 0<t_1<1\} \qquad (n\geqq 2).$$

4.2. Similarly, we define the following sets for Types V and VII. For an integer n, we set

$$T_{2n-1}(t_1, \rho:\mathrm{V})=\{1-(-t_1)^n\rho^{1/2})\}/\{\rho^{1/2}-(-t_1)^n\} \qquad (n\geqq 1)$$

for $-1<t_1<0$ and $\rho>0$, and

$$T_{2n}(t_1, \rho:\mathrm{VII})=\frac{1-(-t_1)^{(2n+1)/2}(-\rho)^{1/2}}{(-\rho)^{1/2}+(-t_1)^{(2n+1)/2}} \qquad (n\geqq 0)$$

for $-1<t_1<0$ and $\rho<0$.

For fixed t_1 with $-1<t_1<0$, we denote by $P_{2n-1}(t_1:\mathrm{V})=(t_1, t_{2,2n-1}(t_1:\mathrm{V}), \rho_{2n-1}(t_1:\mathrm{V}))$ (resp. $P_{2n}(t_1:\mathrm{VII})=(t_1, t_{2n}(t_1:\mathrm{VII}), \rho_{2n}(t_1:\mathrm{VII}))$ the intersection points of the following two curves $K_{\mathrm{V}}^{+}(2n-1)$ and $K_{\mathrm{V}}^{-}(2n-1)$ in $0<t_2<1$ for an integer $n\geqq 2$ (resp. $K_{\mathrm{VII}}^{+}(2n)$ and $K_{\mathrm{VII}}^{-}(2n)$ in $-1<t_2<0$ for an integer $n\geqq 1$):

$$K_{V}^{+}(2n-1): t_2^{1/2}=T_{2n-1}(t_1, \rho:\mathrm{V})$$
$$K_{\mathrm{V}}^{-}(2n-1): t_2^{1/2}=-T_{2n-3}(t_1, \rho:\mathrm{V})$$
$$K_{\mathrm{VII}}^{+}(2n) \quad : (-t_2)^{1/2}=T_{2n}(t_1, \rho:\mathrm{VII})$$

$$K_{\text{VII}}^{-}(2n) \quad : (-t_2)^{1/2} = -T_{2(n-1)}(t_1, \rho : \text{VII}) .$$

We set $P_1(t_1:\text{V})=(t_1, t_{2,1}(t_1:\text{V}), \rho(t_1:\text{V}))$ and $P_0(t_1:\text{VII})=(t_1, t_{2,0}(t_1:\text{VII}), \rho_0(t_1:\text{VII}))$, where $t_{2,1}(t_1:\text{V})=1$, $\rho_1(t_1:\text{V})=1$, $t_{2,0}(t_1:\text{VII})=-\{(1-(-t_1)^{1/2})/(1+(-t_1)^{1/2}\}^2$, $\rho_0(t_1:\text{VII})=-1$.

For fixed t_1 with $-1<t_1<0$, we denote by $P_{-(2n-1)}(t_1:\text{V})=(t_1, t_{2,-(2n-1)}(t_1:\text{V}), \rho_{-(2n-1)}(t_1:\text{V}))$ (resp. $P_{-2n}(t_1:\text{VII})=(t_1, t_{2,-2n}(t_1:\text{VII}), \rho_{-2n}(t_1:\text{VII}))$ the intersection points of the following two curves $K_{\text{V}}^{+}(-(2n-1))$ and $K_{\text{V}}^{-}(-(2n-1))$ in $0<t_2<1$ for an integer $n\geqq 2$ (resp. $K_{\text{VII}}^{+}(-2n)$ and $K_{\text{VII}}^{-}(-2n)$ in $-1<t_2<0$ for an integer $n\geqq 1$):

$$K_{\text{V}}^{+}(-(2n-1)) : t_2^{1/2} = T_{2n-1}(t_1, 1/\rho : \text{V})$$

$$K_{\text{V}}^{-}(-(2n-1)) : t_2^{1/2} = -T_{2n-3}(t_1, 1/\rho : \text{V})$$

$$K_{\text{VII}}^{+}(-2n) \quad : (-t_2)^{1/2} = T_{2n}(t_1, 1/\rho : \text{VII})$$

$$K_{\text{VII}}^{-}(-2n) \quad : (-t_2)^{1/2} = -T_{2(n-1)}(t_1, 1/\rho : \text{VII}) .$$

We note that $t_{2,2n-1}(t_1:\text{V})=t_{2,-(2n-1)}(t_1:\text{V})$, $t_{2,2n}(t_1:\text{VII})=t_{2,-2n}(t_1:\text{VII})$, $\rho_{2n-1}(t_1:\text{V})\rho_{-(2n-1)}(t_1:\text{V})=1$ and $\rho_{2n}(t_1:\text{VII})\rho_{-2n}(t_1:\text{VII})=1$.

We set

$$H_{\text{V}}(2n-1)=\{\tau \mid t_2=0,\ 1/t_1^{2(n-1)}<\rho<1/t_1^{2n},\ -1<t_1<0\} \qquad (n\geqq 1)$$

$$H_{\text{V}}(-(2n-1))=\{\tau \mid t_2=0,\ t_1^{2n}<\rho<t_1^{2(n-1)},\ -1<t_1<0\} \qquad (n\geqq 1)$$

$$F_{\text{V}}^{+}(2n-1)=\{\tau \mid t_2^{1/2}=T_{2n-1}(t_1, \rho:\text{V}),\ \rho_{2n-1}(t_1:\text{V})<\rho<1/t_1^{2n},\ -1<t_1<0\} \qquad (n\geqq 1)$$

$$F_{\text{V}}^{+}(-(2n-1))=\{\tau \mid t_2^{1/2}=T_{2n-1}(t_1, \rho:\text{V}),\ t_1^{2n}<\rho<\rho_{-(2n-1)}(t_1:\text{V}),\ -1<t_1<0\} \quad (n\geqq 1),$$

where $\rho_{-1}(t_1:\text{V})=1$.

$$F_{\text{V}}^{-}(2n-1)=\{\tau \mid t_2^{1/2}=-T_{2n-3}(t_1, \rho:\text{V}),\ 1/t_1^{2(n-1)}<\rho<\rho_{2n-1}(t_1:\text{V}),\ -1<t_1<0\} \qquad (n\geqq 2)$$

$$F_{\text{V}}^{-}(-(2n-1))=\{\tau \mid t_2^{1/2}=-T_{2n-3}(t_1, \rho:\text{V}),\ \rho_{2n-1}(t_1:\text{V})<\rho<t_1^{2(n-1)},\ -1<t_1<0\} \qquad (n\geqq 2).$$

Furthermore for Type VII, we set

$$H_{\text{VII}}(2n)=\{\tau \mid t_2=0,\ 1/t_1^{2n+1}<\rho<1/t_1^{2n-1},\ -1<t_1<0\} \qquad (n\geqq 1)$$

$$H_{\text{VII}}(-2n)=\{\tau \mid t_2=0,\ t_1^{2n-1}<\rho<t_1^{2n+1},\ -1<t_1<0\} \qquad (n\geqq 1)$$

$$H_{\text{VII}}(0)=\{\tau \mid t_2=0,\ 1/t_1<\rho<t_1,\ -1<t_1<0\}$$

$$F_{\text{VII}}^{+}(2n)=\{\tau \mid (-t_2)^{1/2}=T_{2n}(t_1, \rho:\text{VII}),\ 1/t_1^{2n+1}<\rho<\rho_{2n}(t_1:\text{VII}),\ -1<t_1<0\} \qquad (n\geqq 0)$$

$$F^{+}_{\mathrm{VII}}(-2n)=\{\tau\,|\,(-t_2)^{1/2}=T_{2n}(t_1,1/\rho:\mathrm{VII}),$$
$$\rho_{-2n}(t_1:\mathrm{VII})<\rho<t_1^{2n+1},\ -1<t_1<0\}\qquad (n\geqq 0),$$

where $\rho_0(t_1:\mathrm{VII})=\rho_{-0}(t_1:\mathrm{VII})=-1$;

$$F^{-}_{\mathrm{VII}}(2n)=\{\tau\,|\,(-t_2)^{1/2}=-T_{2(n-1)}(t_1,\rho:\mathrm{VII}),$$
$$\rho_{2n}(t_1:\mathrm{VII})<\rho<1/t_1^{2n-1},\ -1<t_1<0\}\qquad (n\geqq 1)$$

$$F^{-}_{\mathrm{VII}}(-2n)=\{\tau\,|\,(-t_2)^{1/2}=-T_{2(n-1)}(t_1,1/\rho:\mathrm{VII}),$$
$$t_1^{2n-1}<\rho<\rho_{2n}(t_1:\mathrm{VII}),\ -1<t_1<0\}\qquad (n\geqq 1).$$

We call the surfaces defined in §§4.1 and 4.2 surfaces of length one.

4.3. There are some relationship among the surfaces of length one. We have the following proposition by straightforward calculations.

PROPOSITION 4.1. *Let N_3 be the Nielsen transformation. Then*

(1) (i) $N_3(F^+_{\mathrm{II}}(n))=F^+_{\mathrm{II}}(n+1)$ $(n\geqq 1)$
$N_3^{-1}(F^+_{\mathrm{II}}(-n))=F^+_{\mathrm{II}}(-(n+1))$ $(n\geqq 1)$
(ii) $N_3(F^-_{\mathrm{II}}(n))=F^-_{\mathrm{II}}(n+1)$ $(n\geqq 2)$
$N_3^{-1}(F^-_{\mathrm{II}}(-n))=F^-_{\mathrm{II}}(-(n+1))$ $(n\geqq 2)$
$N_3(F^-_{\mathrm{II}}(-2))=\{(t_1,-1,1)\,|\,0<t_1<1\}$.

(2) (i) $N_3(F^+_{\mathrm{V}}(2n-1))=F^+_{\mathrm{VII}}(2n)$ $(n\geqq 1)$
$N_3^{-1}(F^+_{\mathrm{V}}(-(2n-1))=F^+_{\mathrm{VII}}(-2n)$ $(n\geqq 1)$
(ii) $N_3(F^-_{\mathrm{V}}(2n-1))=F^-_{\mathrm{VII}}(2n)$ $(n\geqq 2)$
$N_3^{-1}(F^-_{\mathrm{V}}(-(2n-1))=F^-_{\mathrm{VII}}(-2n)$ $(n\geqq 2)$.

(3) (i) $N_3(F^+_{\mathrm{VII}}(2n))=F^+_{\mathrm{V}}(2n+1)$ $(n\geqq 0)$
$N_3(F^+_{\mathrm{VII}}(-0))=\{(t_1,1,1)\,|\,-1<t_1<0\}$
$N_3^{-1}(F^+_{\mathrm{VII}}(-2n))=F^+_{\mathrm{V}}(-(2n+1))$ $(n\geqq 0)$
(ii) $N_3(F^-_{\mathrm{VII}}(2n))=F^-_{\mathrm{V}}(2n+1)$ $(n\geqq 1)$
$N_3(F^-_{\mathrm{VII}}(-2))=\{(t_1,1,1)\,|\,-1<t_1<0\}$
$N_3^{-1}(F^-_{\mathrm{VII}}(-2n))=F^-_{\mathrm{V}}(-(2n+1))$ $(n\geqq 1)$.

Combining Proposition 4.1, (2) with Proposition 4.1, (3), we have the following.

COROLLARY.

(1) (i) $N_3^2(F^+_{\mathrm{V}}(2n-1))=F^+_{\mathrm{V}}(2n+1)$ $(n\geqq 1)$
$N_3^2(F^+_{\mathrm{V}}(-1))=\{(t_1,1,1)\,|\,-1<t_1<0\}$
$N_3^4(F^+_{\mathrm{V}}(-1))=F^-_{\mathrm{V}}(3)$
$N_3^{-2}(F^+_{\mathrm{V}}(-(2n-1))=F^+_{\mathrm{V}}(-(2n+1))$ $(n\geqq 1)$
(ii) $N_3^2(F^-_{\mathrm{V}}(2n-1))=F^-_{\mathrm{V}}(2n+1)$ $(n\geqq 2)$
$N_3^2(F^-_{\mathrm{V}}(-3))=\{(t_1,1,1)\,|\,-1<t_1<0\}$
$N_3^4(F^-_{\mathrm{V}}(-3))=F^+_{\mathrm{V}}(1)$
$N_3^{-2}(F^-_{\mathrm{V}}(-(2n-1))=F^-_{\mathrm{V}}(-(2n+1))$ $(n\geqq 2)$.

(2) (i) $N_3^2(F^+_{\rm VII}(2n))=F^+_{\rm VII}(2(n+1))$ $(n\geqq 0)$

$N_3^2(F^+_{\rm VII}(-0))=F^-_{\rm VII}(2)$

$N_3^{-2}(F^+_{\rm VII}(-2n))=F^+_{\rm VII}(-2(n+1))$ $(n\geqq 0)$

(ii) $N_3^2(F^-_{\rm VII}(2n))=F^-_{\rm VII}(2(n+1))$ $(n\geqq 1)$

$N_3^2(F^-_{\rm VII}(-2))=F^+_{\rm VII}(+0)$

$N_3^{-2}(F^-_{\rm VII}(-2n))=F^-_{\rm VII}(-2(n+1))$ $(n\geqq 1)$.

For simplicity, we introduce the notation $\varepsilon=+$ or $-$. By straightforward calculations, we have the following two propositions.

PROPOSITION 4.2. *Let N_1 be the Nielsen transformation. Then for an integer n,*

(1) $N_1(F^\varepsilon_{\rm II}(n))=F^\varepsilon_{\rm II}(-n)$ $(n=\pm 1, \pm 2, \cdots)$.

(2) $N_1(F^\varepsilon_{\rm V}(2n-1))=F^\varepsilon_{\rm V}(-(2n-1))$ $(n=\pm 1, \pm 2, \cdots)$.

(3) $N_1(F^\varepsilon_{\rm VII}(2n))=F^\varepsilon_{\rm VII}(-2n)$ $(n=\pm 0, \pm 1, \pm 2, \cdots)$.

PROPOSITION 4.3. *Let N_2 be the Nielsen transformation. Then*

(1) $N_2(F^+_{\rm II}(1))=F^+_{\rm V}(1)$, $N_2(F^+_{\rm II}(-1))=F^+_{\rm V}(-1)$.

(2) $N_2(F^+_{\rm V}(1))=F^+_{\rm II}(1)$, $N_2(F^+_{\rm V}(-1))=F^+_{\rm II}(-1)$.

(3) $N_2(F^+_{\rm VII}(0))=F^+_{\rm VII}(0)$, $N_2(F^+_{\rm VII}(-0))=F^+_{\rm VII}(-0)$.

4.4. We will construct many surfaces out of $F^\varepsilon_{\rm II}(\pm n)$, $F^\varepsilon_{\rm V}(\pm(2n-1))$ and $F^\varepsilon_{\rm VII}(\pm 2n)$, where $\varepsilon=+$ or $-$. Let N_2 be the Nielsen transformation defined in §1. For $n_0=1, 2, 3, \cdots$ and $m_0=2, 3, 4, \cdots$, we set

$$F^+_{\rm V}(1, n_0):=N_2(F^+_{\rm II}(n_0)),\quad F^+_{\rm V}(-1, -n_0):=N_2(F^+_{\rm II}(-n_0)),$$
$$F^-_{\rm V}(1, m_0):=N_2(F^-_{\rm II}(m_0)),\quad F^-_{\rm V}(-1, m_0):=N_2(F^-_{\rm II}(-m_0));$$
$$F^+_{\rm II}(1, 2n_0-1):=N_2(F^+_{\rm V}(2n_0-1)),\quad F^+_{\rm II}(-1, -(2n_0-1)):=N_2(F^+_{\rm V}(-(2n_0-1))),$$
$$F^-_{\rm II}(1, 2m_0-1):=N_2(F^-_{\rm V}(2m_0-1)),\quad F^-_{\rm II}(-1, -(2m_0-1)):=N_2(F^+_{\rm V}(-(2m_0-1)));$$
$$F^+_{\rm VII}(1, 2(n_0-1)):=N_2(F^+_{\rm VII}(2(n_0-1))),\quad F^+_{\rm VII}(-1, -2(n_0-1)):=N_2(F^+_{\rm VII}(-(2n_0-1))),$$
$$F^-_{\rm VII}(1, 2(m_0-1)):=N_2(F^-_{\rm VII}(2(m_0-1))),\quad F^-_{\rm VII}(-1, -2(m_0-1)):=N_2(F^-_{\rm VII}(-(2m_0-1))).$$

We define the following surfaces by using the Nielsen transformation N_3. For $n=0, 1, 2, \cdots$, $n_0=1, 2, 3, \cdots$, $m_0=2, ,3, 4, \cdots$, we set

$$F^+_{\rm II}(\varepsilon n, \varepsilon(2n_0-1)):=N_3^{\varepsilon(n-1)}(F^+_{\rm II}(\varepsilon 1, \varepsilon(2n_0-1)),$$
$$F^-_{\rm II}(\varepsilon n, \varepsilon(2m_0-1)):=N_3^{\varepsilon(n-1)}(F^-_{\rm II}(\varepsilon 1, \varepsilon(2m_0-1)));$$
$$F^+_{\rm V}(\varepsilon(2n+1), \varepsilon n_0):=N_3^{\varepsilon 2n}(F^+_{\rm V}(\varepsilon 1, \varepsilon n_0)),\quad F^-_{\rm V}(\varepsilon(2n+1), \varepsilon m_0):=N_3^{\varepsilon 2n}(F^-_{\rm V}(\varepsilon 1, \varepsilon m_0)),$$
$$F^+_{\rm V}(\varepsilon 0, \varepsilon 2(n_0-1)):=N_3^{-\varepsilon 1}(F^+_{\rm VII}(\varepsilon 1, \varepsilon 2(n_0-1))),$$
$$F^-_{\rm V}(\varepsilon 0, \varepsilon 2(m_0-1)):=N_3^{-\varepsilon 1}(F^-_{\rm VII}(\varepsilon 1, \varepsilon 2(m_0-1)));$$
$$F^+_{\rm VII}(\varepsilon(2n+1), \varepsilon 2(n_0-1)):=N_3^{\varepsilon 2n}(F^+_{\rm VII}(\varepsilon 1, \varepsilon 2(n_0-1))),$$
$$F^-_{\rm VII}(\varepsilon(2n+1), \varepsilon 2(m_0-1)):=N_3^{\varepsilon 2n}(F^-_{\rm VII}(\varepsilon 1, \varepsilon 2(m_0-1))),$$
$$F^+_{\rm VII}(\varepsilon 0, \varepsilon n_0):=N_3^{-\varepsilon 1}(F^+_{\rm V}(\varepsilon 1, \varepsilon n_0)),\quad F^-_{\rm VII}(\varepsilon 0, \varepsilon m_0):=N_3^{-\varepsilon 1}(F^-_{\rm V}(\varepsilon 1, \varepsilon m_0)),$$

where $\varepsilon=+$ or $-$, and $-\varepsilon$ denotes $+$ or $-$ according as ε is $-$ or $+$.

Furthermore, for $n=1, 2, 3, \cdots$, $n_0=1, 2, 3, \cdots$ and $m_0=2, 3, 4, \cdots$, we set

$$F_{\mathrm{V}}^{+}(\varepsilon 2n, \varepsilon 2(n_0-1)) := N_3^{\varepsilon 1}(F_{\mathrm{VII}}^{+}(\varepsilon(2n-1), \varepsilon 2(n_0-1))),$$
$$F_{\mathrm{V}}^{-}(\varepsilon 2n, \varepsilon 2(m_0-1)) := N_3^{\varepsilon 1}(F_{\mathrm{VII}}^{-}(\varepsilon(2n-1), \varepsilon 2(m_0-1)));$$
$$F_{\mathrm{VII}}^{+}(\varepsilon 2n, \varepsilon n_0) := N_3^{\varepsilon 1}(F_{\mathrm{V}}^{+}(\varepsilon(2n-1), \varepsilon n_0)), \quad F_{\mathrm{VII}}^{-}(\varepsilon 2n, \varepsilon m_0) := N_3^{\varepsilon 1}(F_{\mathrm{V}}^{-}(\varepsilon(2n-1), \varepsilon m_0)).$$

We call these surfaces surfaces of length two.

4.5. Inductively we now define the following surfaces. Let $\varepsilon=+$ or $-$. We assume that the surfaces $F_l^{\varepsilon}(\varepsilon n_k, \cdots, \varepsilon n_1, \varepsilon n_0)$ ($l=$II, V, VII) have been defined. We let

$$F_{\mathrm{II}}^{\varepsilon}(1, n_k, \cdots, n_1, n_0) := N_2(F_{\mathrm{V}}^{\varepsilon}(n_k, \cdots, n_1, n_0)),$$
$$F_{\mathrm{II}}^{\varepsilon}(n, n_k, \cdots, n_1, n_0) := N_3^{n-1}(F_{\mathrm{II}}^{\varepsilon}(1, n_k, \cdots, n_1, n_0));$$
$$F_{\mathrm{II}}^{\varepsilon}(-1, -n_k, \cdots, -n_1, -n_0) := N_2(F_{\mathrm{V}}^{\varepsilon}(-n_k, \cdots, -n_1, -n_0)),$$
$$F_{\mathrm{II}}^{\varepsilon}(-n, -n_k, \cdots, -n_1, -n_0) := N_3^{-(n-1)}(F_{\mathrm{II}}^{\varepsilon}(-1, -n_k, \cdots, -n_1, -n_0)).$$

Furthermore, for Types V and VII we let

$$F_{\mathrm{V}}^{\varepsilon}(1, n_k, \cdots, n_1, n_0) := N_2(F_{\mathrm{II}}^{\varepsilon}(n_k, \cdots, n_1, n_0)),$$
$$F_{\mathrm{VII}}^{\varepsilon}(1, n_k, \cdots, n_1, n_0) := N_2(F_{\mathrm{VII}}^{\varepsilon}(n_k, \cdots, n_1, n_0)),$$
$$F_{\mathrm{V}}^{\varepsilon}(-1, -n_k, \cdots, -n_1, -n_0) := N_2(F_{\mathrm{II}}^{\varepsilon}(-n_k, \cdots, -n_1, -n_0)),$$
$$F_{\mathrm{VII}}^{\varepsilon}(-1, -n_k, \cdots, -n_1, -n_0) := N_2(F_{\mathrm{VII}}^{\varepsilon}(-n_k, \cdots, -n_1, -n_0));$$

for $n=1, 2, 3, \cdots$,

$$F_{\mathrm{V}}^{\varepsilon}(n+1, n_k, \cdots, n_1, n_0) := N_3(F_{\mathrm{VII}}^{\varepsilon}(n, n_k, \cdots, n_1, n_0)),$$
$$F_{\mathrm{VII}}^{\varepsilon}(n+1, n_k, \cdots, n_1, n_0) := N_3(F_{\mathrm{V}}^{\varepsilon}(n, n_k, \cdots, n_1, n_0)),$$
$$F_{\mathrm{V}}^{\varepsilon}(-(n+1), -n_k, \cdots, -n_1, -n_0) := N_3^{-1}(F_{\mathrm{VII}}^{\varepsilon}(-n, -n_k, \cdots, -n_1, -n_0)),$$
$$F_{\mathrm{VII}}^{\varepsilon}(-(n+1), -n_k, \cdots, -n_1, -n_0) := N_3^{-1}(F_{\mathrm{V}}^{\varepsilon}(-n, -n_k, \cdots, -n_1, -n_0));$$
$$F_{\mathrm{V}}^{\varepsilon}(0, n_k, \cdots, n_1, n_0) := N_3^{-1}(F_{\mathrm{VII}}^{\varepsilon}(1, n_k, \cdots, n_1, n_0)),$$
$$F_{\mathrm{VII}}^{\varepsilon}(0, n_k, \cdots, n_1, n_0) := N_3^{-1}(F_{\mathrm{V}}^{\varepsilon}(1, n_k, \cdots, n_1, n_0)),$$
$$F_{\mathrm{V}}^{\varepsilon}(-0, -n_k, \cdots, -n_1, -n_0) := N_3(F_{\mathrm{VII}}^{\varepsilon}(-1, -n_k, \cdots, -n_1, -n_0)),$$
$$F_{\mathrm{VII}}^{\varepsilon}(-0, -n_k, \cdots, -n_1, -n_0) := N_3(F_{\mathrm{V}}^{\varepsilon}(-1, -n_k, \cdots, -n_1, -n_0)).$$

5. Relationship among surfaces.

5.1. In this section we will consider relationship among the surfaces defined in the previous section. Let $\varepsilon=+$ or $-$. From now on, we use the notation $F_l^{\varepsilon}(n_k, \cdots, n_1, n_0)$ and $F_l^{\varepsilon}(-n_k, \cdots, -n_1, -n_0)$ ($l=$II, V, VII) only when they are defined. We have the following two properties by definition.

PROPOSITION 5.1. *Let N_1 be the Nielsen transformation. For $l=$*II, V, VII, *and for* *integers* $n_j \geqq 0$ $(j=0, 1, 2, \cdots, k)$,

(1) $N_1(F_l^\varepsilon(n_k, \cdots, n_1, n_0)) = F_l^\varepsilon(-n_k, \cdots, -n_1, -n_0)$.

(2) $N_1(F_l^\varepsilon(-n_k, \cdots, -n_1, -n_0)) = F_l^\varepsilon(n_k, \cdots, n_1, n_0)$.

PROPOSITION 5.2. *Let* N_3 *be the Nielsen transformation. For* $l=$ V, VII, *and for integers* $n_j \geqq 0$ $(j=0, 1, 2, \cdots, k)$,

(1) $N_3^2(F_l^\varepsilon(n_k, n_{k-1}, \cdots, n_0)) = F_l^\varepsilon(n_k+2, n_{k-1}, \cdots, n_0)$.

(2) $N_3^{-2}(F_l^\varepsilon(-n_k, -n_{k-1}, \cdots, -n_0) = F_l^\varepsilon(-(n_k+2), -n_{k-1}, \cdots, -n_0)$.

5.2. PROPOSITION 5.3. *Let* N_3 *be the Nielsen transformation. Then the following hold*:

(1) $N_3(F_{\mathrm{II}}^+(-1)) = \{(t_1, -1, 1) \mid 0 < t_1 < 1\}$.

(2) $N_3(F_{\mathrm{II}}^\varepsilon(-1, -(2n_0-1)) = F_{\mathrm{II}}^\varepsilon(1, 2, 2n_0-2)$ $(n_0 \geqq 1)$.

(3) $N_3(F_{\mathrm{II}}^\varepsilon(-1, -1, -n_{k-2}, \cdots, -n_0)) = N_3(F_{\mathrm{II}}^\varepsilon(-n_{k-2}, \cdots, -n_0))$ $(n_{k-2} \geqq 0)$.

(4) $N_3(F_{\mathrm{II}}^\varepsilon(-1, -2, -n_{k-2}, \cdots, -n_0)) = F_{\mathrm{II}}^\varepsilon(1, n_{k-2}+1, n_{k-3}, \cdots, n_0)$ $(n_{k-2} \geqq 1)$.

(5) $N_3(F_{\mathrm{II}}^\varepsilon(-1, -2, -0, -n_{k-3}, \cdots, -n_0) = F_{\mathrm{II}}^\varepsilon(n_{k-3}, n_{k-4}, \cdots, n_0)$ $(n_{k-3} \geqq 0)$.

(6) $N_3(F_{\mathrm{II}}^\varepsilon(-1, -n_{k-1}, -n_{k-2}, \cdots, -n_0)) = F_{\mathrm{II}}^\varepsilon(1, 2, n_{k-1}-1, n_{k-2}, \cdots, n_0)$ $(n_{k-1} \geqq 3)$.

PROOF. (1) is a consequence of straightforward calculations.

(2) By Lemma 1.2, (4),

$$\begin{aligned} N_3(F_{\mathrm{II}}^\varepsilon(-1, -(2n_0-1))) &= N_2N_3N_2N_3^{-1}N_2N_1(F_{\mathrm{II}}^\varepsilon(-1, -(2n_0-1))) \\ &= N_2N_3N_2N_3^{-1}N_2(F_{\mathrm{II}}^\varepsilon(1, 2n_0-1)) = N_2N_3N_2N_3^{-1}(F_{\mathrm{V}}^\varepsilon(2n_0-1)) \\ &= N_2N_3N_2(F_{\mathrm{VII}}^\varepsilon(2n_0-2)) = N_2N_3(F_{\mathrm{VII}}^\varepsilon(1, 2n_0-2)) \\ &= N_2(F_{\mathrm{V}}^\varepsilon(2, 2n_0-2)) = F_{\mathrm{II}}^\varepsilon(1, 2, 2n_0-2)\,. \end{aligned}$$

(3) follows from the equality $N_2^2(F_{\mathrm{II}}^\varepsilon(-1, -1, -n_{k-2}, \cdots, -n_0)) = F_{\mathrm{II}}^\varepsilon(-n_{k-2}, \cdots, -n_0)$ and $N_2^2=1$.

(4), (5) and (6) can be proved similarly. q.e.d.

PROPOSITION 5.4. *Let* N_3 *be the Nielsen transformation. Then the following hold*:

(1) $N_3(F_{\mathrm{II}}^\varepsilon(-0, -(2n_0-1))) = F_{\mathrm{II}}^\varepsilon(2, 2, 2n_0-2)$ $(n_0 \geqq 1)$.

(2) $N_3(F_{\mathrm{II}}^\varepsilon(-0, -1, -n_{k-2}, -n_{k-3}, \cdots, -n_0))$
$= N_3(F_{\mathrm{II}}^\varepsilon(-(n_{k-2}-1), -n_{k-3}, \cdots, -n_0))$ $(n_{k-2} \geqq 1)$.

(3) $N_3(F_{\mathrm{II}}^\varepsilon(-0, -1, -0, -n_{k-3}, \cdots, -n_0))$
$= N_3^2(F_{\mathrm{II}}^\varepsilon(-0, -n_{k-3}, \cdots, -n_0))$ $(n_{k-3} \geqq 0)$.

(4) $N_3(F_{\mathrm{II}}^\varepsilon(-0, -2, -n_{k-2}, -n_{k-3}, \cdots, -n_0))$
$= F_{\mathrm{II}}^\varepsilon(2, n_{k-2}+1, n_{k-3}, \cdots, n_0)$ $(n_{k-2} \geqq 1)$.

(5) $N_3(F_{\mathrm{II}}^\varepsilon(-0, -2, -0, -n_{k-3}, -n_{k-4}, \cdots, -n_0))$
$= F_{\mathrm{II}}^\varepsilon(n_{k-3}+1, n_{k-4}, \cdots, n_0)$ $(n_{k-3} \geqq 0)$.

(6) $N_3(F_{\mathrm{II}}^\varepsilon(-0, -n_{k-1}, -n_{k-2}, \cdots, -n_0))$
$= F_{\mathrm{II}}^\varepsilon(2, 2, n_{k-1}-1, n_{k-2}, \cdots, n_0)$ $(n_{k-1} \geqq 3)$.

PROOF. (1) By Proposition 5.3, (2), we have

$$N_3(F^{\varepsilon}_{\text{II}}(-0, -(2n_0-1)) = N_3(N_3(F^{\varepsilon}_{\text{II}}(-1, -(2n_0-1)))$$
$$= N_3(F^{\varepsilon}_{\text{II}}(1, 2, 2n_0-2)) = F^{\varepsilon}_{\text{II}}(2, 2, 2n_0-2).$$

(2) By Proposition 5.3, (3), we have

$$N_3(F^{\varepsilon}_{\text{II}}(-0, -1, -n_{k-2}, -n_{k-3}, \cdots, -n_0))$$
$$= N_3(N_3(F^{\varepsilon}_{\text{II}}(-1, -1, -n_{k-2}, -n_{k-3}, \cdots, -n_0)))$$
$$= N_3^2(F^{\varepsilon}_{\text{II}}(-n_{k-2}, -n_{k-3}, \cdots, -n_0)) = N_3(F^{\varepsilon}_{\text{II}}(-(n_{k-2}-1), -n_{k-3}, \cdots, -n_0)).$$

(3) $N_3(F^{\varepsilon}_{\text{II}}(-0, -1, -0, -n_{k-3}, \cdots, -n_0))$
$$= N_3(N_3(F^{\varepsilon}_{\text{II}}(-1, -1, -0, -n_{k-3}, \cdots, -n_0)))$$
$$= N_3^2(F^{\varepsilon}_{\text{II}}(-0, -n_{k-3}, \cdots, -n_0)).$$

(4), (5) and (6) can be proved similarly. q.e.d.

5.3. By Lemma 1.2, (1) and Proposition 5.3, we easily see the following:

PROPOSITION 5.5. *Let N_3 be the Nielsen transformation. Then the following hold.*

(1) $N_3^{-1}(F^{+}_{\text{II}}(1)) = \{(t_1, -1, 1) \mid 0 < t_1 < 1\}$.

(2) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(1, 2n_0-1)) = F^{\varepsilon}_{\text{II}}(-1, -2, -(2n_0-2))$ $(n_0 \geqq 1)$.

(3) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(1, 1, n_{k-2}, \cdots, n_0)) = N_3^{-1}(F^{\varepsilon}_{\text{II}}(n_{k-2}, \cdots, n_0))$ $(n_0 \geqq 0)$.

(4) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(1, 2, n_{k-2}, \cdots, n_0))$
$= F^{\varepsilon}_{\text{II}}(-1, -(n_{k-2}+1), -n_{k-3}, \cdots, -n_0)$ $(n_{k-2} \geqq 1)$.

(5) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(1, 2, 0, n_{k-3}, \cdots, n_0)) = F^{\varepsilon}_{\text{II}}(-n_{k-3}, -n_{k-4}, \cdots, -n_0)$ $(n_{k-3} \geqq 0)$.

(6) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(1, n_{k-1}, n_{k-2}, \cdots, n_0))$
$= F^{\varepsilon}_{\text{II}}(-1, -2, -(n_{k-1}-1), -n_{k-2}, \cdots, -n_0)$ $(n_{k-1} \geqq 3)$.

By Lemma 1.2, (1) and Proposition 5.4, we easily see the following:

PROPOSITION 5.6. *Let N_3 be the Nielsen transformation. Then the following hold*:

(1) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(0, 2n_0-1)) = F^{\varepsilon}_{\text{II}}(-2, -2, -(2n_0-2))$ $(n_0 \geqq 1)$.

(2) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(0, 1, n_{k-2}, n_{k-3}, \cdots, n_0))$
$= N_3(F^{\varepsilon}_{\text{II}}(n_{k-2}-1, n_{k-3}, \cdots, n_0)$ $(n_{k-2} \geqq 1)$.

(3) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(0, 1, 0, n_{k-3}, \cdots, n_0)) = N_3^2(F^{\varepsilon}_{\text{II}}(0, n_{k-3}, \cdots, n_0))$ $(n_{k-3} \geqq 0)$.

(4) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(0, 2, n_{k-2}, n_{k-3}, \cdots, n_0))$
$= F^{\varepsilon}_{\text{II}}(-2, -(n_{k-2}+1), -n_{k-3}, \cdots, -n_0)$ $(n_{k-2} \geqq 1)$.

(5) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(0, 2, 0, n_{k-3}, n_{k-4}, \cdots, n_0))$
$= F^{\varepsilon}_{\text{II}}(-(n_{k-3}+1), -n_{k-4}, \cdots, -n_0)$ $(n_{k-3} \geqq 0)$.

(6) $N_3^{-1}(F^{\varepsilon}_{\text{II}}(0, n_{k-1}, n_{k-2}, \cdots, n_0))$
$= F^{\varepsilon}_{\text{II}}(-2, -2, -(n_{k-1}-1), -n_{k-2}, \cdots, -n_0)$ $(n_{k-1} \geqq 3)$.

5.4. The proofs of the following Propositions 5.7 through 5.10 are similar to those of Propositions 5.3 through 5.6. Let δ denote the number $+1$ or -1, and let $-\delta$ denote -1 or $+1$ according as δ is $+1$ or -1. For simplicity, we write

$$\delta(n_k, \cdots, n_0) = \begin{cases} (n_k, \cdots, n_0) & \text{if} \quad \delta = +1 \\ (-n_k, \cdots, -n_0) & \text{if} \quad \delta = -1 . \end{cases}$$

PROPOSITION 5.7.

(1) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(1, n_0))) = F_{\mathrm{VII}}^\varepsilon(\delta(1, 2, n_0 - 1))$ $(n_0 \geqq 2)$.

(2) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(1, 1, n_{k-2}, \cdots, n_0))) = N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(n_{k-2}, \cdots, n_0)))$ $(n_{k-2} \geqq 0)$.

(3) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(1, 2, n_{k-2}, n_{k-3}, \cdots, n_0)))$
$= F_{\mathrm{VII}}^\varepsilon(\delta(1, n_{k-2}+1, n_{k-3}, \cdots, n_0))$ $(n_{k-2} \geqq 1)$.

(4) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(1, 2, 0, n_{k-3}, n_{k-4}, \cdots, n_0)))$
$= F_{\mathrm{VII}}^\varepsilon(\delta(n_{k-3}, n_{k-4}, \cdots, n_0))$ $(n_{k-3} \geqq 0)$.

(5) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(1, n_{k-1}, n_{k-2}, \cdots, n_0)))$
$= F_{\mathrm{VII}}^\varepsilon(\delta(1, 2, n_{k-1}-1, n_{k-2}, \cdots, n_0))$ $(n_{k-1} \geqq 3)$.

PROPOSITION 5.8.

(1) $N_3^\delta(F_{\mathrm{V}}^+(-\delta(0, 0))) = F_{\mathrm{VII}}^-(\delta(2))$.

(2) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(0, 2n_0))) = F_{\mathrm{VII}}^\varepsilon(\delta(2, 2, 2n_0 - 1))$ $(n_0 \geqq 1)$.

(3) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(0, 1, n_{k-2}, n_{k-3}, \cdots, n_0)))$
$= N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(n_{k-2}-1, n_{k-3}, \cdots, n_0)))$ $(n_{k-2} \geqq 1)$.

(4) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(0, 1, 0, n_{k-3}, \cdots, n_0)))$
$= N_3^{\delta 2}(F_{\mathrm{VII}}^\varepsilon(-\delta(0, n_{k-3}, \cdots, n_0)))$ $(n_{k-3} \geqq 0)$,

where $\delta 2$ denotes $+2$ or -2 according as δ is $+1$ or -1.

(5) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(0, 2, n_{k-2}, n_{k-3}, \cdots, n_0)))$
$= F_{\mathrm{VII}}^\varepsilon(\delta(2, n_{k-2}+1, n_{k-3}, \cdots, n_0))$ $(n_{k-2} \geqq 1)$.

(6) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(0, 2, 0, n_{k-3}, n_{k-4}, \cdots, n_0)))$
$= F_{\mathrm{VII}}^\varepsilon(\delta(n_{k-3}+1, n_{k-4}, \cdots, n_0))$ $(n_{k-3} \geqq 0)$.

(7) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(0, n_{k-1}, n_{k-2}, \cdots, n_0)))$
$= F_{\mathrm{VII}}^\varepsilon(\delta(2, 2, n_{k-1}-1, n_{k-2}, \cdots, n_0))$ $(n_{k-1} \geqq 3)$.

PROPOSITION 5.9.

(1) $N_3^\delta(F_{\mathrm{VII}}^+(-\delta(1, 0))) = \{(t_1, 1, 1) \mid -1 < t_1 < 0\}$.

(2) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(1, 2n_0))) = F_{\mathrm{V}}^\varepsilon(\delta(1, 2, 2n_0 - 1))$ $(n_0 \geqq 1)$.

(3) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(1, 1, n_{k-2}, \cdots, n_0))) = N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(n_{k-2}, \cdots, n_0)))$ $(n_{k-2} \geqq 0)$.

(4) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(1, 2, n_{k-2}, \cdots, n_0)))$
$= F_{\mathrm{V}}^\varepsilon(\delta(1, n_{k-2}+1, n_{k-3}, \cdots, n_0))$ $(n_{k-2} \geqq 1)$.

(5) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(1, 2, 0, n_{k-3}, n_{k-4}, \cdots, n_0)))$
$= F_{\mathrm{V}}^\varepsilon(\delta(n_{k-3}+1, n_{k-4}, \cdots, n_0))$ $(n_{k-3} \geqq 0)$.

(6) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(1, n_{k-1}, n_{k-2}, \cdots, n_0)))$
$= F_{\mathrm{V}}^\varepsilon(\delta(1, 2, n_{k-1}-1, n_{k-2}, \cdots, n_0))$ $(n_{k-1} \geqq 3)$.

REMARKS.

(1) (i) $F_{\mathrm{VII}}^+(-1, -0) = F_{\mathrm{VII}}^+(-0)$ and $F_{\mathrm{VII}}^+(1, 0) = F_{\mathrm{VII}}^+(0)$.

(ii) $F_l^\varepsilon(\delta(n_k, \cdots, n_1, 1)) = F_l^\varepsilon(\delta(n_k, \cdots, n_1))$ for $l = \mathrm{II}, \mathrm{V}, \mathrm{VII}$.

(2) The cases $n_{k-1}=0$ in Propositions 5.3, 5.4, 5.6, 5.8 and 5.10 will be treated in §6.

PROPOSITION 5.10.

(1) $N_3^\delta(F_{\mathrm{VII}}^+(-\delta(0,1)))=\{(t_1,1,1)\mid -1<t_1<0\}$.

(2) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(0,n_0)))=F_{\mathrm{V}}^\varepsilon(\delta(2,2,n_0-1))$ $\quad(n_0\geqq 2)$.

(3) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(0,1,n_{k-2},n_{k-3},\cdots,n_0)))$
$=N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(n_{k-2}-1,n_{k-3},\cdots,n_0)))$ $\quad(n_{k-2}\geqq 1)$.

(4) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(0,1,0,n_{k-3},\cdots,n_0)))$
$=N_3^{\delta 2}(F_{\mathrm{V}}^\varepsilon(-\delta(0,n_{k-3},\cdots,n_0)))$ $\quad(n_{k-3}\geqq 0)$,

where $\delta 2$ *denotes* $+2$ *or* -2 *according as* δ *is* $+1$ *or* -1.

(5) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(0,2,n_{k-2},\cdots,n_0)))$
$=F_{\mathrm{V}}^\varepsilon(\delta(2,n_{k-2}+1,n_{k-3},\cdots,n_0))$ $\quad(n_{k-2}\geqq 1)$.

(6) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(0,2,0,n_{k-3},n_{k-4},\cdots,n_0)))$
$=F_{\mathrm{V}}^\varepsilon(\delta(n_{k-3}+1,n_{k-4},\cdots,n_0))$ $\quad(n_{k-3}\geqq 0)$.

(7) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(0,n_{k-1},n_{k-2},\cdots,n_0)))$
$=F_{\mathrm{V}}^\varepsilon(\delta(2,2,n_{k-1}-1,n_{k-2},\cdots,n_0))$ $\quad(n_{k-1}\geqq 3)$.

6. The domains of existence.

6.1. In this section we will determine the shapes of the real classical Schottky spaces $R_{\mathrm{II}}\mathfrak{S}_2^0$, $R_{\mathrm{V}}\mathfrak{S}_2^0$ and $R_{\mathrm{VII}}\mathfrak{S}_2^0$ in $\boldsymbol{R}^3$. Let ε and δ be the same symbols as in §5.

PROPOSITION 6.1. *Let* N_3 *be the Nielsen transformation. Then the following hold*:

(1) $N_3^\delta(F_{\mathrm{II}}^\varepsilon(-\delta(0,n_{k-1},\cdots,n_0)))=F_{\mathrm{II}}^\varepsilon(\delta(1,2,0,0,n_{k-1},\cdots,n_0))$ $\quad(n_{k-1}\geqq 0)$.

(2) $N_3^\delta(F_{\mathrm{V}}^\varepsilon(-\delta(0,n_{k-1},\cdots,n_0)))=F_{\mathrm{VII}}^\varepsilon(\delta(1,2,0,0,n_{k-1},\cdots,n_0))$ $\quad(n_{k-1}\geqq 0)$.

(3) $N_3^\delta(F_{\mathrm{VII}}^\varepsilon(-\delta(0,n_{k-1},\cdots,n_0)))=F_{\mathrm{V}}^\varepsilon(\delta(1,2,0,0,n_{k-1},\cdots,n_0))$ $\quad(n_{k-1}\geqq 0)$.

PROOF. We only prove (1), since the proofs for (2) and (3) are similar. First we will consider the case $\delta=+1$. By Lemma 1.2, (4) we have $N_3\sim N_2N_3N_2N_3^{-1}N_2N_1$. Hence

$$
\begin{aligned}
N_3(F_{\mathrm{II}}^\varepsilon(-0,-n_{k-1},\cdots,-n_0))&=N_2N_3N_2N_3^{-1}N_2N_1(F_{\mathrm{II}}^\varepsilon(-0,-n_{k-1},\cdots,-n_0))\\
&=N_2N_3N_2N_3^{-1}N_2(F_{\mathrm{II}}^\varepsilon(0,n_{k-1},\cdots,n_0))=N_2N_3N_2N_3^{-1}(F_{\mathrm{V}}^\varepsilon(1,0,n_{k-1},\cdots,n_0))\\
&=N_2N_3N_2(F_{\mathrm{VII}}^\varepsilon(0,0,n_{k-1},\cdots,n_0))=N_2N_3(F_{\mathrm{VII}}^\varepsilon(1,0,0,n_{k-1},\cdots,n_0))\\
&=N_2(F_{\mathrm{VII}}^\varepsilon(2,0,0,n_{k-1},\cdots,n_0))=F_{\mathrm{II}}^\varepsilon(1,2,0,0,n_{k-1},\cdots,n_0).
\end{aligned}
$$

In the case $\delta=-1$, we have the desired result by Lemma 1.2, (1) and $N_3\sim N_2N_3N_2N_3^{-1}N_2N_1$ by the same method as above. q.e.d.

6.2. We set

$$
\boldsymbol{R}_{\mathrm{II}}^3=\{(t_1,t_2,\rho)\in\boldsymbol{R}^3\mid 0<t_1<1,\ -1<t_2<0,\ \rho>0\}\,,
$$

$$
\boldsymbol{R}_{\mathrm{V}}^3=\{(t_1,t_2,\rho)\in\boldsymbol{R}^3\mid -1<t_1<0,\ 0<t_2<1,\ \rho>0\}\,,
$$

$$
\boldsymbol{R}_{\mathrm{VII}}^3=\{(t_1,t_2,\rho)\in\boldsymbol{R}^3\mid -1<t_1<0,\ -1<t_2<0,\ \rho<0\}\,.
$$

We denote by $M_l(\delta(n_k,n_{k-1},\cdots,n_0))$ the three-dimensional manifolds in $\boldsymbol{R}_l^3$ bounded by $F_l^+(\delta(n_k,n_{k-1},\cdots,n_0))$ and $F_l^-(\delta(n_k,n_{k-1},\cdots,n_0))$ for $l=\mathrm{II}$, V, VII, where

$$F_{\mathrm{II}}^{-}(1)=F_{\mathrm{II}}^{-}(-1)=\{(t_1, t_2, \rho)\in \boldsymbol{R}_{\mathrm{II}}^3 \,|\, 0<t_1<1,\ -1<t_2<0,\ \rho=1\}\ ,$$

$$F_{\mathrm{V}}^{-}(1)=F_{\mathrm{V}}^{-}(-1)=\{(t_1, t_2, \rho)\in \boldsymbol{R}_{\mathrm{V}}^3 \,|\, -1<t_1<0,\ 0<t_2<1,\ \rho=1\}\ ,$$

$$F_{\mathrm{VII}}^{-}(0)=F_{\mathrm{VII}}^{+}(-0)\ ,\quad F_{\mathrm{VII}}^{-}(-0)=F_{\mathrm{VII}}^{+}(0)\quad \text{and}\quad M_{\mathrm{VII}}(0)=M_{\mathrm{VII}}(-0)\ .$$

REMARK. For each $l=$ II, V, VII,

(1) $F_l^{\varepsilon}(\delta(\cdots, n_{j+1}, 1, 1, n_{j-2}, \cdots))=F_l^{\varepsilon}(\delta(\cdots, n_{j+1}, n_{j-2}, \cdots))$,

(2) $M_l(\delta(\cdots, n_{j+1}, 1, 1, n_{j-2}, \cdots))=M_l(\delta(\cdots, n_{j+1}, n_{j-2}, \cdots))$.

The following proposition is an easy consequence of Propositions 4.1, 4.2, 4.3, 4.4, 5.1 and 6.1.

PROPOSITION 6.2. *Let N_j $(j=1, 2, 3)$ be the Nielsen transformations. Then the following hold*:

(1) $N_1(M_l(\delta(n_k, \cdots, n_0)))=M_l(-\delta(n_k, \cdots, n_0))$ *for* $l=$ II, V, VII, *where* $M_{\mathrm{VII}}(0)=M_{\mathrm{VII}}(-0)$.

(2) (i) $N_2(M_{\mathrm{II}}(\delta(n_k, \cdots, n_0)))=M_{\mathrm{V}}(\delta(1, n_k, \cdots, n_0))$.

(ii) $N_2(M_{\mathrm{V}}(\delta(n_k, \cdots, n_0))=M_{\mathrm{II}}(\delta(1, n_k, \cdots, n_0))$.

(iii) $N_2(M_{\mathrm{VII}}(\delta(n_k, \cdots, n_0)))=M_{\mathrm{VII}}(\delta(1, n_k, \cdots, n_0))$.

(3) (i) $N_3^{\delta}(M_{\mathrm{II}}(-\delta))=M_{\mathrm{II}}(\delta)$,

$N_3^{\delta}(M_{\mathrm{II}}(\delta(n_0))=M_{\mathrm{II}}(\delta(n_0+1))$ $\quad (n_0=\pm 1, \pm 2, \cdots)$.

(ii) $N_3^{\delta}(M_{\mathrm{V}}(-\delta))=M_{\mathrm{VII}}(0)$,

$N_3^{\delta}(M_{\mathrm{V}}(\delta(2n_0-1)))=M_{\mathrm{VII}}(\delta(2n_0))$ $\quad (n_0=\pm 1, \pm 2, \cdots)$.

(iii) $N_3^{\delta}(M_{\mathrm{VII}}(\delta(2n_0))=M_{\mathrm{V}}(\delta(2n_0+1))$ $\quad (n_0=0, \pm 1, \pm 2, \cdots)$.

(4) (i) $N_3^{\delta}(M_{\mathrm{II}}(\delta(n_k, n_{k-1}, \cdots, n_0)))$
$=M_{\mathrm{II}}(\delta(n_k+1, n_{k-1}, \cdots, n_0))$ $\quad (n_k=0, \pm 1, \pm 2, \cdots)$.

$N_3^{\delta}(M_{\mathrm{II}}(-\delta(n_k, n_{k-1}, \cdots, n_0)))$
$=M_{\mathrm{II}}(\delta(1, 2, 0, n_k, n_{k-1}, \cdots, n_0))$ $\quad (n_k=0, \pm 1, \pm 2, \cdots)$.

(ii) $N_3^{\delta}(M_{\mathrm{V}}(\delta(n_k, n_{k-1}, \cdots, n_0)))$
$=M_{\mathrm{VII}}(\delta(n_k+1, n_{k-1}, \cdots, n_0))$ $\quad (n_k=0, \pm 1, \pm 2, \cdots)$.

$N_3^{\delta}(M_{\mathrm{V}}(-\delta(n_k, n_{k-1}, \cdots, n_0)))$
$=M_{\mathrm{VII}}(\delta(1, 2, 0, n_k, n_{k-1}, \cdots, n_0))$ $\quad (n_k=0, \pm 1, \pm 2, \cdots)$.

(iii) $N_3^{\delta}(M_{\mathrm{VII}}(\delta(n_k, n_{k-1}, \cdots, n_0)))$
$=M_{\mathrm{V}}(\delta(n_k+1, n_{k-1}, \cdots, n_0))$ $\quad (n_k=0, \pm 1, \pm 2, \cdots)$.

$N_3^{\delta}(M_{\mathrm{VII}}(-\delta(n_k, n_{k-1}, \cdots, n_0)))$
$=M_{\mathrm{V}}(\delta(1, 2, 0, n_k, n_{k-1}, \cdots, n_0))$ $\quad (n_k=0, \pm 1, \pm 2, \cdots)$.

6.3 Noting that $R_{\mathrm{II}}\mathfrak{S}_2^0=(\bigcup_\phi \phi(R_{\mathrm{II}}\mathfrak{S}_2^{00}))\cap \boldsymbol{R}_{\mathrm{II}}^3$, $R_{\mathrm{V}}\mathfrak{S}_2^0=(\bigcup_\phi \phi(R_{\mathrm{V}}\mathfrak{S}_2^{00}))\cap \boldsymbol{R}_{\mathrm{V}}^3$ and $R_{\mathrm{VII}}\mathfrak{S}_2^0=(\bigcup_\phi \phi(R_{\mathrm{VII}}\mathfrak{S}_2^{00})\cap \boldsymbol{R}_{\mathrm{VII}}^3$, where ϕ runs through the Schottky modular group of genus two $\mathrm{Mod}(\mathfrak{S}_2)$, we have the following theorem by Theorem 2 and Proposition 6.2.

THEOREM 3.

$$R_{\mathrm{II}}\mathfrak{S}_2^0=\bigcup M_{\mathrm{II}}(n_k, n_{k-1}, \cdots, n_0)\ ,$$

$$R_{\mathrm{V}}\mathfrak{S}_2^0=\bigcup M_{\mathrm{V}}(n_k, n_{k-1}, \cdots, n_0)$$

and

$$R_{\mathrm{VII}}\mathfrak{S}_2^0=\bigcup M_{\mathrm{VII}}(n_k, n_{k-1}, \cdots, n_0)\,,$$

where $M_l(n_k, n_{k-1}, \cdots, n_0)$ $(l=\mathrm{II}, \mathrm{V}, \mathrm{VII})$ *are as defined in* §6.2.

7. Fundamental regions.

7.1. In this section we will determine fundamental regions for $[\Phi_2]$ and $\mathrm{Mod}(\mathfrak{S}_2)$ acting on $R_{\mathrm{II}}\mathfrak{S}_2^0$, $R_{\mathrm{V}}\mathfrak{S}_2^0$ and $R_{\mathrm{VII}}\mathfrak{S}_2^0$, respectively. We denote by $\mathrm{Mod}(R_l\mathfrak{S}_2^0)$ (resp. $[R_l\Phi_2]$) the restriction of $\mathrm{Mod}(\mathfrak{S}_2^0)$ (resp. $[\Phi_2]$) to $R_l\mathfrak{S}_2^0$, that is, the set of all equivalence classes of orientation preserving automorphisms (resp. the set of all equivalence classes of automorphisms) in $R_l\mathfrak{S}_2^0$ for $l=\mathrm{II}, \mathrm{V}, \mathrm{VII}$.

Throughout this section, let N_j $(j=1, 2, 3)$ be the Nielsen transformations. We denote by $[\phi]$ the equivalence class of $\phi\in\Phi_2$. We write ϕ for an element $[\phi]$ in $[\Phi_2]$ or $\mathrm{Mod}(\mathfrak{S}_2)$ when there is no fear of confusion. We denote by $W(\phi_1, \phi_2, \cdots, \phi_n)$ a word in $\phi_1, \phi_2, \cdots, \phi_n$. We denote by $SW(\phi_1, \phi_2, \cdots, \phi_n)$ (resp. $S[W(\phi_1, \phi_2, \cdots, \phi_n)]$) the set of all words in $\phi_1, \phi_2, \cdots, \phi_n$ (resp. the set of all equivalence classes of words in $\phi_1, \phi_2, \cdots, \phi_n$).

We easily see the following two lemmas.

LEMMA 7.1. *If* $\phi\in\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0)$ *(resp.* $\phi\in[R_{\mathrm{V}}\Phi_2]$*), then* $N_2\phi N_2\in\mathrm{Mod}(R_{\mathrm{II}}\mathfrak{S}_2^0)$ *(resp.* $N_2\phi N_2\in[R_{\mathrm{II}}\Phi_2]$*) and* $N_3^{-1}\phi N_3\in\mathrm{Mod}(R_{\mathrm{VII}}\mathfrak{S}_2^0)$ *(resp.* $N_3^{-1}\phi N_3\in[R_{\mathrm{VII}}\Phi_2]$*).*

LEMMA 7.2. (1) *If* $\psi\in\mathrm{Mod}(R_{\mathrm{II}}\mathfrak{S}_2^0)$ *(resp.* $\psi\in[R_{\mathrm{II}}\Phi_2]$*), then there exists* $\phi\in\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0)$ *(resp.* $\phi\in[R_{\mathrm{V}}\Phi_2]$*) with* $\psi=N_2\phi N_2$.

(2) *If* $\psi\in\mathrm{Mod}(R_{\mathrm{VII}}\mathfrak{S}_2^0)$ *(resp.* $\psi\in[R_{\mathrm{VII}}\Phi_2]$*), then there exist* $\phi\in\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_0^2)$ *(resp.* $\phi\in[R_{\mathrm{V}}\Phi_2]$*) with* $\psi=N_3^{-1}\phi N_3$.

PROPOSITION 7.1. (1) $\mathrm{Mod}(R_{\mathrm{II}}\mathfrak{S}_2^0)=N_2(\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0))N_2$ *and* $[R_{\mathrm{II}}\Phi_2]=N_2[R_{\mathrm{V}}\Phi_2]N_2$.

(2) $\mathrm{Mod}(R_{\mathrm{VII}}\mathfrak{S}_2^0)=N_3^{-1}(\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0))N_3$ *and* $[R_{\mathrm{VII}}\Phi_2]=N_3^{-1}[R_{\mathrm{V}}\Phi_2]N_3$.

PROOF. This follows from Lemmas 7.1 and 7.2. q.e.d.

By straightforward calculations, we have:

LEMMA 7.3. (1) $N_1(R_{\mathrm{V}}\mathfrak{S}_2^0)=R_{\mathrm{V}}\mathfrak{S}_2^0$.

(2) $(N_2W(N_1, N_3)N_2)(R_{\mathrm{V}}\mathfrak{S}_2^0)=R_{\mathrm{V}}\mathfrak{S}_2^0$.

(3) $(N_3^{\pm 1}W(N_1, N_2)N_3^{\pm 1})(R_{\mathrm{V}}\mathfrak{S}_2^0)=R_{\mathrm{V}}\mathfrak{S}_2^0$.

PROPOSITION 7.2. *The set* $[R_{\mathrm{V}}\Phi_2]$ *consists of all equivalence classes of words in* N_1, $N_2W_\alpha N_2$, $N_3^{\pm 1}W_\beta N_3^{\pm 1}$ *with* $W_\alpha\in SW(N_1, N_3)$, $W_\beta\in SW(N_1, N_2)$.

PROOF. This follows from Lemma 7.3. q.e.d.

LEMMA 7.4. *The group* $\{[N_2W_\alpha N_2] \mid W_\alpha \in SW(N_1, N_3)\}$ *is generated by* $[N_1]$ *and* $[N_2N_3N_2]$.

PROOF. Let $W(N_1, N_3)=\phi_1\phi_2\cdots\phi_n$, where ϕ_j $(j=1, 2, \cdots, n)$ are N_1 or N_3. By noting that $N_2^2=1$, we have $N_2W(N_1, N_3)N_2=N_2\phi_1\phi_2\cdots\phi_nN_2=N_2\phi_1N_2N_2\phi_2N_2\cdots N_2\phi_nN_2$. Since $N_2N_1N_2\sim N_1$, we have the desired result. q.e.d.

LEMMA 7.5. (1) *The group* $\{[N_3W_\alpha N_3] \mid W_\alpha \in SW(N_1, N_2)\}$ *is generated by* $[N_1N_3^{-2}N_1]$ $(=[N_3^{-2}])$, $[N_3N_1N_3]$ $(=[N_1])$ *and* $[N_3N_2N_3]$.

(2) *The group* $\{[N_3^{-1}W_\alpha N_3] \mid W_\alpha \in SW(N_1, N_2)\}$ *is generated by* $[N_3^{-1}N_1N_3]$ *and* $[N_3^{-1}N_2N_3]$.

PROOF. (1) First we note that $W(N_1, N_2)$ is 1, N_1, N_2 or N_1N_2 $(\sim N_2N_1)$. Since $N_31N_3=N_3^2$, $N_3N_1N_3\sim N_1$, and $N_3N_1N_2N_3=N_3N_1N_3N_3^{-2}N_3N_2N_3$, we have the desired result.

(2) Since $N_3^{-1}1N_3=1$ and $N_3^{-1}N_1N_2N_3=N_3^{-1}N_1N_3N_3^{-1}N_2N_3$, we have the desired result. q.e.d.

7.3. PROPOSITION 7.3. (1) $[R_{\mathrm{V}}\Phi_2]$ *is generated by* $[N_1]$, $[N_3^2]$, $[N_2N_3N_2]$ *and* $[N_3^{-1}N_1N_3]$.

(2) $\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0)$ *is generated by* $[N_3^2]$ *and* $[N_2N_3N_2]$.

PROOF. (1) First we note that $N_3N_1N_3=N_3^2N_3^{-1}N_1N_3$. Since $N_2N_3N_2\sim N_3N_1N_2N_3=N_3N_1N_3N_3^{-2}N_3N_2N_3$ by Lemma 1.2, (6) we have $N_3N_2N_3\sim N_3^2(N_3N_1N_3)^{-1}(N_2N_3N_2)$. Hence noting that $N_3^{-1}N_2N_3=N_3^{-2}N_3N_2N_3$ and $N_3W(N_1, N_2)N_3^{-1}=N_3^2(N_3^{-1}W(N_1, N_2)N_3)N_3^{-2}$, we have the desired result by Proposition 7.2, and Lemmas 7.4 and 7.5.

(2) We have the following by Lemma 1.2: (i) $N_1^2=1$, $N_3^{-1}N_1N_3N_1\sim N_3^{-2}$, $N_1N_3^{-1}N_1N_3\sim N_3^2$, $(N_3^{-1}N_1N_3)^2=1$; (ii) $N_1N_3^2N_1\sim N_3^{-2}$, $N_1N_3^2N_3^{-1}N_1N_3\sim 1$, $N_1N_2N_3N_2N_1\sim(N_2N_3N_2)^{-1}$, $N_1N_2N_3N_2N_3^{-1}N_1N_3\sim(N_2N_3N_2)^{-1}N_3^2$, $N_3^{-1}N_1N_3N_3^2N_1\sim N_3^{-4}$, $N_3^{-1}N_1N_3N_3^2N_3^{-1}N_1N_3\sim N_3^{-2}$, $N_3^{-1}N_1N_3N_2N_3N_2N_1\sim N_3^{-2}(N_2N_3N_2)^{-1}$, $N_3^{-1}N_1N_3N_2N_3N_2N_3^{-1}N_1N_3\sim N_3^{-2}(N_2N_3N_2)^{-1}N_3^2$.

Since an element of $\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0)$ is an orientation preserving automorphism, the cardinality of totality of N_1 and $N_3^{-1}N_1N_3$ contained in each element of $\mathrm{Mod}(R_{\mathrm{V}}\mathfrak{S}_2^0)$ is even. Therefore by noting $N_1^2=1$, we have the desired result. q.e.d.

PROPOSITION 7.4. (1) (i) $[R_{\mathrm{II}}\Phi_2]$ *is generated by* $[N_1]$, $[N_2N_3^2N_2]$ *and* $[N_3]$.

(ii) $\mathrm{Mod}(R_{\mathrm{II}}\mathfrak{S}_2^0)$ *is generated by* $[N_3]$ *and* $[N_2N_3^2N_2]$.

(2) (i) $[R_{\mathrm{VII}}\Phi_2]$ *is generated by* $[N_1]$, $[N_1N_2]$ *and* $[N_3^2]$ $(=[N_1N_3^{-2}N_1])$.

(ii) $\mathrm{Mod}(R_{\mathrm{VII}}\mathfrak{S}_2^0)$ *is generated by* $[N_3^2]$ *and* $[N_1N_2]$.

PROOF. (1) (i) First we note that $N_2(N_3^{-1}N_1N_3)N_2=N_2N_3^{-2}N_3N_1N_3N_2=N_2N_3^{-2}N_2N_2N_3N_1N_3N_2\sim(N_2N_3^2N_2)^{-1}N_1$. Since $N_2N_1N_2\sim N_1$ and $N_2(N_2N_3N_2)N_2=N_3$, we have the desired result by Propositions 7.1, (1) and 7.3, (1).

(ii) is seen by Propositions 7.1, (1) and 7.3, (2).

(2) (i) We have $N_3^{-1}N_1N_3 \sim N_1N_3N_3 = N_1N_3^2$; $N_3^{-1}N_3^2N_3 = N_3^2$; $N_3^{-1}(N_2N_3N_2)N_3 \sim N_3^{-1}(N_3N_1N_2N_3)N_3 = N_1N_2N_3^2$; $N_3^{-1}(N_3^{-1}N_1N_3)N_3 = N_3^{-2}N_1N_3^2$. Therefore noting that $N_3\phi N_3 = N_3^2N_3^{-1}\phi N_3$ for $\phi \in [R_V\Phi_2]$, we have the desired result by Propositions 7.1, (2) and 7.3, (1).

(ii) is a consequence of Propositions 7.1, (2) and 7.3, (2). q.e.d.

7.4. We set

$$B_1 = \{(t_1, t_2, \rho) \in M_{VII}(0) \mid t_2 = t_1, \rho \leqq -1\}$$

$$B_2 = \{(t_1, t_2, \rho) \in M_{VII}(0) \mid t_2 = t_1, \rho \geqq -1\} .$$

By Lemma 1.1, (1), we have:

LEMMA 7.6. $N_1N_2(B_1) = B_2$.

LEMMA 7.7. *The set* $F_{VII}(\mathfrak{S}_2^0) = \{\tau = (t_1, t_2, \rho) \in M_{VII}(0) \mid t_2 < t_1\}$ *is a fundamental region in* $M_{VII}(0)$ *for the group* $\langle N_1N_2 \rangle$ *generated by* N_1N_2.

Proof. Since $M_{VII}(0) \subseteq \{F_{VII}(\mathfrak{S}_2^0)\}^- \cup \{N_1N_2(F_{VII}(\mathfrak{S}_2^0))\}^-$ and $N_1N_2(F_{VII}(\mathfrak{S}_2^0)) \cap F_{VII}(\mathfrak{S}_2^0) = \varnothing$, we have the desired result, where $\{S\}^-$ denotes the closure of a set S. q.e.d.

PROPOSITION 7.5. $F_{VII}(\mathfrak{S}_2^0)$ *is a fundamental region in* $R_{VII}\mathfrak{S}_2^0$ *for* $\mathrm{Mod}_{VII}(\mathfrak{S}_2^0)$.

PROOF. This follows from Lemma 7.7, Proposition 7.4 (2), (ii) and Corollary (2) to Proposition 4.1. q.e.d.

PROPOSITION 7.6. *The set* $F_{VII}([\Phi_2]) = \{(t_1, t_2, \rho) \in M_{VII}(0) \mid t_2 < t_1, \rho < -1\}$ *is a fundamental region in* $R_{VII}\mathfrak{S}_2^0$ *for* $[R_{VII}\Phi_2]$.

PROOF. Set $S := N_1N_3^{-2}N_1$ $(\sim N_3^2)$ and $T := N_1N_2$. Then $\mathrm{Mod}_{VII}(\mathfrak{S}_2^0)$ (resp. $[R_{VII}\Phi_2]$) is generated by S and T (resp. S, T and N_1) by Proposition 7.4, (2). Since

$$F_{VII}(\mathfrak{S}_2^0) = N_1(F_{VII}([\Phi_2])) \cup F_{VII}([\Phi_2]) \cup (\{(t_1, t_2, \rho) \mid \rho = -1\} \cap F_{VII}(\mathfrak{S}_2^0)) ,$$

we have the desired result. q.e.d.

We denote by $F_V(\mathfrak{S}_2^0)$ the set in $M_V(1)$ bounded by $F_V^+(1)$ and $F_V(1/2)$, where $F_V(1/2)$ is the set

$$\{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2 = (1 - \rho t_1)/(\rho - t_1), 1 < \rho < 1/t_1^2, -1 < t_1 < 0\} .$$

Set $F_V^* = N_3(\{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2 = t_1\} \cap R_{VII})$. Let $F_V([\Phi_2])$ be the set in $M_V(1)$ bounded by $F_V^+(1)$, $F_V(1/2)$ and F_V^*. Similarly we define the sets $F_{II}(\mathfrak{S}_2^0)$ and $F_{II}([\Phi_2])$ as follows: $F_{II}(\mathfrak{S}_2^0)$ is the set in $M_{II}(1)$ bounded by $F_{II}^+(1)$ and $F_{II}(1/2)$, where

$$F_{II}(1/2) = \{(t_1, t_2, \rho) \in \boldsymbol{R}^3 \mid t_2 = (1 - \rho t_1^{1/2})/(\rho - t_1^{1/2}), 1 < \rho < 1/t_1, 0 < t_1 < 1\} ;$$

$F_{II}([\Phi_2])$ is the set in $M_{II}(1)$ bounded by $F^+_{II}(1)$, $F_{II}(1/2)$ and $F^*_{II}=N_2(F^*_V)$. Then we have the following:

THEOREM 4. (1) *For each* $l=$ II, V, VII, $F_l(\mathfrak{S}^0_2)$ *is a fundamental region in* $R_l\mathfrak{S}^0_2$ *for* $\mathrm{Mod}(R_l\mathfrak{S}^0_2)$.

(2) *For each* $l=$ II, V, VII, $F_l([\Phi_2])$ *is a fundamental region in* $R_l\mathfrak{S}^0_2$ *for* $[R_l\Phi_2]$.

PROOF. The case $l=$ VII follows from Propositions 7.5 and 7.6. We only prove this theorem for $l=$ V, since the proof for $l=$ II is similar.

We denote by $F^*_{VII}(\mathfrak{S}^0_2)$ the set in R_{VII} bounded by $F^+_{VII}(0)$ and $\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid \rho=-1\}\cap M_{VII}(0)$. Then $F^*_{VII}(\mathfrak{S}^0_2)$ is also a fundamental region in $R_{VII}\mathfrak{S}^0_2$ for $\mathrm{Mod}(R_{VII}\mathfrak{S}^0_2)$, since the image of the set $F_{VII}(\mathfrak{S}^0_2)\cap\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid \rho\geqq -1\}$ under the mapping N_1N_2 is the set in $M_{VII}(0)$ bounded by $\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid \rho=-1\}$, $\{(t_1, t_2, \rho)\in \boldsymbol{R}^3 \mid t_2=t_1\}$, $F^+_{VII}(0)$ and $H_{VII}(0)$. Since $N_3(F^*_{VII}(\mathfrak{S}^0_2))=F_V(\mathfrak{S}^0_2)$, $F_V(\mathfrak{S}^0_2)$ is a fundamental region in $R_V\mathfrak{S}^0_2$ for $\mathrm{Mod}(R_V\mathfrak{S}^0_2)$ by Proposition 7.1, (2). We can similarly prove that $F_V([\Phi_2])$ is a fundamental region in $R_V\mathfrak{S}^0_2$ for $[R_V\Phi_2]$. q.e.d.

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